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THE SUPPORT OF MATHEMATICAL RESEARCH BY THE NATIONAL SCIENCE FOUNDATION

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1. **The present terms of support of mathematics by the Foundation.** The writer has recently completed a short term as Acting Program Director for Mathematics in the National Science Foundation. One residue of this experience is the realization that the National Science Foundation Act of 1950 did not in itself create a new era in which a non-military arm of the Government will support *basic* research in theoretical mathematics without the demand of direct military or even physical science relatedness. It will take a great deal of building within the framework of the Act to realize this objective. The mathematicians, who are without doubt the poorest politicians in the world, do not seem to realize that a non-military arm of the Government must show the actual need and must justify the requests for appropriations for mathematics in itself instead of asking for appropriations for the defense of the country, a much stronger appeal. Hence we must express our needs for support from the National Science Foundation in the form of proposals. These can be submitted to more than one agency if both agencies are so informed.

The submission of these proposals should be regarded as a form of public service or service to mathematics. Inevitably some good proposals will have to be rejected until adequate appropriations are secured to permit the support of mathematics in ways not now covered by the military agencies. We must accept this outcome without wounded feelings, for the principles which govern the decisions on what to support and what not to support, or even what can be supported, are not formulated. Moreover one cannot expect a full explanation of the reasons for rejection of any particular proposal while another is granted. This is too difficult.

To date, the National Science Foundation has granted a number of predoctoral and postdoctoral fellowships in mathematics, has made a few individual research grants and one group research grant, has provided for the support of a Summer Research Institute in mathematics, has set up a provision for assisting in travel to foreign scientific meetings which has benefitted some mathematicians and has made an arrangement with the National Bureau of Standards for making high-speed digital computation available for non-military research in universities. It is supporting a survey, jointly with other agencies, of Applied Mathematics. It is also considering the support of mathematics publication, a research conference in mathematics, and a Summer Institute for college teachers of mathematics.

2. **The individual project.** To submit a proposal for individual research, which is ordinarily granted only to scientists of proved ability in research, one should describe the proposed research in such a way that it can be understood by reviewers who are expert in the field of the investigation. The proposal must be submitted through the proper administrative channels in the institution. The budget ordinarily calls for the payment of the academic salary of the principal investigator and his junior associates, who may be either predoctoral or postdoctoral research men. This mechanism provides for the support of young postdoctoral research men when they can be associated with some senior investigator who applies for the grant. The senior investigator need not participate in the budget himself; in fact he usually does not do so in experimental science grants. An Assistant Professor who wishes to apply for full time research at some large center would do well to apply from his own university if he comes from a smaller school, rather than from the place where he wants to work. These forms of individual research grants can be used to achieve somewhat the same objectives as postdoctoral fellowships, which are now few in number. Fiscal details are described in a "Guide for the Submission of Research Proposals" obtainable from the National Science Foundation, Mathematical, Physical and Engineering Sciences Division, Washington 25, D.C.

3. **A plan for a continuing group research project.** Next, I should like to describe a kind of continuing group project which can be used to support a department active in graduate work and research without asking for a "block grant," which would almost certainly be rejected. This group project is designed primarily to support a secondary center which does not already have substantial Government support in the form of fellows, individual grants or special contracts. It may be expected that, to be approved, such a proposal for a group research contract must come from a department which can show evidence of initiative from within and achievements in research prior to submitting the proposal; especially it must have *students* and a proper environment with some assurance of stability and of support by its own university. The proposal should describe the contributions of the university in the form of tenure appointments to men of professorial rank (still a contribution little appreciated by government administrators whose grants and contracts cannot provide tenure), graduate teaching, libraries, secretarial staff, office space, etc.

A mathematical subject of the proposal for a group research grant is chosen so as to be fundamental enough to lie in the core of graduate training of any mathematician and broad enough to cover the researches of a small group of mature mathematicians with their junior research associates. Normally, these are recent Ph.D.s or men who have passed the preliminary examination for the doctorate, but don't call them fellows or students! The work is characterized by research seminars rather than by the traditionally solitary studies of the mathematician. Formal courses are, of course, outside the project. The proposal will describe the general methods of study so that the project can be reviewed for scientific merit in the same way as an individual proposal. Once approved, the continuation of the grant cannot legally be guaranteed and will undoubtedly depend upon the achievements of the group. Thus the judgment of the consultants remains as an assurance of the quality of the performance.

It is not unrealistic to conceive that there might be ten to twenty such group research projects in the United States, supported by the Foundation or ONR, or perhaps other military agencies, in departments which are doing basic graduate training, each having an annual budget of the order of $30,000. (For a large department a budget of $5,000 is more in keeping with present limitations.) This might include $12,000 for summer research, $12,000 for research associates during the regular year and $6,000 for travel, preparation of manuscripts, consultants' expenses for short-term visitors who may lecture, as well as "consult," publication costs *by all means*, and "indirect costs" not exceeding 15 per cent of the total. The publication item might cover estimated page costs of publication done under the project.

4. Other possibilities. The National Science Foundation has a clear charge to support basic research and it may work by *grants* instead of contracts. This liberalization of principle in comparison with military agencies could be explored to determine if there are not other good ways to promote the welfare of mathematical research and training, and the service of mathematics to the sciences with or without grants of funds. Some things which are now difficult, if not impossible, are: support of the writing of books to be published commercially, non-scientific equipment or books for universities, extra jobs of research which will increase the rate of pay of the scientist, foreign travel except to foreign scientific meetings, sabbaticals (in the free sense), and any project calling for support in perpetuo.

These suggestions are by no means an official statement from the National Science Foundation and may disagree with its policies. The

GRADIENT MAPPINGS

E. H. ROTHE

1. **Introduction.** A gradient field in a finite-dimensional Euclidean space is a field of vectors $g(x)$ for which there exists a scalar function $I(x)$ such that $g(x) = \text{grad } I(x)$. A classical example is a conservative force field in which case $I(x)$ is the potential. It has been known for a long time that the treatment of such fields offers a considerable number of simplifications and special properties not shared by arbitrary vector fields. On the other hand those boundary value and integral equation problems which can be derived from variational problems likewise offer simplifications and special properties not shared by more general problems of this type. Now the relation between an integral in the calculus of variations which is to be made an extremum and the corresponding Euler-Lagrange equation, or rather the operator given by the Euler-Lagrange expression, is quite analogous to the relation between the scalar $I(x)$ and its gradient. It seems therefore reasonable to expect that with a proper definition of the term gradient one will be able to obtain a theory which encompasses the finite-dimensional as well as the function space case.[1]

2. **The definition of the term gradient mapping.** If $I(x)$ is a differentiable scalar defined for points $x = (x_1, x_2, \cdots, x_n)$ of a Euclidean n-space E, then the differential dI corresponding to the increment $h = (h_1, h_2, \cdots, h_n)$ is given by

$$dI = \sum_{\nu=1}^{n} \partial I/\partial x_\nu h_\nu = (\text{grad } I, h) \tag{2.1}$$

where the parentheses indicate the scalar product. Thus $g(x) = \text{grad } I(x)$ assigns to the point x of E the linear form $l_x(h) = dI = (g(x), h)$, that is, an element of the conjugate space E^* of E. This remark motivates the following definition.

DEFINITION 2.1. Let E be a real Banach space.[2] Let $I = I(x)$ be a scalar function defined in E or part of E. Suppose that I possesses a continuous Fréchet differential, that is, that there exists a linear

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[1] The use of gradients in Hilbert space for the treatment of functional equations seems to occur first in [9]. (Numbers in brackets refer to the bibliography at the end of the paper.)

[2] For the definition of a Banach space see, for example, [14, p. 10].

bounded functional of h, $l_x(h) = D(x, h)$, depending continuously on x such that

$$I(x + h) - I(x) = D(x, h) + R(x, h) \tag{2.2}$$

where with the usual notation $\|h\|$ for the norm of h

$$\lim_{\|h\| \to 0} R(x, h)/\|h\| = 0. \tag{2.2'}$$

If E^* denotes the conjugate space [14, p. 21] of E, then the Fréchet differential $D(x, h) = l_x(h)$ induces a mapping of E (or part of E) into E^* which maps the point x of E into the point $l_x(h) = l_x$ of E^*. Any mapping of E (or part of E) into E^* which is thus induced by the Fréchet differential of a scalar is called a gradient mapping.

If $l_x = l_x(h)$ is a given map of E (or part of it) into E^* which has a continuous Fréchet differential (with respect to x) $\delta_x(h, k)$, then the symmetry of the bilinear form (in h and k) $\delta_x(h, k)$ is a necessary and sufficient condition that l_x is a gradient map. This follows immediately from a theorem by Kerner [15, Satz 2′].

If E is especially a Hilbert space,[3] it is well known that to a given linear bounded functional $l(h)$ there exists a uniquely determined element g of E such that $l(h) = (g, h)$, and that one may consider E as identical with E^* by identifying l with g. In this sense we consider in a Hilbert space as the gradient of $I(x)$ the element $g(x)$ of E which is uniquely determined by

$$D(x, h) = l_x(h) = (g(x), h).$$

Thus $dI = (g(x), h)$ in obvious generalization of (2.1).

If $G(x)$ is a given map of the Hilbert space E or part of it into itself which has a continuous Fréchet differential (with respect to x) $L_x(k)$, then a necessary and sufficient condition that $G(x)$ is a gradient is that the bounded linear operator (in k)$L_x(k)$ be symmetric, that is, that $(L_x(k), h) = (k, L_x(h))$. This is a nearly immediate consequence[4] of the criterion for general Banach spaces stated above. In the special case that the given $G(x)$ is a bounded linear operator, its Fréchet differential $L_x(k) = G(k)$. Thus the necessary and sufficient condition for a bounded linear operator to be a gradient is its symmetry.[5] From this point of view gradients appear as a natural generalization of symmetric linear operators.[6]

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[3] A Hilbert space is a Banach space in which for any two elements x, y a scalar (x, y) satisfying the usual rules is defined such that $(x, x)^{1/2}$ is the norm of x.

[4] See [26, Lemma 2.3]; compare also [22, Lemma 3.1 and the "Remark to Lemma 3.1," p. 585], where slightly different definitions were used.

[5] For the treatment of eigenvalue and boundary value problems connected with such operators, we refer to the classical treatment in [7].

3. **Examples.** We list a few examples of scalars and their gradients. To most of these examples we shall have to refer later on.

EXAMPLE 1. Let $A(x)$ be a linear bounded symmetric operator defined in a Hilbert space E, and let

$$(3.1)\qquad I(x) = (1/2)(A(x), x).$$

Then grad $I = A(x)$. Note that the remainder term $R(x, h)$ of (2.2) becomes in our case

$$(3.2)\qquad R(x, h) = (A(h), h)/2$$

and therefore due to the supposed boundedness of A certainly satisfies (2.2'). As a special case we mention

$$(3.3)\qquad I(x) = \frac{1}{2}\int_D \int_D K(s, t)x(s)x(t)dsdt,$$

$$(3.4)\qquad \text{grad}\, I = A(x) = \int_D K(s, t)x(t)dt \qquad (K(s, t) = K(t, s)),$$

where D is a bounded domain in a Euclidean n-space, $s = (s_1, s_2, \cdots, s_n)$ a point of D, and the integration is extended over D. The Hilbert space E is the space of all functions $x(s)$ square integrable over D. Under certain assumptions about the kernel $K(s, t)$ the symmetric operator (3.4) is indeed bounded.[7] Under somewhat more restrictive conditions, e.g., if

$$(3.5)\qquad \int_D \int_D K(s, t)^2 dsdt < \infty,$$

the operator is completely continuous.[8]

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[6] Thus, the operators $G(x)$ in a Hilbert space called by Lusternik [17] symmetric and homogeneous of order $p-1$ are in the terminology of the present paper gradients for which the corresponding scalars $I(x)$ are homogeneous, that is, satisfy equation (4.5) of the present paper. In generalization of the classical case of linear positive definite symmetric operators ($p=2$), the existence of infinitely many positive eigenvalues λ_i with $\lambda_1 \geqq \lambda_2 \geqq \lambda_3 \geqq \cdots$ and $\lim_{i\to\infty} \lambda_i = 0$ is proved in [17] for operators G which are homogeneous with even p, completely continuous and positive, that is, for which $(G(x), x) > 0$ for $x \neq 0$. Soboleff [32] obtains a similar result for symmetric, completely continuous, positive definite Hilbert space operators $G(x)$ with $G(-x) = -G(x)$.

[7] See, for example, [30, pp. 32 and 36].

[8] A map of a Banach space E (or part of it) into the Banach space E_1 (which may coincide with E) is called completely continuous if (i) it is continuous, and (ii) the image of every bounded set is compact in E_1 (in the sense of the definition given in [2, p. 84]).

EXAMPLE 1a. With the same notations as in example 1, let $A(x)$ be an unbounded linear symmetric operator defined in a linear subset of the Hilbert space E. In this case we can no longer conclude that the remainder term (3.2) satisfies (2.2′), and therefore $A(x)$ is in general not the gradient of the scalar (3.1). Consider, for example, the linear symmetric operator

$$A(y) = B(y) + y \tag{3.6}$$

with

$$B(y) = -\sum_{\nu=1}^{n} \frac{\partial}{\partial s_\nu}\left(p\,\frac{\partial y}{\partial s_\nu}\right). \tag{3.7}$$

Here $y = y(s)$ is a function of the point $s = (s_1, s_2, \cdots, s_n)$ of the domain D which vanishes on the boundary of D and for which the derivatives occurring in B exist and are continuous while $p = p(s)$ is a given positive differentiable function of s. Application of Green's formula shows that the scalar (3.1) becomes the generalized Dirichlet integral

$$\frac{1}{2}\,(A(y), y) = \frac{1}{2}\int_D \left\{p \sum_{\nu=1}^{n} (\partial y/\partial s_\nu)^2 + y^2\right\} ds. \tag{3.8}$$

EXAMPLE 2. Let E be the space of all functions $x = x(t)$ which together with their first derivative are defined and continuous in $0 \leqq t \leqq 1$. With the norm

$$\|x\| = \max |x(t)| + \max |x'(t)|$$

E becomes a Banach space [4, pp. 11 and 54]. Let

$$I(x) = \int_0^1 f(t, x(t), x'(t))dt.$$

Then under proper assumptions about the given function f the Fréchet differential dI corresponding to the increment $h = h(t)$ is

$$\begin{aligned} l_x(h) &= \int_0^1 \{h(t) f_x(t, x(t), x'(t)) + h'(t) f_{x'}(t, x(t), x'(t))\} dt \\ &= h(t) f_{x'}(t, x(t), x'(t)) \Big|_{t=0}^{t=1} + \int_0^1 h(t)\left\{-\frac{d}{dx} f_{x'} + f_x\right\} dt. \end{aligned}$$

This is a linear functional on E, i.e., an element of E^* and the gradient mapping $g(x)$ is the map which maps $x = x(t)$ into the element $l_x(h)$ of E^*.

EXAMPLE 3. The Hammerstein scalar. Let D, $x(s)$, and $K(s, t)$ have the same meaning as in example 1. Moreover we suppose that $K(s, t)$ satisfies (3.5), is symmetric and positive definite. Finally let $f(t, u)$ be a continuous function defined for $t \in D$ and for $-\infty < u < \infty$. The Hammerstein scalar $i = i(x)$ is then defined by

$$i(x) = \frac{1}{2}\int_D \int_D K(s, t)x(s)x(t)ds dt + I(x) \tag{3.9}$$

where

$$I(x) = \int_D \left\{ \int_0^{y(t)} f(t, u) du \right\} dt \tag{3.10}$$

with

$$y(t) = \int_D K(s, t)x(s)ds. \tag{3.11}$$

The significance of this scalar lies in its connection with the so-called nonlinear integral equation of the Hammerstein type

$$y(s) + \int_D K(s, t)f(t, y(t))dt = 0. \tag{3.12}$$

It can be shown[9] that grad $i(x)$ equals the left member of (3.12) (with y expressed in terms of x by means of (3.11)) if in addition to the above assumptions concerning K and f at least one of the following conditions (A), (B) is satisfied:

(A) $|K(s, t)|$ is bounded;

(B) $|f|$ is dominated by a linear function of u with coefficients independent of t.

For later reference we note that if f satisfies (B) and has in addition a continuous partial derivative with respect to u, and if $d^2(x, h, h_1)$ denotes the second Fréchet differential of $i(x)$ corresponding to the increments $h = h(t)$ and $h_1 = h_1(t)$, then

$$\begin{aligned} d^2(x, h, h) = {} & \frac{1}{2}\int_D \int_D K(s, t)h(s)h(t)ds dt \\ & + \int_D \int_D \partial f/\partial u(t, y(t)) \left\{ \int_D K(s, t)h(s)ds \right\}^2 dt, \end{aligned} \tag{3.13}$$

where again $y(t)$ is given by (3.11) [26, equation (7.24)].

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[9] [25, Theorem 3.1]. See [25] also for a more precise definition of the Hammerstein scalar; in the terminology of §7 of the present paper the Hammerstein scalar should be defined in the extension E_1 of E.

4. The existence of an extremum. The examples of the previous section will make clear the role of gradients in the treatment of boundary value problems or integral equations: if the scalar $i(x)$ has an extremum in an interior point of the solid sphere $V = V_R$ given by $\|x\| \leq R$ of the Banach space E, then

$$\text{grad } i = g(x) = 0; \tag{4.1}$$

if x is an extremum point on the boundary $S = S_R$ of V_R, then it is easy to see (at least in the case that E is a Hilbert space) that for some λ

$$g(x) + \lambda x = 0. \tag{4.2}$$

The "variational" method of finding solutions of (4.1) and (4.2) consists in finding extrema of $i(x)$. In a finite-dimensional space $i(x)$ reaches a maximum and minimum in every bounded closed set if $i(x)$ is continuous, and at least a minimum if $i(x)$ is lower semicontinuous. As is well known, these statements are no more true in function spaces. If, for example, E is the Hilbert space of points $x = (x_1, x_2, \cdots)$ with $\sum_{i=1}^{\infty} x_i^2 < \infty$, and $\lambda_1, \lambda_2, \cdots$ a sequence of positive nonincreasing constants converging to zero, then the scalar

$$I(x) = \sum_{\nu=1}^{\infty} \lambda_\nu x_\nu^2 \tag{4.3}$$

is continuous and positive on the unit sphere S_1 given by $\|x\| = 1$. On the other hand the greatest lower bound of $I(x)$ on S_1 is 0, as is seen easily by considering the sequence of points x^n on S_1 whose coordinates are all 0 except for the nth, which is 1. Thus $I(x)$ has no minimum on S_1.

This situation is due to the fact that a closed bounded set in E is not necessarily compact. However, it follows from a well known theorem of Alaoglu [1] that the solid sphere V is compact in the "weak topology" if E is reflexive,[10] where the weak topology is the one induced by the following neighborhood definition: let K be a number greater than R; then a neighborhood of a point x_0 of V_K is determined by a positive number ϵ and a finite number of continuous linear functionals $l_i(x)$ $(i = 1, 2, \cdots, n)$ and consists of all points x of V_K for which

$$|l_i(x) - l_i(x_0)| < \epsilon \text{ for } i = 1, 2, \cdots n.$$

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[10] For the definition of a reflexive Banch space, see [14, p. 22]. Any Hilbert space is reflexive. More generally any uniformly convex space is reflexive; see [19].

As a consequence the scalar $I(x)$ takes a maximum and a minimum in V_R if it is weakly continuous (that is, continuous in the weak topology) and it takes a minimum in V_R if it is weakly lower semicontinuous. It follows that for any integer n

$$i(x) = \|x\|^n + I(x), \tag{4.4}$$

with weakly continuous $I(x)$, takes a minimum in V since $\|x\|^n$ is weakly lower semicontinuous [24, §2].

If $I(x)$ in addition to being weakly continuous is homogeneous, that is, if for some non-negative p

$$I(\alpha x) = \alpha^p I(x), \tag{4.5}$$

statements about the existence of extrema on the boundary S_R of V can be made. If, for example, $I(x)$ takes positive values at all in V_R, it takes a positive maximum in V_R. It then follows easily from (4.5) that this positive maximum must be taken on S_R, and consideration of $-I(x)$ shows that in any case an extremum is taken on S_R. Thus the scalar (4.3) takes the maximum λ_1 in the point

$$(1, 0, 0, \cdots)$$

of S_1, but no minimum on S_1. It takes its minimum 0 in V_1 in the center 0 of this solid sphere.

5. **Weak continuity of $I(x)$ and complete continuity of grad $I(x)$.** The next task is to find sufficient conditions for the weak continuity of $I(x)$. In this respect the following theorem holds in any Banach space E [24, Theorem 3.2]: sufficient for the weak continuity of $I(x)$ is the complete continuity of the gradient map $G(x)$ (that is, that $G(x)$ is continuous, and that the image of V under G is a compact set of E; compare the definition of a gradient map in §2). More explicitly: if $G(x)$ is completely continuous, then there exists to a given positive η linear continuous functionals $l_1, l_2, \cdots, l_n$ and an $\epsilon>0$ such that $|I(x+h)-I(x)|<\eta$ for all h with $x+h\in V_K$ for which $|l_i(h)|<\epsilon$ $(i=1, 2, \cdots, n)$ where V_K has the same meaning as in §4. Under the same assumption for complete continuity of G it is also true that to given $\eta>0$ there exist linear continuous functionals l_i $(i=1, \cdots, n)$ such that

$$|I(x+h) - I(x)| < \eta\|h\|/2 \text{ if } \|l_i(h)\| < \eta\|h\|/2 \quad (i = 1, 2, \cdots, n).$$

If E is a Hilbert space, the converse of the last statement holds [24, Theorem 3.3].

As a special application we draw the conclusion that the scalar (3.1) is weakly continuous if the operator $A(x)$ is completely continuous.[11] Since this scalar is homogeneous (the p of (4.5) being 2), it follows from the previous section that this scalar reaches an extremum on the surface of the unit sphere. This is especially true of the double integral (3.4) (under the assumption (3.5)), a well known fact which is basic for the variational treatment of the theory of linear integral equations.[12]

6. Extension of the domain of applicability of the preceding theory.

The application of the preceding theory seems to be confined to those scalars which have a completely continuous gradient, and therefore to have no bearing, for example, on problems connected with the differential operator A defined by (3.6); for (3.7) is not completely continuous quite apart from the fact already pointed out in example 1a that this operator is not the gradient of the scalar (3.8) in the sense of the definition given in §2.

However, it is sometimes possible by introducing a new "independent variable" and a new norm[13] to make the gradient completely continuous and therefore the corresponding scalar weakly continuous. We illustrate this procedure for the operator A defined by (3.6). It is well known that, under certain conditions concerning the differentiability of p and of the boundary of the bounded domain D, there exists a Green's function $K(s, t)$ belonging to the operator B defined in (3.7) and the boundary conditions zero.[14] For every continuously differentiable function the unique solution of the boundary value problem

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[11] That the complete continuity of the linear operator $A(x)$ is a necessary and sufficient condition for the weak continuity of the scalar (3.1) was already recognized by F. Riesz [20, pp. 96–97, footnote]. As regards the terminology, it should be noted that Riesz (following Hilbert) calls the quadratic form $(A(x), x)$ completely continuous if it is weakly continuous in the sense explained in §4 of the present paper. The terminology which calls a nonlinear (not necessarily quadratic) scalar in Hilbert space completely rather than weakly continuous if its gradient is completely continuous is still used in [22, Definition 5.1 and Theorem 5.1] and in [23]. A scalar which is completely continuous in the sense of this latter definition is actually weakly continuous (as noticed by the referee of [23]; compare Lemma 2.1 of that paper). The connection between weak continuity of a scalar and the complete continuity of its gradient has also been noticed by E. S. Citlanadze in the Hilbert space case [5, Theorem 2] and in the case of a reflexive Banach space with basis in [6].

[12] See, for example, [7, vol. 1, chap. 3].

[13] The method of renorming a Hilbert space E (or subspace of E) was introduced and applied to the treatment of second order boundary value problems by K. Friedrichs [8]. For higher order problems, compare [3, §10]. For "introducing a new variable," see also [12, p. 68].

[14] Schauder [29, p. 279] showed that the use of the Green's function is not necessary for the method in question.

$$B(y) = x, \qquad y = 0 \text{ on the boundary of } D, \tag{6.1}$$

is then given by (3.11) for which we write shortly

$$y = K(x). \tag{6.2}$$

Moreover, if E is the Hilbert space of all functions $x(t)$ for which

$$\int_D x^2(t)dt \tag{6.3}$$

exists with the square root of this integral as norm, then (6.2) represents a completely continuous linear map of E into itself mapping x into y.

We now introduce x as a "new variable" instead of y in the operator A and obtain from (3.6), (6.1), and (6.2) (at least for a certain subspace of E)

$$A(y) = x + K(x). \tag{6.4}$$

We now renorm our space as follows: with the scalar product corresponding to the norm given by (6.3), we have from (6.1), (6.2), (3.7), and Green's formula

$$(x, K(x)) = (B(y), y) = \int_D p \sum_{\nu=1}^{n} (\partial y/\partial s_\nu)^2 ds > 0.$$

We therefore can define a norm $\|x\|$ given by

$$\|x\|^2 = (x, K(x)) \tag{6.5}$$

instead of (6.3). $K(x)$ is defined for all x in E. However, E is not necessarily complete in the new norm given by (6.5). But E can be completed.[15] We denote the completed space with the new norm by E_1. If we denote the scalar product corresponding to the new norm $\|x\|$ by square brackets such that $\|x\|^2 = [x, x]$, we obtain from (6.4) and (6.2) for the scalar (3.8), which we now denote by $i(x)$,

$$\begin{aligned} 2i(y) &= (A(y), y) = (x, K(x)) + (K(x), K(x)) \\ &= \|x\|^2 + [x, K(x)]. \end{aligned} \tag{6.6}$$

Now $K(x)$ is completely continuous in the norm given by (6.3), and it can be seen easily that it remains completely continuous in the new norm $\|x\|$. Consequently, the scalar $I(x)$ defined by

$$I(x) = [x, K(x)]/2 \tag{6.7}$$

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[15] Concerning the completion of E, see [8]. For a more concrete method applicable if the eigenfunctions of the positive definite kernel are known, see [25].

is weakly continuous by the last paragraph of §5, and the scalar (3.8) is by (6.6) of the form (4.4) that is,

$$i(x) = \|x\|^2 + I(x) \tag{6.8}$$

with weakly continuous I, and it takes, therefore, a minimum in every sphere $\|x\| \leq R$ while the scalar (6.7) takes not only a minimum in such a sphere but also a maximum on its boundary $\|x\| = R$ (compare the last paragraph of §4).[16]

7. Application to the Hammerstein integral equation. The method illustrated in the previous section in the case of the linear operator A can also be applied to nonlinear problems. We shall illustrate this for the nonlinear integral equation (3.12) with the symmetric positive definite kernel $K(s, t)$ satisfying the assumptions made in §3, example 3. We define the spaces E and E_1 as in §6 with K now being the kernel of (3.12).[17] From (6.5) the Hammerstein scalar $i(x)$ defined by (3.9), (3.10) can then be written

$$i(x) = \frac{\|x\|^2}{2} + I(x), \tag{7.1}$$

where $I(x)$ is defined by (3.10), (3.11) with continuous f. Then, as has been mentioned already in §3, either of the conditions (A), (B) (§3, example 3) is sufficient for the left member of (3.12) (expressed by means of (3.11) in terms of x) to be equal to grad $i(x)$. It can also be proved that either of these conditions is sufficient for

$$\operatorname{grad} I(x) = \int_D K(s, t) f(t, y(t)) dt$$

to be completely continuous [25, Theorem 3.2]. Thus, the Hammerstein scalar (7.1) has the same properties as (4.4) and therefore reaches a minimum in any solid sphere V_R of E_1. If the minimum is taken in an interior point x_0 of V_R, the grad i will be zero in x_0, that is, the y connected with $x = x_0$ by (3.11) will be a solution of (3.12). The minimum point x_0 will certainly, at least for large enough R, be an interior point of V_R if $i(x) \to \infty$ as $\|x\| \to \infty$. Now it can be shown that this latter condition is satisfied if

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[16] The point (or points) x in which the extremum is attained is in E_1 but not necessarily in E. For the applications it is therefore of importance to extend the operator (6.2) from E to E_1 in such a way that y is in the original function space. Compare the following footnote.

[17] For a more precise description of the linear operator (6.2) (that is, (3.11)) in the extended space E_1, see [25, §§2 and 3].

$$\int_0^y f(t, u)du > -k/2y^2 - C, \tag{7.2}$$

where C and k are constants, the latter subject to the condition $0 \leqq k < 1/\lambda_1$ with λ_1 being the greatest eigenvalue of the kernel K. Thus, the conditions of this section taken together (with either A or B) are sufficient for the existence of a solution of the Hammerstein equation (3.12).[18]

8. **Critical points and Leray-Schauder index.** So far the existence theorems for boundary value or integral equation problems which can be written in form (4.1) have been based on the proof that the corresponding scalar $i(x)$ has a maximum or minimum. However, maxima or minima are rather special types of critical points, i.e., of points x satisfying (4.1). Let x_0 be a point which is not a limit of critical points. Then a sufficient condition for x_0 to be a critical point is that at least one of the (Morse)-type numbers at x_0 are different from zero. These numbers are defined as follows:[19] let U be a neighborhood of x_0 containing no critical point except possibly x_0. Let C be the intersection of U with the set of those x for which $i(x) \leqq i(x_0)$, and C' the set C minus the point x_0; the rth type number m^r of $i(x)$ at x_0 is then defined as the singular rth relative Betti number of C mod C' $(r=0, 1, \cdots)$. In what follows we shall always assume that E is a Hilbert space, and that $g(x) = \operatorname{grad} i(x)$ is of the form

$$g(x) = x/2 + G(x) \tag{8.1}$$

with completely continuous G. By the result mentioned at the beginning of §5, $i(x)$ is then of the form (4.4) with $n=2$ and weakly continuous $I(x)$ encountered several times in our examples. Now for mappings of the form (8.1) the mapping degree is defined. More precisely: if D is any open bounded convex domain in E, and if y is a point of E which does not lie on the boundary of the image of D under g, then the degree $j=j(D, g, y)$ of the map $g: D \rightarrow E$ at y is defined [16]. Therefore, especially $j=j(g, U, 0)$ is defined where 0 denotes the zero point of E, and U is any neighborhood of the point

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[18] This is one of the principal results of Hammerstein's paper [13]. Since then more general results concerning existence and uniqueness of solutions of equation (3.12) and systems of such equations have been obtained by M. Golomb in [9] and [10] without the assumption of positive definiteness and of symmetry. See also [21, §5]. The method described in the present paper can also be used to obtain generalizations (in a somewhat different direction) pertaining to not necessarily positive and symmetric kernels; see [25] and [26, §7].

[19] See [18] or [31].

x_0 of E having the property described above. This number (which can be shown to be the same for all such neighborhoods U) is called the (Leray-Schauder) index of g at x_0. It plays the role of the "multiplicity" of x_0 as root of the equation (4.1). In particular, $j \neq 0$ is a sufficient condition that x_0 is a solution of (4.1), that is, a critical point. In the latter case, j is also called the index of the singularity x_0 of the "vector field" $g(x)$. Under certain conditions the following relation[20] holds between the index j and the type numbers m^r:

$$j = \sum_r (-1)^r m^r. \tag{8.2}$$

Now let $V = V_R$ be the solid sphere $\|x\| \leq R$ of the Hilbert space E, and $S = S_R$ its boundary. It follows easily from the complete continuity of $G(x)$ that there are at most a finite number of critical points of the scalar $i(x)$ in V_R. We denote these by a_σ $(\sigma = 1, 2, \cdots, s)$, the index of g at a_σ by j_σ, and the rth type number of $i(x)$ at a_σ by m^r_σ. We suppose that none of the critical points a_σ lies on S_R. Then under the mapping g the zero point 0 of E is not on the image of the boundary S_R of V_R, and therefore the mapping degree $\chi = d(g, V_R, 0)$ is defined. This number is known to depend only on the behavior of g on S_R, and it is therefore also called the order of the image (under g) of S_R with respect to 0, or also the characteristic of the gradient field $g(x)$ on S_R.[21] If we now apply (8.2) to each a_σ and sum in σ, we obtain immediately the relation

$$\chi = \sum_{\sigma=1}^{s} \sum_r (-1)^r m^r \tag{8.3}$$

by applying the well known fact [21, Satz 5] that $\chi = \sum_{\sigma=1}^{s} j_\sigma$.

To show that this formula sometimes yields information about the number of critical points, we consider the case that the differential $L(x, h)$ of $G(x)$ exists, is completely continuous in h, and that moreover the second differential $d^2(x, h, h)$ of $i(x)$ is uniformly positive definite, that is, that there exists a positive constant μ such that

$$d^2(x, h, h) \geq \mu \|h\|^2. \tag{8.4}$$

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[20] For the proof in the finite-dimensional case, see [27]. While this seems to be the only published proof, the formula in this case was known independently to R. Bott and to M. Morse. For the proof in the Hilbert space case, see [28].

We remark (as R. Bott also did) that the well known theorem of Bendixson, according to which the index of an isolated singularity of a plane gradient field is never greater than 1, is a nearly immediate consequence of (8.2).

[21] For the concepts of order and characteristic, see [2, chap. 12] in the finite-dimensional case, and [21] for the Hilbert space case.

We claim that in this case V_R (for large enough R) contains exactly one critical point, and that this critical point is a minimum [26, Theorem 6.2].

Indeed, that every eventual critical point a_r of V_R is a minimum, and that $m_r^0=1$, $m_r^r=0$ for $r \geqq 1$ follows immediately from (8.4). The relation (8.3) therefore specializes to

$$\chi = \sum_{r=1}^{s} m_r^0, \tag{8.5}$$

and since $m_r^0 \geqq 1$ it will be sufficient to prove that

$$\chi = 1, \tag{8.6}$$

for then (8.5) implies $s=1$. To prove (8.6) we remark that since $d^2(x, h, k)$ is the differential of the first differential $d(x, h)$ of $i(x)$, we obtain from the mean value theorem [11, Theorem 5].

$$d(x, k) - d(0, k) = \int_0^1 d^2(tx, x, k)dt,$$

and therefore from (8.4), with $k=x$,

$$\begin{aligned} d(x, x) &\geqq \mu\|x\|^2 - |d(0, x)| \\ &= \|x\|\{\mu\|x\| - |d(0, x)|/\|x\|\}. \end{aligned} \tag{8.7}$$

Recalling that, by the definition of a gradient in a Hilbert space E (last paragraph of §2), $d(x, x) = (g(x), x)$, we see immediately from (8.7) that to each large enough R there exists a positive $\mu_1=\mu_1(R)$ such that $(g(x), x) \geqq R^2\mu_1(R)>0$ for x on S_R. But this inequality means that the vector field $g(x)$ on S_R is "exteriorly directed." Therefore its characteristic χ on S is $+1$ [21, part b of Satz 7].

As an application, we remark that the Hammerstein scalar (§3) is uniformly positive definite if, in addition to B and the assumptions made in §7, it satisfies the following condition: f has a first continuous derivative with respect to u satisfying the inequality $(\partial f/\partial u) \geqq -c$ where the constant c is subject to the condition that c^{-1} is greater than the greatest eigenvalue of the kernel $K(s, t)$. This implication follows easily from (3.13).[21] Consequently, the Hammerstein equation (3.12) has one and only one solution in this case.[22] Moreover, it follows from (3.12) that $x(s) = -f(s, y(s))$ satisfies (3.11). Consequently, this $x(s)$ minimizes the Hammerstein scalar.

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[21] For details, see [26, §7].

[22] Concerning more general uniqueness statements, see the papers quoted in footnote 18.

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GENERALIZED AXIALLY SYMMETRIC POTENTIAL THEORY[1]

ALEXANDER WEINSTEIN

1. Introduction. The subject of this paper is the theory of a special class of linear partial differential equations with variable coefficients which are rational functions of the independent variables. The type of differential equation considered here is of particular interest because it arises in a number of seemingly disconnected fields in pure and applied mathematics both classical and modern.

The simplest elliptic differential equation with constant coefficients is the Laplace equation. Our class may be considered as the simplest among those with variable coefficients. Like Bessel's equation, which plays such a prominent role among ordinary differential equations with rational coefficients, our class is derived from the Laplace equation by introduction of a particular type of symmetry.

We shall deal mainly with elliptic equations, but in a natural way our subject will touch upon a class of hyperbolic equations of the Euler-Poisson type and on mixed equations of Tricomi's type.

Let us begin by considering a solution $f(x_1, x_2, \cdots, x_n)$ of the Laplace equation

$$\frac{\partial^2 f}{\partial x_1^2}+\frac{\partial^2 f}{\partial x_2^2}+\cdots+\frac{\partial^2 f}{\partial x_n^2}=0, \qquad n>2. \tag{1}$$

We are interested only in axially-symmetric solutions of (1), namely, in those which depend on two variables $x=x_1$, and $y=(x_2^2+x_3^2+\cdots x_n^2)^{1/2}$, the x-axis being the axis of symmetry. Under these restrictions f becomes a function $\phi(x, y)$ of the two variables x and y, and satisfies the equation

$$\frac{\partial}{\partial x}\left(y^{n-2}\frac{\partial\phi}{\partial x}\right)+\frac{\partial}{\partial y}\left(y^{n-2}\frac{\partial\phi}{\partial y}\right)=0. \tag{2}$$

The case $n=3$ is classical and was investigated already in Laplace's time. Later, fundamental progress was made by Stokes in 1842 and above all by Beltrami [1] in his celebrated papers on symmetric potentials published about 1880. While these investigations were concerned only with ordinary three-dimensional space, it was Arndt [2]

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who in 1915 first pointed out the importance of the case $n=5$ for the theory of torsion of shafts of revolution.

Let us consider in a more general way the equation

$$\frac{\partial}{\partial x}\left(y^p\,\frac{\partial\phi}{\partial x}\right)+\frac{\partial}{\partial y}\left(y^p\,\frac{\partial\phi}{\partial y}\right)=0, \qquad y \geqq 0, \tag{3}$$

for all positive values of the parameter p. For all such values of p we shall call ϕ an axially symmetric potential in a space of $n=p+2$ dimensions. For nonintegral values of p this space is of course fictitious, but the corresponding equation (3) can be studied in the "meridian plane" (x, y), where it still retains some of the properties of equation (2). The theory of our equations for general values of the parameter p will be called generalized axially symmetric potential theory and will be abbreviated as GASPT. The integral values of p, especially $p=1$ $(n=3)$, will serve as a guide in the study of these generalized potentials. There is usually no great difficulty in rewriting for any positive value of p the familiar and classical formulas of ordinary axially symmetric potential theory. For instance, the function $(x^2+y^2)^{-1/2}$ is replaced by $(x^2+y^2)^{-p/2}$. The limiting case $p=0$ leads us back to the theory of harmonic functions in the (x, y)-plane. It will be seen that most of our results apply in this case provided the harmonic function is symmetric about the x-axis.

The introduction of nonintegral values of p is not done for the sake of formal generalization. On the contrary, it will be shown that some interesting problems can be investigated by the method of GASPT.

2. The stream function and the generalized Stokes-Beltrami equations. Our differential equation (3) indicates the existence of a stream function $\psi(x, y)$ which is defined by the equations

$$y^p\phi_x = \psi_y; \qquad y^p\phi_y = -\psi_x. \tag{4}$$

We shall call (4) the generalized Stokes-Beltrami equations. Beltrami, who considered only the classical case $p=1$, was the first to point out the similarity to the Cauchy-Riemann equations. He showed by a process of repeated differentiation and integration that every potential or stream function generates a descending and ascending sequence of pairs of functions satisfying the same equations (4). For instance, in order to obtain Beltrami's first "derivative" (ϕ_1, ψ_1) of the pair (ϕ, ψ) we put

$$\phi_1 = \phi_x; \qquad \psi_1 = \psi_x. \tag{5}$$

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On the other hand, the pair (ϕ_*, ψ_*),

$$(6)\qquad \phi_* = \int (\phi dx - y^{-p}\psi dy); \qquad \psi_* = \int (\psi dx + y^p \phi dy),$$

will give us the "primitive function" of the pair (ϕ, ψ). In this way Beltrami founded a generalized theory of analytic functions which was later developed in a systematic way by Bers and Gelbart [3]. They considered equations of a more general type than (4) but restricted themselves exclusively to regular solutions, which do not concern us in this paper. Further details of this generalized theory of analytic functions can be found in the literature and will not be discussed further.

It follows from (4) that the stream function satisfies the equation

$$(7)\qquad \frac{\partial}{\partial x}\left(y^{-p}\frac{\partial\psi}{\partial x}\right) + \frac{\partial}{\partial y}\left(y^{-p}\frac{\partial\psi}{\partial y}\right) = 0.$$

Obviously the equation for ψ is obtained from the equation for ϕ by replacing p by $-p$. Therefore, both ϕ and ψ satisfy the equation

$$(8)\qquad \frac{\partial}{\partial x}\left(y^{k}\frac{\partial z}{\partial x}\right) + \frac{\partial}{\partial y}\left(y^{k}\frac{\partial z}{\partial y}\right) = 0,$$

or

$$(9)\qquad z_{xx} + z_{yy} + ky^{-1}z_y = 0,$$

for $k = p$ and $k = -p$ respectively. It is, however, convenient in some applications to retain the notations ϕ and ψ and to write $\phi\{p\}$ and $\psi\{p\}$ in order to emphasize the dependence upon the parameter p. In the same sense we shall use the notation $z\{k\}$. Since the line $y=0$ is a singular line for the coefficients of our equations, we shall usually restrict our considerations to the half-plane $y \geqq 0$.

3. The correspondence principle. The following simple but fundamental correspondence principle will be of great importance. It can be easily shown (see Weinstein [4]) that

$$(10)\qquad \psi\{p\} = Cy^{p+1}\phi\{p+2\}, \qquad p \geqq 0.$$

This result means that, given a $\psi\{p\}$, we can find a function $\phi\{p+2\}$ in two more dimensions, which is defined up to a constant factor C, and vice versa. In the notation of equation (9), equation (10) can be written as follows:

$$(11)\qquad z\{k\} = Cy^{1-k}z\{2-k\}, \qquad -\infty < k < +\infty.$$

In the following paragraphs we shall apply our principle to certain boundary value problems.

4. The hydrodynamical flow problem and the method of generalized electrostatics. The classical boundary value problem in axially-symmetric hydrodynamics of incompressible perfect fluids is the determination of the flow about a body, the flow having a uniform velocity at infinity. It is an exterior Neumann problem for the potential Φ. This potential can be written for any index p in the form

$$\Phi\{p\} = x - \phi\{p\}, \tag{12}$$

where the normal derivative $\partial\Phi/\partial\nu$ vanishes on the meridian profile of the body. At infinity we have for Φ a development of the type

$$\Phi\{p\} = x + \frac{ax}{(x^2 + y^2)^{(p+3)/2}} + \cdots . \tag{13}$$

The boundary condition $\partial\Phi/\partial\nu = 0$ means that $\partial\phi/\partial\nu = \partial x/\partial\nu$ on the profile. The introduction of the stream function

$$\Psi\{p\} = (p + 1)^{-1}y^{p+1} - \psi\{p\} \tag{14}$$

reduces our problem to an exterior Dirichlet problem. The stream function $\Psi\{p\}$ is single-valued and has at infinity the expansion

$$\Psi\{p\} = (p + 1)^{-1}y^{p+1} - \frac{ay^{p+1}}{(x^2 + y^2)^{(p+3)/2}} + \cdots . \tag{15}$$

It vanishes on the x-axis and assumes a constant value Ψ_0 on the profile. If the profile and the x-axis form a single streamline, Ψ_0 will be zero. We shall discuss here only this last case in which the domain exterior to the profile is simply connected. The Dirichlet problem for $\Psi\{p\}$ is actually a Dirichlet problem for $\psi\{p\}$ which is of a rather complicated nature, since $\psi\{p\}$ must assume the value $(p+1)^{-1}\, y^{p+1}$ on the profile. For this reason the following new procedure (see Weinstein [5]) is more successful. We make use of (10) and replace $\psi\{p\}$ in (14) by $(p+1)^{-1}y^{p+1}\phi\{p+2\}$. In this way we obtain in place of (14) the following expression:

$$\Psi\{p\} = (p + 1)^{-1}y^{p+1}[1 - \phi\{p + 2\}]. \tag{16}$$

The condition $\Psi\{p\} = 0$ means that $\phi\{p+2\} = 1$ on the profile. Moreover, it follows from (15) that $\phi\{p+2\}$ admits the expansion

$$\phi\{p + 2\} = \frac{(p + 1)a}{(x^2 + y^2)^{(p+3)/2}} + \cdots \tag{17}$$

at infinity. This means that formula (16) reduces the determination of $\Psi\{p\}$ to the determination of the electrostatic potential $\phi\{p+2\}$ of the body with the same meridian profile but in a space of two more dimensions. This simple but fundamental remark is based on the fact that we have the same factor y^{p+1} in our correspondence principle (10) and in equation (14).

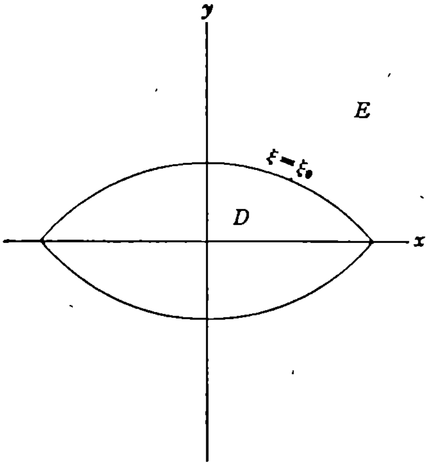

FIG. 1

A trivial example of our procedure is the determination of the flow about a sphere of unit radius which is usually obtained by putting a dipole at the origin. In contrast to this classical method we must merely find the electrostatic potential of a five-dimensional sphere, which is obviously $(x^2+y^2)^{-3/2}$. Thus we obtain immediately the familiar formula

$$\Psi\{1\} = \frac{y^2}{2}\,[1 - (x^2 + y^2)^{-3/2}]. \tag{18}$$

Naturally, the applications of our method of generalized electrostatics are not confined to such simple examples. In fact L. E. Payne [6] has recently obtained in this way the flow about a spindle (Fig. 1) which, despite its obvious importance, has not been treated previously.

In the case of the spindle we use dipolar coordinates defined by the equation

$$x + iy = ic \cot (\xi + i\eta)/2, \tag{19}$$

where c is an arbitrary positive constant. The range of coordinates is chosen as $-\infty < \eta < +\infty$, $0 < \xi \leq \pi$. The profile of a spindle in the upper half-plane $y \geqq 0$ is a circular arc given by $\xi = \xi_0 < \pi$. The region exterior to the profile is defined by $0 < \xi < \xi_0$. The electrostatic potential of the three-dimensional spindle was first given by Mehler [7] in 1881. G. Szegö [8] in his address to the American Mathematical Society attributes it to C. Neumann. We need here, however, the electrostatic potential of a spindle in five-dimensional space. The generalization of Mehler's solution to any odd-dimensional space, a generalization which is far from trivial, is given by L. E. Payne [6]. By using equation (16), Payne obtains finally the following formula for the stream function for the flow about a spindle in a space of $n = p + 2$ dimensions (n odd):

$$\Psi\{p\} = \frac{(c \sin \xi)^{2(q+1)}}{2(q+1)(s-t)^{2(q+1)}}\left[1 - \frac{2^{1/2}\Gamma(1/2)(s-t)^{q+3/2}}{\Gamma(q+3/2)} \cdot \int_0^\infty \frac{K_\alpha^{(q+1)}(-t_0)K_\alpha^{(q+1)}(t) \cos \alpha\eta d\alpha}{K_\alpha^{(q+1)}(t_0) \cosh \alpha\pi}\right]. \tag{20}$$

In this formula

$$s = \cosh \eta, \quad t = \cos \xi, \quad \text{and} \quad q = (p-1)/2.$$

The symbol $K_\alpha^{q+1}(t)$ denotes the $(q+1)$-derivative of the conal function $K_\alpha(t)$ [9]. The classical case is of course given by $p = 1$, and by inserting this value for p in (20) we obtain the solution for a three-dimensional spindle.

In order to evaluate the advantages of the method of generalized electrostatics, let us consider now the flow problem about a lens (Fig. 2). This flow was determined for the first time in 1947 by M. Shiffman and D. C. Spencer [10], who applied an ingenious and difficult procedure involving the method of images in a multi-sheeted Riemann-Sommerfeld space. In a much simpler way Payne obtains the solution [6] by a straightforward generalization to five dimensions of the electrostatic problem for the ordinary three-dimensional lens solved already in 1868 by Mehler [11]. Payne uses in place of (19) the peripolar transformation

$$x + iy = -c \cot (\xi + i\eta)/2, \qquad c > 0. \tag{21}$$

The profile of a lens is given by two intersecting circular arcs $\xi = \xi_1$ and $\xi = \xi_2$. We assume that $0 < \xi_1 < \xi_2 < 2\pi$. The external region is given by the inequalities $\eta > 0$, $\xi_2 - 2\pi < \xi < \xi_1$. The stream function obtained by Payne in the most interesting case $n = 3$ ($p = 1$) is given by the formula

$$\Psi\{1\} = \frac{c \sinh^2 \eta}{2(\cosh \eta - \cos \xi)^2}\left[1 + 2^{1/2}(\cosh \eta - \cos \xi)^{3/2} \cdot \int_0^\infty \frac{F(\alpha, \xi)K_\alpha^{(1)}(s)}{\cosh \alpha\pi}\, d\alpha\right], \tag{22}$$

where

$$F(\alpha, \xi) = \frac{\sinh \alpha(\xi_1-\xi) \cosh \alpha(\pi-\xi_2)+\cosh \alpha(\pi-\xi_1) \sinh \alpha(2\pi-\xi_2+\xi)}{\sinh \alpha(2\pi-\xi_2+\xi_1)}$$

and $K_\alpha^{(1)}(s)$ denotes the first derivative of the Mehler function [9].

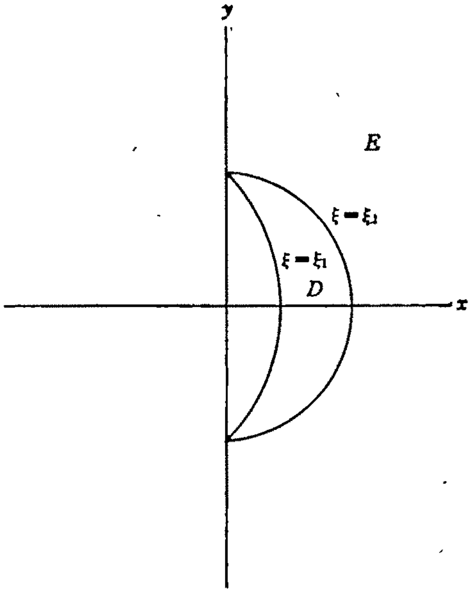

FIG. 2

There seems to be no elementary computational way of identifying Payne's results with those of Shiffman and Spencer. Of course, the identity of the two solutions is guaranteed by the uniqueness theorem. According to an oral communication from Professor Spencer, it would be difficult if not impossible to obtain the flow about a spindle by the method of images.

5. Torsion of shafts of revolution. Let us discuss here briefly another interesting application of the method of generalized electrostatics, namely, an application to the problem of the torsion of shafts, which has already been mentioned in the Introduction. This problem has been reduced by Arndt to the determination of two functions Φ and Ψ defined in the meridian plane and satisfying equations (4) for $p=3$. The profile of the shaft is defined by lines $\Psi=\Psi\{3\}=$constant.

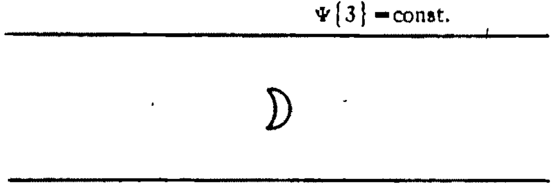

FIG. 3

Therefore, the problem of torsion is essentially equivalent to the flow problem in a five-dimensional space. For instance, the flow about a closed profile on which $\Psi\{3\}=0$ (see Fig. 3) can be interpreted as the solution of a torsion problem for a shaft with a cavity, the outer boundary of the shaft being given by lines $\Psi\{3\}=$constant. For moderately large values of this constant the outer boundary of the profile deviates only slightly from a parallel to the x-axis. According to the correspondence principle (10) the five-dimensional flow problem may be interpreted as an electrostatics problem in a space of seven dimensions. In this way several problems in the torsion of shafts in ordinary three-dimensional space have recently been solved by an application of the theory of electrostatics in a seven-dimensional space (see Weinstein [12]).

6. Capacity, virtual mass, and generalized symmetrization. The most important functional in electrostatics is the capacity of a conductor. For bodies of revolution in a space of any number of dimensions the capacity can be defined as a Dirichlet integral in the (x, y)-plane taken over the region E exterior to the profile of the body. As always, we restrict ourselves to non-negative values of y.

In the following we shall often use notations like $\phi[n]$ in place of $\phi\{p\}$ to indicate the number of dimensions. In this notation the capacity $C[n]$ corresponding to a body $B[n]$ with a meridian section D is given by the following formula, which generalizes the classical case $n=3$:

$$C[n]=\omega_{n-1}\omega_n^{-1}\iint_E y^{n-2}(\operatorname{grad}\phi[n])^2\,dx\,dy. \tag{23}$$

As in previous paragraphs, we take the same section D for all values of n. The ω's in (23) denote areas of unit hyperspheres. The function

$\phi[n]$ represents the electrostatic potential already introduced in section 4.

For the hydrodynamical problem the most important quantity is the virtual mass $M[n]$ of $B[n]$ for a flow in the x-direction. Putting the density and the velocity at infinity equal to unity, we have

$$M[n] = \omega_{n-1} \iint_E y^{n-2}(\operatorname{grad} \phi[n])^2 dxdy. \tag{24}$$

Let us note that the potential $\phi[n]$ in (24) is not the electrostatic potential but the velocity potential introduced by equation (12). Using the associate function $\psi[n]$, we may replace formula (24) by

$$M[n] = \omega_{n-1} \iint_E y^{-(n-2)}(\operatorname{grad} \psi[n])^2 dxdy. \tag{25}$$

By the correspondence principle we can introduce in place of $\psi[n]$ the electrostatic potential $\phi[n+2]$. An integration by parts leads then to the following fundamental formula, given by Payne and Weinstein [13],

$$M[n] + V[n] = \pi^{n/2}\left[(n-1)\Gamma\left(\frac{n}{2}+1\right)\right]^{-1} C[n+2], \tag{26}$$

where

$$V[n] = \omega_{n-1} \iint_D y^{n-2} dxdy. \tag{27}$$

Obviously $V[n]$ is the n-dimensional volume of the body $B[n]$. For $n=3$ formula (26) was given already by G. I. Taylor [14] and mentioned by Schiffer and Szegö [15]. In these papers the right-hand side of (26) was called a dipole coefficient. However, it is advantageous to interpret the dipole coefficient as a capacity in $(n+2)$ dimensions because numerous known results for the capacity of ordinary three-dimensional bodies of revolution remain valid for any value of n. Thus, the fundamental formula (26) may be used for the study of the virtual mass. In this connection let us now say a few words about generalized symmetrization.

The influence of symmetrization on the capacity has been recently investigated by numerous authors, the foremost of whom are Pólya and Szegö [16]. Some particular results for the virtual mass have been obtained by Garabedian and Spencer [17]. It is interesting to note that the method of generalized electrostatics not only gives simpler proofs of several known results but also leads to a natural extension of the concept of symmetrization.

Let us consider an ordinary body of revolution $B[3]$ in a three-dimensional space. A line $x=\text{constant}$, $y \geqq 0$ will intersect the profile L of B in m points, $y_1(x) > y_2(x) > \cdots > y_m(x) > 0$. It is obvious that the number of intersections m depends upon x. (See, for example, Fig. 2.) This body can be symmetrized with respect to its axis of symmetry in a number of ways. One of these is the Schwarz symmetrization, which preserves the volume of $B[3]$. Another is the Steiner symmetrization of the meridian section of $B[3]$, which preserves the area of the section but in general decreases the volume. Under Schwarz symmetrization the profile L is replaced by a new profile L_2 given by the equation $y^2(x) = y_1^2(x) - y_2^2(x) + \cdots$. In the case of Steiner symmetrization the profile L is replaced by another curve L_1 defined by the equation $y(x) = y_1(x) - y_2(x) + \cdots$. This leads to the following generalization. Let us define a profile L_q by the equation

$$y^q(x) = \sum_{k=1}^{m} (-1)^{k-1} y_k^q(x), \tag{28}$$

where q is any positive integral or nonintegral number. We say that the profile L_q given by the function $y(x)$ has been obtained from L by a generalized symmetrization S_q. From now on let us assume, as in section 4, that for any $n \geqq 3$ all profiles are boundaries of bodies of revolutions in an n-dimensional space. We can then prove the following three theorems (see Payne and Weinstein [13]):

I. $V[n]$ does not increase under S_q for $0 < q \leqq n-1$ and does not decrease under S_q for $q \geqq n-1$. In particular, $V[n]$ remains invariant under S_{n-1}.

II. $C[n]$ does not increase under S_q for $0 < q \leqq n-1$.

III. $M[n]$ does not increase under S_q for $n-1 \leqq q \leqq n+1$. Let us observe that, by (26), Theorem III follows immediately from Theorems I and II.

The main tools for the proofs of these theorems are the inequalities

$$\left[\sum_{k=1}^{m} (-1)^{k-1} y_k^{q+s} \right]^{1/(q+s)} \leqq \left[\sum_{k=1}^{m} y_k^q \right]^{1/q} \tag{29}$$

and

$$\left[\sum_{k=1}^{m} (-1)^{k-1} y_k^{q} \right]^{1/q} \leqq \left[\sum_{k=1}^{m} (-1)^{k-1} y_k^{q+s} \right]^{1/(q+s)}, \tag{30}$$

which are valid for positive q and s when $y_1 > y_2 > \cdots > y_m > 0$. The first of these inequalities is classical. The proof of the second inequality is due to H. F. Weinberger and can be found in [13].

Theorem I follows immediately from (29) and (30). As to Theorem II concerning the capacity, we proceed following the pattern given in [16]. We introduce a function $z(x, y)$ which is constant over the domain D and vanishes on the boundary of a large half-circle A. We find, using (29), that the integral generalizing the surface area

$$I = \iint_A y^{n-2}[1 + z_x^2 + z_y^2]^{1/2} dxdy \tag{31}$$

does not increase under a symmetrization S_{n-1}. Let us now replace $z(x, y)$ by $\epsilon\chi(x, y)$ where ϵ is a small positive number tending to zero. Then we find, by using Theorem I, that the integral

$$\iint_A y^{n-2}(\chi_x^2 + \chi_y^2) dxdy \tag{32}$$

does not increase under S_{n-1}. By letting the radius of A tend to infinity, we find that we can replace χ in (32) by the electrostatic potential $\phi[n]$ and the domain of integration A by the exterior E of D. In this way it is shown that $C[n]$ does not increase under S_{n-1}. Let us now consider the influence of the symmetrization S_q for $0 < q \leqq n-1$. According to the inequality (30) no point of the profile L_q lies outside of L_{n-1}. The well known monotone property of the capacity remains valid in any number of dimensions. We see, therefore, that under S_q the capacity is a nondecreasing function of q. This remark concludes the proof of Theorem II.

7. Existence and uniqueness theorems for data on the singular line. In the previous paragraphs we have been chiefly concerned with certain classical boundary value problems in GASPT. Let us now turn to the general theory of our associated equations (3) and (7), which we may replace by the single equation (9) considered for all real values of k. For positive values of y this equation is elliptic with regular analytic coefficients. Therefore, as long as y is positive, the general theory of such equations applies. However, the x-axis is a singular line for the coefficients of (9). In a certain sense there is an analogy between our equation and ordinary differential equations with a singular end point. As is well known, equations such as Bessel's equation admit some solutions which remain regular at this point. Similarly we have some functions z which remain regular on the x-axis, for example, $z = x$. For such solutions the following *unique-*

ness and existence theorems are of importance.

If $k \neq 0, -1, -2, \cdots$, then there is one and only one function $z\{k\}$ which is regular analytic in a region containing a segment of the x-axis in its interior, and which takes prescribed analytic values $g(x)$ on this segment. This function z is necessarily an even function of y. If $k = 0, -2, -4, \cdots$, the solution $z\{k\}$ always exists for arbitrary analytic values on the x-axis but is determined only up to an additive function of the type $y^{1-k}z\{2-k\}$, where the values of $z\{2-k\}$ on the x-axis are arbitrary. If, however, $k = -1, -3, -5, \cdots$, then $z\{k\}$ exists only if the analytic values on the x-axis are given as an arbitrary polynomial in x of degree not exceeding $-k$. Again, in this case, the function $z\{k\}$ is determined only up to an additive function $y^{1-k}z\{2-k\}$. This indetermination is to be expected in view of the correspondence principle.

In this generality the existence and uniqueness theorems were given by Hyman [18], who used a power series expansion of $z\{k\}$. For positive k a uniqueness theorem had already been given by Weinstein [4]. It should be noted also that for $k > 0$, $z\{k\}$ can be explicitly given as a Laplace integral,

$$z\{k\} = C_k \int_0^{\pi} g(x + iy \cos \alpha) \sin^{k-1} \alpha d\alpha \tag{33}$$

where $C_k^{-1} = 2^{k-1}\Gamma^2(k/2)\Gamma^{-1}(k)$. A generalization of (33) for $k \leq 0$ will be given in a forthcoming paper by Diaz and Weinstein.

8. The fundamental solution. The uniqueness theorem of section 7 may be used as an *identification principle* permitting us to identify a complicated solution by its values on the x-axis. The simplest example is the Lipschitz integral

$$\phi_0\{p\} = \frac{2y^{-q}\Gamma(q + 1)}{\Gamma(2q + 1)} \int_0^{\infty} t^q e^{-|x|t} J_q(yt) dt = (x^2 + y^2)^{-p/2}. \tag{34}$$

This integral may be identified by its values on the x-axis, namely, $|x|^{-p}$. In (34) we have $p > 0$ and $q = (p-1)/2$. This solution has a singularity on the x-axis. The first step of the general theory, already clearly recognized by Beltrami for $p = 1$, is to find a fundamental solution with a logarithmic singularity at an arbitrary point above the x-axis, say at $x = 0$, $y = b > 0$. A fundamental solution has the form

$$\phi\{p\} = u(x, y) \log [x^2 + (y - b)^2] + v(x, y), \tag{35}$$

where u and v are regular analytic at the singular point. In the classical case $p = 1$ ($n = 3$) Beltrami points out that such a solution would correspond to the potential of a uniform ring with the point b as the trace in the meridian half-plane. By our identification principle this solution can be easily constructed for $p=1$ due to the fact that it takes on the x-axis the value $(x^2+b^2)^{-1/2}$. Beltrami's result suggests that for $p>0$ the fundamental solution defined in the entire half-plane $y\geqq 0$ will be the solution taking the value $(x^2+b^2)^{-p/2}$ on the x-axis. There is a bewildering variety of forms of such a solution, all of which coincide by the identification principle. The formula [4]

$$(36) \qquad \phi\{p\} = \pi C_p (by)^{-q} \int_0^\infty e^{-|x|t} J_q(yt) J_q(bt)\,dt, \qquad q = \frac{p-1}{2}$$

is a direct generalization of Beltrami's formula for $p=1$. The constant C_p has been introduced in (33). Another expression (see Weinstein [4]), which shows more easily that $\phi\{p\}$ is actually of the form (35), is given by

$$(37) \qquad \phi\{p\} = C_p \int_0^\pi (x^2 + y^2 + b^2 - 2by\cos\alpha)^{-p/2} \sin^{p-1}\alpha\, d\alpha.$$

9. **The stream function and the integrals of Weber-Shafheitlin.** It may be surmised that Beltrami chose the expression (36) for the fundamental solution because it was then easy to compute the associate function ψ. As $|x|$ is equal to x or $-x$ according to whether x is positive or negative, we obtain, generalizing the computations of Beltrami, the following expressions for ψ:

$$(38) \qquad \psi^+ = -\pi C_p b^{-q} y^{q+1} \int_0^\infty e^{-xt} J_{q+1}(yt) J_q(bt)\,dt, \qquad x \geqq 0,$$

$$(39) \qquad \psi^- = \pi C_p b^{-q} y^{q+1} \int_0^\infty e^{xt} J_{q+1}(yt) J_q(bt)\,dt = -\psi^+, \qquad x \leqq 0.$$

At this point Beltrami, who considers only the case $q=0$, makes a fundamental mistake in not realizing that the associate function ψ is a many-valued function and that ψ^+ and ψ^- represent two of its branches which do not coincide on a part of the y-axis. Beltrami's error has often been repeated. We shall show that this error is more than a historic curiosity as the investigation of the situation leads to an interesting extension of the theory of discontinuous integrals. As $\partial\phi/\partial x = 0$ on the y-axis, we see that ψ^+ takes constant values on the interval $0 \leqq y < b$ and the interval $b < y < \infty$. The first of the constants is zero as ψ^+ is zero at the origin. Let us now apply the relationship

$$\int y^p \frac{\partial \phi}{\partial \nu} ds = \int \frac{\partial \psi^+}{\partial s} ds \tag{40}$$

to an infinitesmal arc of a circle with center at b. The computation of the residue of the integral on the left shows that ψ^+ is locally proportional to the angle β (Fig. 4). This leads to the fundamental formula (see Weinstein [4])

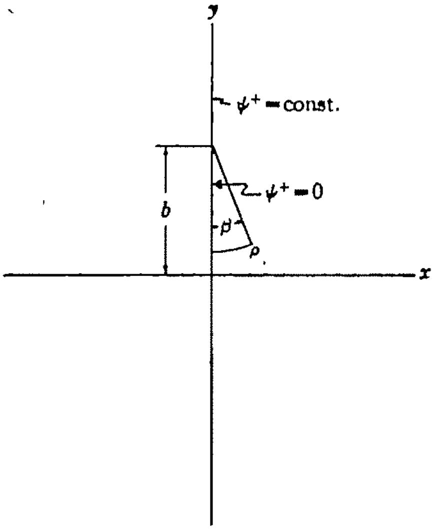

FIG. 4

$$\lim y^{p+1} \int_0^\infty e^{-xt} J_{p+1}(yt) J_p(bt) dt = \frac{\beta}{\pi} b^p, \tag{41}$$

where $P \rightarrow (0, b)$ along a ray. Let us note that, for integral values of p, Van Tuyl [19] and also Sadowsky and Steinberg [20] expressed the many-valued function ψ, defined by (38) and (39), in terms of elliptic integrals. The formula (41) contains as a special case the results of Weber and Schafheitlin. In fact, by taking $\beta = \pi$, we obtain not only the limit but the constant value of ψ^+ on the interval $b < y < \infty$, namely $-\pi C_p$. Choosing $\beta = \pi/2$, we obtain $\psi^+ = -(\pi/2) C_p$ at the branch point. These results are obviously equivalent to the classical Weber-Shafheitlein formula

$$\int_0^\infty J_{p+1}(yt) J_p(bt) dt = \begin{cases} 0, & y < b, \\ (2b)^{-1}, & y = b, \\ b^p y^{-(p+1)}, & y > b. \end{cases} \tag{42}$$

Thus, our theory establishes a connection between discontinuous integrals and many-valued functions.

10. The Green's function for a half-plane and Poisson's integral. Let us use again the notation $z\{k\}$ instead of $\phi\{p\}$ and $\psi\{p\}$. In section 8 we gave several expressions which represent a fundamental solution $z\{k\}$, $k>0$, in the half-plane $y\geqq 0$. The equation (11), $z\{k\}=Cy^{1-k}z\{2-k\}$, permits us to find new fundamental solutions in the large which are of particular interest in the interval $-\infty<k<1$. They are obtained by taking for $z\{2-k\}$ the fundamental solutions corresponding, in the presentation notation, to (36) or (37). As these new fundamental solutions vanish on the x-axis, we have thus established an explicit formula for the Green's function belonging to the half-plane $y\geqq 0$ for the equation $z_{xx}+z_{yy}+(k/y)z_y=0$, for $-\infty<k<1$. Corresponding to this Green's function one has a Poisson's integral formula

$$z\{k\}=Ay^{1-k}\int_{-\infty}^{\infty}\frac{f(\xi)d\xi}{[(x-\xi)^2+y^2]^{(2-k)/2}}, \tag{43}$$

which solves the Dirichlet problem for the half-plane $y\geqq 0$, for given bounded continuous boundary values $f(\xi)$. The numerical constant A is determined by putting $f(\xi)\equiv 1$. As it will be shown in a forthcoming paper, Green's function does not exist for $k\geqq 1$.

11. Tricomi's equation. One of the most interesting applications of GASPT is a new approach, given by Weinstein [22], to the theory of Tricomi's equation

$$\eta z_{\xi\xi}+z_{\eta\eta}=0, \qquad z=z(\xi,\eta). \tag{44}$$

This is the simplest equation of mixed type. It is elliptic in the upper half-plane $\eta>0$ and hyperbolic in the lower half-plane $\eta<0$.

In the hyperbolic half-plane $\eta<0$ let us put

$$x=\xi, \qquad t=2(-\eta)^{3/2}/3. \tag{45}$$

In this way we get as a normal form for Tricomi's equation an Euler-Poisson equation

$$z_{xx}-z_{tt}-\frac{1}{3t}z_t=0, \tag{46}$$

which belongs to the type of Euler's equation treated by Riemann in his celebrated memoir on gas dynamics. Equation (46) was used by Tricomi and his successors as a main road of access to the general study of the mixed equation (44). More recently, the following new approach has been proposed [21; 22]. Let us first consider the elliptic half-plane $\eta > 0$. Tricomi's equation can be reduced by the transformation

$$x = \xi, \qquad y = 2\eta^{3/2}/3, \qquad \eta > 0, \tag{47}$$

to the equation

$$z_{xx} + z_{yy} + \frac{1}{3y} z_y = 0 \tag{48}$$

which is an axially symmetric potential in $2\frac{1}{3}$ dimensions. Therefore, the results of GASPT can be applied immediately.

Before proceeding, let us make the following simple remark. To any solution of the equation

$$z_{xx} + z_{yy} + \frac{k}{y} z_y = 0, \qquad k \gtrless 0, \tag{49}$$

there corresponds a solution of the equation

$$z_{xx} - z_{tt} - \frac{k}{t} z_t = 0, \qquad k \gtrless 0, \tag{50}$$

which is obtained from (48) by putting $y = it$. For instance, to the Laplace integral (33) corresponds the following solution of Euler's equation:

$$z\{k\} = C_k \int_0^\pi g(x + t\cos\alpha)\sin^{k-1}\alpha\, d\alpha, \qquad k > 0,\ z_t(x, 0) = 0, \tag{51}$$

which is a celebrated formula given by Poisson.

For certain rational values of the parameter k the equations (49) and (50) are connected by an equation of Tricomi's type (44) in which the factor η is replaced by η^{2m+1}, $m = 0, \pm 1, \pm 2, \cdots$.

Let us now return to Tricomi's equation. A question of paramount interest, widely discussed in recent years, is the determination of a fundamental integral which is defined in the entire plane and has a logarithmic singularity in the elliptic half-plane.

The simplest way to obtain such an integral is to take the fundamental solution (37) corresponding to a ring and rewrite it in the ξ, η coordinates for $\eta > 0$. This solution takes the value

$$z(\xi, 0) = (\xi^2 + (4/9)\beta^3)^{-1/6}, \qquad (4/9)\beta^3 = b^2, \tag{52}$$

on the ξ-axis while $z_\eta = 0$. Then, using Poisson's formula (51) for

Euler's equation, we obtain an analytic continuation for all $\eta \leqq 0$ given by

$$(53)\quad s(\xi, \eta) = C_{1/3} \int_0^{\pi} [(\xi + (2/3)(-\eta)^{3/2} \cos \alpha)^2 + (4/9)\beta^2]^{-1/6} \sin^{-2/3} \alpha d\alpha.$$

This procedure indicates the fruitful correspondence between GASPT, the Euler-Poisson-Riemann equation, and Tricomi's equation.

12. **Concluding remarks.** The review of GASPT given in the preceding paragraphs is by no means complete. We have omitted numerous applications such as transonic flow (Weinstein [22]), the method of sources and sinks (Weinstein [23; 24]), the theory of dislocations (Weinstein [12]), and turbulent diffusion [25]. Neither did we mention the extension of the Riemann's integration method as given by Martin [26] and later by Diaz and Martin [27] in which a pair of associate Euler-Poisson equations plays an important part in the Cauchy problem for the wave equation.

Finally, there is an important extension of GASPT to equations for functions $s(x_1, \cdots, x_m)$ of $m > 2$ variables. The equation (9) is here replaced by

$$(54)\qquad \frac{\partial^2 s}{\partial x_1^2} + \cdots + \frac{\partial^2 s}{\partial x_{m-1}^2} + \frac{\partial^2 s}{\partial x_m^2} + \frac{k}{x_m} \frac{\partial s}{\partial x_m} = 0,$$

which is the Beltrami-Laplace equation on a Riemannian manifold defined by the metric

$$(55)\qquad ds^2 = \frac{dx_1^2 + \cdots + dx_m^2}{(x_m)^{2k/(2-m)}}.$$

This metric is of particular interest, since a special case $(k = 2 - m)$ of (55) is the ds^2 of a Poincaré manifold of constant negative curvature -1. We note finally that there is a formal correspondence between (54) and the Euler-Poisson-Darboux equation in several variables which plays such an important role in the integration of the wave equation. These questions, however, would lead us into some problems concerning hyperbolic equations, which lie beyond the scope of the present address [28].

Added in proof (*January* 27, 1953). In a paper presented to the American Mathematical Society, A. Huber extended considerably the results of §7 by proving the following uniqueness theorem: If $s\{k\}$ is a solution of (54) assuming the boundary value 0 on an open subset S of the hyperplane $x_m=0$, then, for $k\geqq 1$, $z\{k\}\equiv 0$, and for $k<1$, $z\{k\}\equiv x_m^{1-k}z\{2-k\}$, where $z\{2-k\}$ is analytic also on S. A fundamental solution in the large for (54) has been recently given by Diaz and Weinstein in a paper to be published in the Anniversary Volume for R. von Mises.

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THE OCTOBER MEETING IN NEW HAVEN

The four hundred eighty-fourth meeting of the American Mathematical Society was held at Yale University on Saturday, October 25, 1952, in the Osborn Zoological Laboratory. There were about 200 persons in attendance including the following 169 members of the Society:

C. R. Adams, R. B. Adams, D. B. Ames, Joseph Andrushkiw, W. G. Bade, R. B. Barrar, J. Y. Barry, R. G. Bartle, G. E. Bates, J. D. Baum, Anatole Beck, E. G. Begle, P. H. Belisle, R. R. Bernard, Benjamin Bernholtz, D. L. Bernstein, L. F. Boron, C. J. Bouwkamp, J. W. Bower, Richard Brauer, F. E. Browder, A. B. Brown, A. R. Brown, L. W. Cohen, H. S. Collins, R. M. Conkling, T. F. Cope, V. F. Cowling, J. B. Crabtree, Frederic Cunningham, M. D. Darkow, M. D. Davis, R. B. Davis, Allen Devinatz, D. J. Dickinson, H. L. Dorwart, T. C. Doyle, G. R. Duke, Nelson Dunford, W. L. Duren, William H. Durfee, Aryeh Dvoretzky, J. E. Eaton, Samuel Eilenberg, Joanne Elliott, Trevor Evans, J. M. Feld, Chester Feldman, William Feller, W. E. Ferguson, Alvaro Ferlini, R. M. Foster, W. C. G. Fraser, J. B. Freier, Bernard Friedman, M. P. Gaffney, Murray Gerstenhaber, H. A. Giddings, Oscar Goldman, L. W. Green, Laura Guggenbuhl, F. C. Hall, Duncan Harkin, Melvin Hausner, G. A. Hedlund, Henry Helson, S. B. Hobbs, S. P. Hoffman, Alfred Horn, W. A. Howard, J. L. Howell, D. H. Hyers, H. G. Jacob, Nathan Jacobson, R. E. Johnson, R. V. Kadison, Shizuo Kakutani, Aida Kalish, M. E. Kellar, Fred Klokemeister, George Klein, W. J. Klimczak, A. T. Kovitz, P. H. Kratz, Serge Lang, Solomon Leader, Jean Leray, Marie Lesnick, D. J. Lewis, W. G. Lister, A. E. Livingston, Eugene Lukacs, B. H. McCandless, G. W. Mackey, H. M. MacNeille, L. F. Markus, H. G. Mazurkiewicz, L. F. Meyers, E. J. Miles, W. H. Mills, Don Mittleman, R. E. Montgomery, J. C. Moore, Morris Morduchow, G. D. Mostow, Simon Mowshowitz, D. J. Newman, E. N. Nilson, Katsumi Nomizu, E. T. Onat, R. H. Owens, F. P. Pedersen, F. W. Perkins, R. S. Pierce, H. O. Pollak, Harry Pollard, M. H. Protter, F. A. Quigley, R. A. Raimi, Helene Reschovsky, H. G. Rice, C. E. Rickart, R. A. Roberts, G. B. Robison, Hartley Rogers, I. H. Rose, J. P. Russell, Charles Salkind, J. E. Sammet, Arthur Sard, W. K. Saunders, R. D. Schafer, Robert Schatten, Abraham Schwartz, J. T. Schwartz, C. H. W. Sedgewick, C. B. Seligman, V. L. Shapiro, O. K. Smith, N. E. Steenrod, Marvin Stern, R. L. Sternberg, F. M. Stewart, M. H. Stone, Walter Strodt, M. M. Sullivan, R. L. Taylor, D. L. Thomsen, J. I. Tracey, H. G. Tucker, Bryant Tuckerman, D. F. Votaw, J. V. Wehausen, H. F. Weinberger, Morris Weisfeld, Paul Weiss, David Wellinger, John Wermer, A. L. Whiteman, Hassler Whitney, Albert Wilansky, C. H. Wilcox, A. B. Willcox, N. Z. Wolfsohn, K. G. Wolfson, Arthur Wouk, Fumio Yagi, D. M. Young, Oscar Zariski.

An address *The linear hyperbolic differential equation*, was presented at the General Session by Professor Jean Leray of the Collège de France and The Institute for Advanced Study by invitation of the Committee to Select Hour Speakers for Eastern Sectional Meetings. Professor William Feller presided.

Sessions for contributed papers were held in the morning, Professor W. L. Duren presiding, and in the afternoon, Professor T. F. Cope presiding.

Abstracts of the papers presented at the meeting follow, those with "*t*" after the abstract number having been read by title. Of the papers having joint authorship paper 4 was presented by Dr. Taylor, paper 11 by Professor Cowling, and paper 12 by Dr. Weinberger. Mr. Chehata was introduced by Dr. B. H. Neumann, Professor Zassenhaus by Professor S. A. Jennings, and Mrs. Dickinson by Professor R. L. Wilder.

ALGEBRA AND THEORY OF NUMBERS

1*t*. Leonard Carlitz: *Some sums analogous to Dedekind sums.*

Analogues of the Dedekind sums are constructed in the field $GF(q, x)$. The most interesting property of the analogues is a reciprocity theorem analogous to the well known Dedekind reciprocity theorem. In addition analogues are obtained for Theorems 1 and 2 of Rademacher and Whiteman's paper (Amer. J. Math. vol. 63 (1941) pp. 377–407). (Received August 11, 1952.)

2*t*. Leonard Carlitz: *The reciprocity theory for Dedekind sums.*

By means of the representation $s(h, k)=1/4k+(1/k)\sum_{\zeta\neq 1}(\zeta^{-1}-1)^{-1}(\zeta^{h}-1)^{-1}$, where ζ runs through the kth roots of unity and $s(h, k)$ denotes the Dedekind sum, a proof of the reciprocity theorem is given. The proof was suggested by Redei's proof (Acta Scientarium Mathematicarum vol. 12, part B (1950) pp. 236–239). The method also applies to Apostol's generalization (Duke Math. J. vol. 17 (1950) pp. 147–157). (Received August 11, 1952.)

3*t*. C. G. Chehata: *On an ordered semigroup.*

The semigroup of Malcev (Math. Ann. vol. 113 (1937) pp. 686–691), which is known not to be embeddable in a group, can be turned into a (fully) ordered semigroup. This answers negatively the question whether an ordered semigroup can always be embedded in a group, ordered or not. An incidental consequence is that the ring considered by Malcev (loc. cit.), which can not be embedded in a division ring, can nevertheless be embedded in a formal power series ring. (Received September 8, 1952.)

4. Anne P. Cobbe and R. L. Taylor: *On Q-kernels with operators.*

Let H be an aditive abelian group with a multiplicative group of operators B. A *domain* for (B, H) is a set D with an additive, simply transitive operation of H on D and a multiplicative operation of B on D, such that $b(h+d)=bh+bd$. The isomorphism-classes of domains form a group, isomorphic in a natural way to $H^1(B, H)$. The *deviation* of a domain D is the corresponding element of $H^1(B, H)$; it is zero if and only if D has a fixed point under B. Let $\theta: Q\rightarrow A(K)/I(K)$ be an extendible Q-kernel; let B denote the group of all automorphisms b of Q such that $\theta b=\theta$. If (E, ρ) is an extension of $\theta: Q\rightarrow A(K)/I(K)$, then so is $(E, b\rho)$, for every $b\in B$. Thus B operates on the set $\mathcal{E}$ of all isomorphism-classes of extensions of $\theta: Q\rightarrow A(K)/I(K)$. B also operates on $H^2(Q, Z_K)$; $\mathcal{E}$ forms a domain for $[B, H^2(Q, Z_K)]$, whose deviation $\in H^1[B, H^2(Q, Z_K)]$ is called the *domain-deviation* of $\theta: Q\rightarrow A(K)/I(K)$. An example shows that the B-orbits of $\mathcal{E}$ cannot be determined from the operation of B on $H^2(Q, Z_K)$, without knowledge of the domain-deviation; the latter is computed explicitly for cyclic Q. The *equivariant obstruction*, an element of $H^3_B(Q, Z_K)$, is defined whether

θ: $Q \to A(K)/I(K)$ is extendible or not; it determines, but is not determined by, the Eilenberg-MacLane obstruction; in the extendible case, it determines, but is not determined by, the domain-deviation. The ideas are generalized to the case of a Q-kernel with a group B operating on both Q and K. (Received August 5, 1952.)

5. Serge Lang: *The theory of real places.*

Let K and L be two fields. An L-valued place ϕ of K is a homomorphism of K into $\{L, \infty\}$. We say that ϕ is a real place if L is a real field. If ϕ is a real place of K taking its values in a real closed field R, then K is real and ϕ can be extended to an R-valued place of a real closure of K. The real places of K can be completely classified in terms of the possible orderings of K. Let K be a function field (in several variables) over a real constant field F and suppose K is ordered. Let $\xi_1 < \xi_2 < \cdots < \xi_n$ be a finite set of elements of K. Then there exists a zero-dimensional real place ϕ of K (i.e. ϕ takes its values in the real closure R of F and is identity on F) such that all $\phi(\xi_i)$ are finite and $\phi(\xi_1) < \phi(\xi_2) < \cdots < \phi(\xi_n)$. This generalizes the Hilbert Problem solved by Artin. Sturm's Theorem remains an essential step in the argument. (Received July 30, 1952.)

6t. G. N. Raney: *A subdirect-union representation for completely distributive complete lattices.* Preliminary report.

Let L be a complete lattice. Define a semi-ideal of L to be a subset M of L such that if $y \in M$ and $x \leqq y$, then $x \in M$. For every $y \in L$, let $K(y)$ be the intersection of the family of semi-ideals M such that $\bigcup M \geqq y$. It is shown that L is completely distributive if and only if for every $y \in L$, $\bigcup K(y) = y$. Let $x\rho y$ mean $x \in K(y)$; then the relation ρ has the property that $\rho\circ\rho = \rho$. Moreover, it is shown that with every relation ρ such that $\rho\circ\rho = \rho$ there is associated a competely distributive complete lattice L, in such a way that if ρ is reflexive (and hence a quasi-ordering) then L is isomorphic with a complete ring of sets. This generalizes a result of G. Birkhoff. In terms of these notions it is proved that every completely distributive complete lattice is isomorphic with a closed sublattice of the direct union of a family of complete chains. (Received September 5, 1952.)

7t. H. J. Zassenhaus: *On the representations of Lie algebras of prime characteristic.*

It is shown that the enveloping algebra $A(L)$ of a Lie algebra L of finite dimension n over a field F of prime characteristic p is a maximal order of a division algebra K of dimension p^{2m} over its center $Q(Z)$, the quotientfield of the center Z of $A(L)$. Z is an algebraic variety of dimension n. Each specialization Θ of Z onto F over F is extendible to a specialization Θ^* of $A(L)$ onto a hypercomplex system $\Theta^*A(L)$ over F. Each absolutely irreducible representation Δ^* of $\Theta^*A(L)$ over F induces the absolutely irreducible representation Δ of L given by $\Delta(a) = \Delta^*\Theta^*(a)$ and each absolutely irreducible representation of L is obtained this way. The degree of Δ is not greater than p^m and smaller than p^m only for the genuine subvariety of specializations of Z mapping the discriminant ideal of $A(L)$ over Z onto zero. The absolutely irreducible constituents of an absolutely indecomposable representation of L over F all belong to the same specialization Θ of Z onto F. For each Θ there are absolutely indecomposable representations of arbitrary high degree. Application is made to nilpotent Lie algebras and the Lie algebras of *Witt* where some of the results had been known before. (Received September 8, 1952.)

ANALYSIS

8. Joseph Andrushkiw: *On the positivity of a polynomial for positive values of the variable.*

Let $z_1, z_2, \cdots, z_n$ denote the zeros of the real polynomial $f(z)$, $s_k=\sum_{j=1}^{n} z_j^k$, the D_k and $\bar{D}_k$ the determinants $|s_{i+j}|$, $i, j=0, 1, \cdots, k-1$, and $|s_{i+j-1}|$, $i, j=1, 2, \cdots, k$, respectively. Denoting by d and $\bar{d}$ the differences $p-v$ and $\bar{p}-\bar{v}$ of the sign permanences and sign variations in the sequences $1, D_1, \cdots, D_n$ and $1, \bar{D}_1, \cdots, \bar{D}_n$ respectively, one obtains, following the method of Hermite (E. Netto, *Vorlesungen ueber Algebra*, vol. 1, p. 263–270), that $d+\bar{d}=0$ is necessary and sufficient condition that $f(z)>0$, $z\geqq 0$, in case, if $D_k\neq 0$ and $\bar{D}_k\neq 0$, $k=1, 2, \cdots, n$ (regular case). Based on the results obtained in a former paper (Bull. Amer. Math. Soc. Abstract 58-3-289) another criterion, without any restriction, can be obtained. Putting $f(-z^2)=g(z)$, $F(z)=g(z+x)$, the Hurwitzian determinant (Hurwitz, *Werke*, vol. 2, p. 534) $H_{2n}(x)$ of $F(z)$ is constructed. Let $H_j^{(i)}(x)$ denote the ith derivative of its principal minor of order j. If $H_j(0)=H_j'(0)=\cdots=H_j^{(i-1)}(0)=0$, $H_j^{(i)}(0)=h_j\neq 0$, let v denote the number of sign variations in the sequence $1, h_1, h_2/h_1, \cdots, h_{2n}/h_{2n-1}$. Then $v=n$ is necessary and sufficient condition that $f(z)>0$, $z\geqq 0$. In the regular case it is shown that $h_{2k-1}=D_k$, $h_{2k}=\bar{D}_k$. (Received September 10, 1952.)

9. F. E. Browder: *Completeness of eigenfunctions and eigenfunction expansions for non self-adjoint linear elliptic differential equations of arbitrary order.*

Let K be a suitably differentiable linear elliptic differential operator of order $2m$ on the bounded domain D of E^n, B a suitably differentiable positive self-adjoint differential operator on D of order less than $2m$. If $\phi\in C^{2m}(D)$ and vanishes with all its derivatives of order less than m on the boundary of D, ϕ is said to be an nth order eigenfunction of K with respect to B and with B-eigenvalue λ provided that $K\phi-\lambda B\phi=B\psi$ where ψ is an eigenfunction of order less than n of K with respect to B and with B-eigenvalue λ. For $n=1$ these are the ordinary eigenfunctions of K with respect to B. It is shown that there exists a set of eigenfunctions of K with respect to B which is complete and forms a basis in $L^2(D)$ and $H_m(D)$. If K commutes with B, the eigenfunctions of order n satisfy the equation $(K-\lambda B)^n\phi=0$. Use is made in the proofs of the operational calculus for analytic functions of linear operators on a Banach space developed by Gelfand, Lorch, and Dunford. (Received September 11, 1952.)

10. H. S. Collins: *Completeness and weak compactness in linear topological spaces.*

Let X be a locally convex Hausdorff linear topological space over the reals R. X has property A [property B] if the bounded weak-* and weak-* closures for linear [maximal linear] subspaces of X^* coincide. A set M in X is mean compact if given any sequence of points $\{x_n\}$ in X there is a point x in X such that $\lim\inf_n f(x_n)\leqq f(x)\leqq \lim\sup_n f(x_n)$ for all f in X^*. The results are: 1. If M is mean compact, then $\bar{M}$ is contained in the adjoint of (X^*, bounded weak-*), where $\bar{M}$ denotes the closure of M in the product topology of R^{X^*}. If X has property B, it follows that mean compactness and the various well known weak compactness notions are all equivalent, a result which generalizes earlier work of Smulian, Phillips, and Eberlein, and later work of Grothendieck. 2. Property B is equivalent to completeness. 3. Property A is, in general, stronger than property B, and if X has property A and L is any closed linear subspace of X, then L, X/L, any linear-open image of X, and any finer topology for X giving the same adjoint all have property A. In virtue of these results, property A is suggested as a more desirable definition of completeness in locally convex spaces. (Received May 26, 1952.)

11. V. F. Cowling and W. J. Thron: *Zero-free regions of polynomials.*

In this paper we obtain certain results about the location of the zeros of a polynomial. We use a method which was developed by Leighton and Thron (Bull. Amer. Math. Soc. vol. 50) to study the convergence of continued fractions and was then adapted by Cowling (Proceedings of the American Mathematical Society vol. 2) to obtain results on the distribution of values of the partial sums of a Taylor series. This latter method has now been further modified to make it as suitable as possible for our present purpose. A typical theorem which is proved is the following: *Let* $P(z)=a_0+a_1z^{r_1}+\cdots+a_nz^{r_n}$ *where all* a_k, $k=0,\cdots,n$, *are different from zero and where* $r_1<r_2<\cdots<r_n$. *Then* $P(z)$ *assumes all its zeros for* $|z|\leq\max_{1\leq k\leq n}(((1+r_{k-1})/r_k)|a_{k-1}/a_k|)^{1/(r_k-r_{k-1})}$ *where* $r_0=0$, $r_n=1$, *and the remaining* r_k *are arbitrary positive numbers*. A well known result (see Dieudonné, *La théorie analytique des polynomes d'une variable*, p. 18) is obtained as a corollary of this theorem if we set $r_k=1$, $k=1,\cdots,n$. (Received August 15, 1952.)

12. J. B. Diaz and H. F. Weinberger: *A solution of the singular initial value problem for the Euler-Poisson-Darboux equation.*

A solution is found for the initial value problem (*) $\Delta u=u_{tt}+(K/t)u_t$ with $u(x_1\cdots x_m,0)=f(x_1,\cdots,x_m)$, $u_t(x_i,0)=0$ where K is a real constant and f is given. (Another solution for this initial value problem was recently given by A. Weinstein, C. R. Acad. Sci. Paris vol. 234 (1952) pp. 2584–2585.) It is based on the known fact that for $K=m-1$, the solution of (*) is $M(x_i,t;f)$, the mean value of f over the surface of the sphere of radius t about the point $(x_1,\cdots,x_m)$. For $K>m-1$, the solution of (*) is given by an integral formula which coincides with the solution obtained by Hadamard's method of descent for $K=m,m+1,m+2,\cdots$. For $K\leq m-1$, this integral does not converge, but the function of K represented by it is continued analytically. In this manner, the following solution of (*) is obtained for $K>m-2p-3$ (p a non-negative integer, $K\neq-1,-3,\cdots$): $A\int_0^1N_p(x_i,\alpha,t;f)(1-\alpha^2)^{(K-m-1)/2}\alpha^{m-1}d\alpha+B\sum_{n=0}^{p}(t^n/n!)(\partial^nM/\partial t^n)(x_i,t;f)\cdot S_n$ where A, B, and S_n are known constants, and N_p is $M(x_i,\alpha t;f)$ minus the first $p+1$ terms of its Taylor series in α, about $\alpha=1$. For $m-k$ a positive odd integer, the first term disappears and one obtains a Huygens' principle. (Received September 8, 1952.)

13. Joanne Elliott: *The boundary value problems and semi-groups associated with certain integro-differential operators.*

We are concerned with initial value problems for the pair of equations (1) $u_t(x,t)=\pi^{-1}P\cdot\int_{-1}^{+1}(u_\xi(\xi,t)/(\xi-x))d\xi=\Omega u$ and (2) $u_t(x,t)=\pi^{-1}(d/dx)P\cdot\int_{-1}^{+1}((u(\xi,t)/(\xi-x))d\xi=\Omega^*u$ in $C[-1,+1]$ and $L[-1,+1]$ respectively. Probabilistically, these equations bear the same relation to the Cauchy process as the backward and forward heat equations on a finite interval bear to homogeneous diffusion on the infinite interval. Hence, it is surprising that each equation behaves analytically like a certain parabolic differential equation with non-constant coefficients. The solutions are treated as semi-groups of transformations on the initial values. Following the method of Feller

(Ann. of Math. vol. 55), we start with (1) and study the resolvent equation $\lambda f - \Omega f = h$ in $C[-1, +1]$. This is reduced to an integral equation, and the existence of positive, norm-not-increasing solutions is proved. It is shown, by use of the Hille-Yosida criterion for infinitesimal generators, that the most general boundary conditions for the parabolic equations have analogues in this case. The adjoint semi-groups are then studied, and it is shown that in certain important cases they are generated by Ω^*, but that in general the generators will contain additional terms. (Received September 10, 1952.)

14t. William Feller: *On the generation of strongly continuous semi-groups.*

Let $\{T_t\}$ be a semi-group of bounded linear operators from a Banach space X to itself which is (strongly measurable and therefore) strongly continuous for $t>0$. Under the restrictive conditions (1) $T_t x \to x$ as $t \to 0$ for each $x \in X$, (2) $\|T_t\| < 1 + ct$ for $t < 1$, Hille and Yosida defined an infinitesimal generator Ω and succeeded in characterizing all infinitesimal generators in terms of the resolvent $(\lambda I - \Omega)^{-1}$. In this paper an analogous result is obtained for the general case where both (1) and (2) are dropped. Then $\|T_t\|$ may be unbounded near $t=0$. The theory of the "resolvent" goes through although $(\lambda I - \Omega)^{-1}$ is an *unbounded operator* and the entire right half of the λ-plane may belong to the *continuous spectrum* of Ω. By specialization one obtains a characterization of Ω for the case of strong continuity at $t=0$ (condition 1). (Received September 10, 1952.)

15t. William Feller: *Semi-groups of transformations in general weak topologies.*

For semi-groups $\{T_t\}$ of bounded linear operators from a real Banach space X to itself usually strong measurability is assumed. Translations in $BV(-\infty, -\infty)$ are not even weakly measurable, translations in $C(-\infty, -\infty)$ not strongly measurable. A more general approach avoids these difficulties. Let X^* be the conjugate space of X and T_t^* the adjoint transformation to T_t. Let $Y \subset X^*$ be invariant under $\{T_t^*\}$ (e.g. $Y =$ subspace generated by $T_t^* x_0^*$). Call $\{T_t\}$ *weakly* (Y) *measurable* if the real function $x^*(T_t x)$ is L-measurable whenever $x \in X$, $x^* \in Y$. Considering $y \in Y$ as operator from X to the set of L-measurable functions leads to a definition of "*essential norm*" $\|T_t\|_e$. It is L-measurable and bounded in $\delta < t < \delta^{-1}$ (while, according to R. S. Phillips, neither statement is true of $\|T_t\|$, even if $\{T_t\}$ is weakly measurable, i.e. $Y = X^*$). It is shown that $\{T_t\}$ can be extended to spaces $X_e \supset X$, $Y_e \supset Y$ in which $\{T_t\}$ *is closed under Pettis integration*, and has a *weak continuity* property expressible in terms of Banach's transfinite limits. Each of these spaces contains a subspace in which $\{T_t\}$ is *strongly continuous* with a norm $\leq \|T_t\|_e$. The notions of *infinitesimal generator* and *resolvent* are generalized. Let X_0 be the space of those $x \in X$ for which $yT_t x = 0$ a.e. for each $y \in Y$. The quotient space x/X_0 can be remetrized and completed under a finer topology so as to make $\{T_t\}$ weakly continuous with bounded true norm. (Received September 10, 1952.)

16t. A. E. Livingston: *The Lebesgue constants for the Euler methods of summability.*

The Euler method (E, t), $0 < t \leq 1$, is defined by the infinite matrix (a_{nk}) where $a_{nk} = {}_nC_k t^k (1-t)^{n-k}$ for $k = 0, \cdots, n$ and $a_{nk} = 0$ for $k = n+1, n+2, \cdots$. The Lebesgue constants for the method (E, t) are then given by $L_n(t)=(2/\pi)\int_0^\pi(|\mathrm{Im}[e^{ix/2}(1-t+te^{ix})^n]|/2\sin 2^{-1}x)dx$. The author shows that $L_n(t)=(2/\pi^2)\log n+O(1)$ for $0<t<1$. (Received October 25, 1952.)

17t. G. W. Mackey: *A Frobenius reciprocity theorem for noncompact subgroups.* Preliminary report.

Let $\mathcal{G}$ be a separable unimodular locally compact group and let G be a unimodular closed subgroup of $\mathcal{G}$. Let the regular representations of G and $\mathcal{G}$ be of type I and let their canonical decompositions into factor representations be $\int_X^\oplus F d\zeta(x)$ and $\int_Y^\oplus H d\eta(y)$ where ${}^xF$ is a multiple of the irreducible representation ${}^xL$ of G and ${}^yH$ is a multiple of the irreducible representation ${}^yM$ of $\mathcal{G}$. Then there exist functions $x\to\beta_x$ and $y\to\gamma_y$ where each β_x is a measure on Y and each γ_y is a measure on X such that the measures $\int\gamma_y d\eta(y)$ and $\int\beta_x d\zeta(x)$ on $X\otimes Y$ are identical. Moreover if α denotes this measure, then there exists an α measurable function n from $X\otimes Y$ to the at most countable cardinals such that for almost all x the representation of $\mathcal{G}$ induced by ${}^xL$ is equivalent to $\int_Y n(x, y)\,{}^yM d\beta_x(y)$ and for almost all y the restriction to G of ${}^yM$ is equivalent to $\int_X n(x, y)\,{}^xL d\gamma_y(x)$. When G is compact the result takes on a simpler form and the hypothesis on the regular representation of $\mathcal{G}$ can be omitted. This case has been treated by Mautner by what seem to be rather different methods. The hypothesis of unimodularity can be weakened and probably eliminated altogether. (Received September 10, 1952.)

18t. J. L. Massera: *Total stability and approximately periodic vibrations.*

Let $\dot{x}=X(x, t)$, $X(0, t)=0$, be a system of differential equations. The solution $x=0$ is said to be totally stable if given $\epsilon>0$ there is a $\delta>0$ such that the solutions $x(t)$ of a varied system $\dot{x}=X_1$ satisfy $\|x(t)\|<\epsilon$ for $t\geqq 0$, provided that $\|x(0)\|<\delta$ and $\|X-X_1\|<\delta$. A sufficient condition for the total stability is the existence of a Lyapunov function $V(x, t)$ with the properties: (a) V is positive definite; (b) dV/dt is negative definite; (c) the partials of V are bounded. It is possible to prove similar sufficient conditions for the total stability of a solution in the "orbital" sense. A function $x(t)$ is said to be ϵ^+-periodic of period T if there is a sequence t_n, $t_0=0$, $0<T-\epsilon<t_{n+1}-t_n<T+\epsilon$, such that $|x(t_n+\tau)-x(\tau)|<\epsilon$ for $0\leqq\tau\leqq t_{n+1}-t_n$. It is then possible to show that systems of differential equations which are close to systems having periodic totally stable (in the strong or in the orbital sense) solutions, have ϵ^+-periodic solutions. (Received September 15, 1952.)

19. M. H. Protter: *The Cauchy problem with data on the parabolic line.*

Consider the equation $K(y)P(x, y)u_{xx}-u_{yy}+a(x, y)u_x+b(x, y)u_y+c(x, y)u=f(x, y)$ where $K(y)$ behaves like y^α, $\alpha>0$, in a neighborhood of $y=0$, $P(x, y)>0$ for all x, y, and all coefficients are, say, twice differentiable. The Cauchy problem for the above equation is treated when the initial data are prescribed along a segment of the x-axis (which is a parabolic line). It is shown that the Cauchy problem is correctly set if $ya(x, y)/(K(y))^{1/2}\to 0$ as $y\to 0$. This generalizes previous results of Berezin and Bers for the Cauchy problem. The method consists of reducing the above equation to a system of the first order and then solving the corresponding system of integral equations obtained by integrating along characteristics. (Received September 10, 1952.)

20. V. L. Shapiro: *An extension of results in the uniqueness of double trigonometric series.*

Let $T=\sum a_{mn}e^{i(mx+ny)}$ be a double trigonometric series whose coefficients $a_{mn}=o(1)$. Call T a series of type (U) if the circular partial sums of the series $\sum (a_{mn}/(m^2+n^2))e^{i(mx+ny)}$ converge uniformly. Suppose the circular partial sums of T are $(C, 1)$ summable to a function $f(x, y)$. It is known (M. Cheng, Ann. of Math. vol. 52 (1950) pp. 403–416) that if $f(x, y)$ is mean continuous everywhere and the Fourier series of $f(x, y)$ is of type (U), then T is the Fourier series of $f(x, y)$. In this paper it is proved that T is also the Fourier series of $f(x, y)$ if $f(x, y)$ is an essentially bounded function. In case T is a double sine series of type (U), then it is shown in this paper that $f(x, y)$ need only be an integrable function for T to be its Fourier series. (Received September 10, 1952.)

21*t*. Bertram Yood: *Complex Banach algebras with an involution.*

Let B, B_1 be two complex Banach algebras with an involution. Call B a ρ^*-algebra if there exists an auxiliary norm $|x|$ (in which B is a not necessarily complete normed algebra) and a number $k>0$ such that $\rho(x)\geq k|x|$ for all self-adjoint (s.a.) x, where $\rho(x)$ is the spectral radius of x. Call B a ρ_c^*-algebra if the auxiliary norm can be chosen with B complete in it. An important distinction between these algebras and the algebras with involution usually treated is that $x^*x=0$ does not imply $x=0$. It is shown that if B is a ρ^*-algebra in which every s.a. element has real spectrum, then B has the uniqueness of norm property discussed by Rickart [Ann. of Math. vol. 51 (1950) pp. 615–628]. If B and B_1 are two ρ_c^*-algebras where every s.a. element in B has real spectrum and the involution in B_1 is continuous, then any algebraic $*$-isomorphism of B into B_1 is bi-continuous. Conditions are given under which a ρ_c^*-algebra is $*$-isomorphic to a C^*-algebra where the $*$-isomorphism is bi-continuous. (Received September 10, 1952.)

22*t*. Bertram Yood: *Homomorphisms defined on regular Banach algebras.*

Let T be an algebraic homomorphism of a complex commutative Banach algebra B_1 into another such algebra B_2. It is assumed that each B_i has an identity element e_i, $T(e_1)=e_2$, and that B_1 is regular in the sense of Šilov [Trav. Inst. Math. Stekloff vol. 21 (1947)]. It is shown that $\overline{T(B_1)}$ is always regular, and that for each maximal ideal M, $T(M)$ is contained in a maximal ideal of B_2 if and only if $T^{-1}(\overline{T(R)})=R$ where R is the radical of B_1. If B_1 is semi-simple, T one-to-one, and the mapping $x\rightarrow x(M)$ of B_1 into the algebra of continuous functions on its space of maximal ideals has closed range, then T is continuous and $T(B_1)$ is closed. Some results are also obtained for the situation where T is not an isomorphism. (Received September 12, 1952.)

23*t*. H. J. Zimmerberg: *On the Green's matrix of a differential system.*

Let $G(x, t)$ be the Green's matrix for an incompatible homogeneous differential system $\mathcal{L}[y]=y'-A(x)y=0$, $My(a)+Ny(b)=0$, M and N complex constant matrices such that the boundary conditions are linearly independent. A theorem of Miller and Schiffer [Proceedings of the American Mathematical Society vol. 3 (1952) p. 436] is extended to show how n of the columns of the $n\times 2n$ matrix $\|G(x, a)\ G(x, b)\|$

may be selected as the columns of a fundamental matrix solution of $\mathcal{L}[y]=0$. (Received August 21, 1952.)

APPLIED MATHEMATICS

24. T. C. Doyle: *Topological and dynamical invariant theory of an electrical network.*

If s is a row matrix of the n oriented segments of a network G, the expression $g_i = s\xi_i$, ξ_i a column matrix of integers, is interpreted as an oriented subgraph g_i whose segment s_k has multiplicity ξ^k. Any transformation $g = s\xi$ of an infinite group G_ξ is regarded as a change of basis in the vector space V_n of subgraphs from the primitive basis of segments to a general basis of oriented subgraphs g_i. The topology of the network is its invariant theory under G_ξ. The dynamics of the network is the invariant theory of complex vector space under basis transformations G_ξ which replace the n segment currents J^i by component currents J^{i*} flowing through each segment of the subgraph g_i in the direction of its orientation heedless of Kirchoff's law. J^i is then the composite effect of the component contributions from each basis subgraph overlapping the segment. The fundamental theorem of electrical networks appears as a reduction to a canonical basis. (Received September 8, 1952.)

25*t*. H. F. MacNeish: *A symmetric method for determining the coefficients of a Fourier series.*

a_r, b_r represent the coefficients of the Fourier series $y=f(x)=\sum_{r=0}^{n}a_r \cos rx + \sum_{r=1}^{n-1} b_r \sin rx$, where $f(x)$ is defined by a table of $2n$ values (x_i, y_i), for $i=1, 2, 3, \cdots, 2n-1$. The Runge solution, called the Runge diagrammatic method, is based on (a) an elaborate diagram involving sums s_i and differences d_i of the values of y_i and further sums and differences of s_i and d_i, etc., taken in an erratic manner, and (b) a set of $2n$ opaque formulas for a_r and b_r, so that for the simple case $2n=12$, 26 auxiliary letters and 12 formulas are necessary, also (a) and (b) differ unsymmetrically with n. (See Scarborough, *Numerical mathematical analysis*, chap. 17.) In this paper a simple general scheme involving no diagram gives formulas for the coefficients a_r and b_r for all values of n. Only seven symmetric formulas are required involving $2n$ auxiliary letters $s_{i,i+n}=y_i+y_{i+n}$ and $d_{i,i+n}=y_i-y_{i+n}$, for which the notation is self-evident. It is assumed that the table of values represents a periodic function for which the period may be taken as 2π, from which it follows that $x_i=(180°/n)i$, $i=0, 1, 2, \cdots, 2n-1$. (Received August 26, 1952.)

26*t*. D. M. Young: *On Richardson's method for solving linear systems with positive definite matrices.*

For the linear system $\sum_{j=1}^{N}a_{i,j}u_j+d_i=0$ $(i=1, 2, \cdots, N)$ with positive definite matrix, L. F. Richardson (Roy. Soc. Philos. Trans. A 210 (1910)) presented an iterative method $u_i^{(n+1)}=u_i^{(n)}+\beta_{n+1}\{\sum_{j=1}^{N}a_{i,j}u_j^{(n)}+d_i\}$ $(i=1, 2, \cdots, N)$, where $u_1^{(0)}, u_2^{(0)}, \cdots, u_N^{(0)}$ are arbitrary and where the relaxation factors β_{n+1} are to be chosen. Using a theorem on Tschebyscheff polynomials given by Flanders and Shortley (Journal of Applied Physics vol. 21 (1950) pp. 1326–1332) the author has shown that if a good lower bound for the eigenvalues of $(a_{i,j})$ is known, the relaxation factors can be chosen so that the average rate of convergence is asymptotically proportional to $R^{1/2}$ as $R\to 0$ where R is the rate of convergence of the method using a fixed β. For a given integer m, the relaxation factors which yield the optimum convergence are given by

$\beta_n = 2[(b-a)t_n-(b+a)]^{-1}$ $(n=1, 2, \cdots, m)$ where $t_n = \cos[(2n-1)\pi/2m]$ and b and a are upper and lower bounds respectively for the eigenvalues of $(a_{i,j})$. Although stronger restrictions on $(a_{i,j})$ were required to prove the same order-of-magnitude gain in convergence rate for the successive overrelaxation method (Bull. Amer. Math. Soc. Abstract 56-4-322) Richardson's method is not as well adapted for large automatic computing machines and is slower in many cases by a factor of at least two. (Received September 5, 1952.)

GEOMETRY

27. L. F. Markus: *Orientability of line element fields.*

A continuous field of line elements on a simply-connected differentiable manifold is orientable. A similar statement holds for a regular curve family filling a simply-connected topological manifold. (Received October 18, 1952.)

28. Katsumi Nomizu: *Group of affine transformations of an affinely connected manifold.*

It is a well known theorem that the group of all isometries of a Riemannian manifold is a Lie group (S. B. Meyers and N. E. Steenrod, Ann. of Math. vol. 40 (1939)). The main purpose of the present paper is to establish a similar theorem for the group of affine transformations of an affinely-connected manifold. Given a manifold M with affine connection $dp = \sum_i \theta^i e_i$, $de_i = \sum_j \theta_i^j e_j$, where θ^i and θ_i^j are linear differential forms on the bundle of frames B of M which define the affine connection, a differentiable homeomorphism of M onto itself is called an affine transformation of M if the induced homeomorphism of B leaves the forms θ^i and θ_i^j invariant. Theorem. The group $A(M)$ of all affine transformations of M, provided with the compact-open topology, is a Lie group. For the proof use is made of a theorem of S. Bochner and D. Montgomery concerning locally compact groups of differentiable transformations (Ann. of Math. vol. 47 (1946)). (Received September 10, 1952.)

29t. Walter Prenowitz: *Partially ordered geometries.* Preliminary report.

A *partially ordered geometry* (p.o. geometry) is defined to be a system consisting of a set S of elements (*points*) $a, b, c, \cdots$, and a 3-term relation (*order*) expressed by the notation (abc) which satisfies the following postulates: 01. (aaa); 02. (abc) implies (cba); 03. (ap_1q_1), (ap_2q_2) imply the existence of x such that (p_1xq_2), (p_2xq_1); 04. (ap_1q_1), (p_2aq_2) imply the existence of a' such that $(a'p_1q_2)$, $(p_2a'q_1)$; 05. For any a, b there exists x such that (abx). A p.o. geometry is *simply ordered* if it satisfies 06. (abx), (aby), $x \neq y$ imply (axy) or (ayx). The notion p.o. geometry is introduced with the object of unifying the theory of classical geometries and their generalizations. Any descriptive geometry becomes a p.o. geometry if we define (abc) to mean either b is an interior point of segment ac or $a=b=c$. A spherical geometry may be converted into a p.o. geometry by the adjunction of an "ideal" point bearing appropriate order relations to the given points. Any projective geometry also can be converted into a p.o. geometry by adjoining properly an "ideal" point. Any Boolean algebra becomes a p.o. geometry if (abc) is defined to mean $a \cap c \leq b \leq a \cup c$. There exist p.o. geometries which are not simply ordered and whose points and "lines" form an affine geometry (Prenowitz, *Partially ordered fields and geometries*, Amer. Math. Monthly vol. 53 (1946) pp. 439–449). (Received September 15, 1952.)

LOGIC AND FOUNDATIONS

30. M. D. Davis: *Note on a conjecture of Post.*

$|\pi_\alpha|$, $|\bar{\pi}_\alpha|$, $\alpha<\omega^2$, are defined to be sequences of quantifiers as follows: $|\pi_1|$ is (Ex_1), $|\bar{\pi}_1|$ is (x_1), $|\pi_{\alpha+1}|$ is $(Ex_{\alpha+1})|\bar{\pi}_\alpha|$, $|\bar{\pi}_{\alpha+1}|$ is $(x_{\alpha+1})|\pi_\alpha|$, $|\pi_{\omega p}|$ is $(Ey_p)(k_p)_{k_p\leq y_p}(En_p)|\pi_{\omega(p-1)+n_p}|$, and $|\bar{\pi}_{\omega p}|$ is $(y_p)(Ek_p)_{k_p\leq y_p}(n_p)|\bar{\pi}_{\omega(p-1)+n_p}|$. Let the set of integers K_α, $\alpha<\omega^2$, be defined by taking K_0 to be empty, and setting $K_{\alpha+1}=(K_\alpha)'$, $K_{\omega p}=\{x|(x)_1\in K_{\omega(p-1)+(x)_2}\}'$. (The ′ notation follows Post's abstract, Bull. Amer. Math. Soc. vol. 54-7-269.) For $1\leq\alpha<\omega^2$, let P_α be the class of all predicates which are many-one reducible to K_α; Q_α the class of negations of members of P_α. Then, it is shown that a predicate W belongs to P_α (Q_α) if and only if there is a recursive predicate N [some of whose variables may range over the set of *all finite* sequences of natural numbers] such that $W\equiv|\pi_\alpha|N$ ($W\equiv|\bar{\pi}_\alpha|N$). This extends a theorem of Post (loc. cit.) into the transfinite, as conjectured by him, making use of an extension of Kleene's hierarchy of predicates developed in the author's dissertation (Princeton, 1950). (Received October 16, 1952.)

31. Hartley Rogers, Jr.: *Some results on definability and decidability in elementary theories.* I.

In Journal of Symbolic Logic vol. 16 (1951) pp. 239–240, Church and Quine show that a general binary relation is definable from a symmetric relation in the sense that an expression of quantification theory exists, in one binary function variable and two free individual variables, such that as symmetric relations are assigned to the function variable, the value of the entire expression runs through all binary relations. The general decision problem for quantification theory thus reduces to that for a single symmetric relation. As intermediate degrees of undecidability may possibly exist, this is at present somewhat stronger than an undecidability result alone. Using *relation-sets*—elementary theory models which are values for higher variables in third-order functional calculus—the writer develops a theoretical framework for results of this general kind. The theory accommodates results somewhat beyond the immediate definability method of the Church-Quine paper. The theory is applied to obtain reduction-undecidability-definability for the elementary theory of a single binary disjoint relation (domain and co-domain disjoint), of a single symmetric reflexive relation, and of a single transitive reflexive relation. Reduction-undecidability results are obtained for the cases of two equivalences, one equivalence and one total order, and two total orders. (Received August 6, 1952.)

32*t*. Hartley Rogers, Jr.: *Some results on definability and decidability in elementary theories.* II.

A decision method for the elementary theory of a single equivalence is known, also for monadic quantification theory with identity, i.e., the elementary theory of a single equivalence R and any number of singulary relations F_i which are well-defined with respect to that equivalence ($R(x, y)\supset_{xy}\cdot F_i(x)\supset F_i(y)$ is true). These results are here extended by deriving decision methods for the elementary theory of a single symmetric and transitive relation, and for the elementary theory of a single equivalence together with any number of singulary relations, well-defined or not. By the *elementary theory of rectangular arrays* we shall mean that theory obtained when we consider matrix arrays of 0's and 1's, use two binary constants C and R interpreted as "in the same column as" and "in the same row as," consider occurrences of 1's as individuals, and construct formulas within quantification theory. Results of this paper and the preceding paper are used together to show that the theories of $m\times$*finite*, $m\times\infty$, as well as $m\times n$ arrays are decidable, while the theory for $\infty\times\infty$ arrays is undecidable though axiomatizable. The theories for *finite*$\times$*finite* and *finite*$\times\infty$ arrays are shown equivalent, though decision and axiomatizability here remain undetermined. (Received August 6, 1952.)

STATISTICS AND PROBABILITY

33*t*. Herbert Robbins: *On the equidistribution of the sequence of cumulative sums of independent random variables.*

Let $\{X_n\}$ be independent random variables with a common distribution F and let $S_n=X_1+\cdots+X_n$. The author is interested in the general problem of proving that, under certain hypotheses on F, the sequence $\{S_n\}$ is with probability 1 "equidistributed" (i.e. "uniformly dense") on the line $-\infty<x<\infty$ in some sense to be made precise. For example, if $g(x)$ is any almost periodic function, then, with probability 1, $\lim_{n\to\infty} n^{-1}g(S_j)=\bar{g}=$ mean value of $g(x)$; also, with probability 1 the sequence $\{S_n\}$ is equidistributed in the usual sense modulo I simultaneously for all finite intervals I. These results hold for *any* nonlattice distribution F. Deeper equidistribution properties hold, involving limits of ratios $\sum_1^n g(S_j)/\sum_1^n h(S_j)$ for a wider class of functions $g(x)$, $h(x)$ (e.g. the characteristic functions of any two finite intervals) provided F is suitably restricted. (Received September 26, 1952.)

TOPOLOGY

34. Alice B. Dickinson: *Compactness conditions and uniform structures.*

Let a uniform structure which is less fine than all the other uniform structures compatible with the topology of the space be called the crude structure of the space. In a locally compact space the uniform structure induced by the uniquely defined one-point compactification is the crude structure. A unique uniform structure implies a unique compactification, in particular the one-point compactification of a locally compact space. The proof of this second theorem demonstrates a method of construction of distinct uniform structures. If a space has a unique uniform structure it is countably compact. A paracompact space is complete with respect to its universal structure. This last theorem answers questions raised by André Weil (*Sur les espaces à structure uniforme et sur la topologie générale*, Actualités Scientifiques et Industrielles, no. 551, Paris, Hermann, 1937) and J. Dieudonné (*Une généralisation des espaces compacts*, J. Math. Pures Appl. vol. 23 (1944) pp. 65–76). Relative to paracompactness, it is shown that the three properties, paracompactness, superior compactness from $\aleph_1$, and the property that every covering has a star finite refinement, are equivalent in a space which is locally countably compact and has at most a countable number of quasi components. (Received July 14, 1952.)

35*t*. W. H. Gottschalk: *Intersection and closure.*

Let X be a uniform space and let $A, B\subset X$. The sets A and B are *uniformly separable* provided that if U is a uniform neighborhood of $A\cap B$, then $A-U$ and $B-U$ have disjoint uniform neighborhoods. It is proved that: (1) If A and B are uniformly separable, then $\text{Cl}\ (A\cap B)=\text{Cl}\ (A)\cap\text{Cl}\ (B)$; (2) If A or B is conditionally compact and if Y is a dense subset of X such that $A\cup B\subset Y$, then $\text{Cl}\ (A\cap B)=\text{Cl}\ (A)\cap\text{Cl}\ (B)$

(closures in X) if and only if A and B are uniformly separable in Y. (Received August 4, 1952.)

36*t*. G. D. Mostow: *Factor spaces of solvable groups.*

Definition. A solvmanifold is a manifold in which a solvable Lie group of transformations acts transitively. In the special case that the group is nilpotent, one has a *nilmanifold* (introduced by Malcev, A.M.S. Translation, no. 39). The two-dimensional solvmanifolds are the plane, cylinder, torus, Klein bottle, Mobius band. Theorem 1. *Two compact solvmanifolds having isomorphic fundamental groups are homeomorphic* (Bull. Amer. Math. Soc. Abstract 58-2-261). Corollary. *A compact solvmanifold whose fundamental group is abelian is homeomorphic to a toroid.* Theorem 2. *Any solvmanifold is regularly covered a finite number of times by a direct product of a compact solvmanifold and a euclidean space. The covering group is abelian.* In certain special cases, one can assert that the covering space is univalent. As a special case one obtains: Corollary (Chevalley). *The factor space of a solvable Lie group by a connected subgroup is homeomorphic to a direct product of circles and straight lines.* (Received August 4, 1952.)

L. W. Cohen,
Associate Secretary

THE NOVEMBER MEETING IN RALEIGH

The four hundred eighty-fifth meeting of the American Mathematical Society was held at North Carolina State College of Agriculture and Engineering of the University of North Carolina, Raleigh, N. C., on Friday and Saturday, November 28–29, 1952. The total attendance was 142, including the following 94 members of the Society:

Milton Abramowitz, R. L. Anderson, N. C. Ankeny, A. T. Brauer, E. T. Browne, R. C. Bullock, E. A. Cameron, Leonard Carlitz, Virginia Carlock, C. L. Carroll, J. W. Cell, J. M. Clarkson, Haskell Cohen, J. C. Currie, D. F. Dawson, B. V. Dean, W. J. Dixon, F. G. Dressel, B. M. Drucker, W. L. Duren, Jr., S. E. Dyer, J. C. Eaves, W. W. Elliott, D. O. Ellis, M. E. Estill, H. A. Fisher, J. W. Gaddum, J. R. Garrett, I. C. Gentry, J. J. Gergen, Seymour Ginsburg, Wallace Givens, Herbert Goertzel, H. H. Goldstine, D. B. Goodner, E. E. Grace, H. C. Griffith, E. H. Hadlock, R. R. Hare, Jr., O. G. Harrold, E. A. Hedberg, P. S. Herwitz, A. T. Hind, O. H. Hoke, Harold Hotelling, A. S. Householder, G. B. Huff, Ernest Ikenberry, J. R. Isbell, F. B. Jones, John Jones, Jr., R. J. Koch, F. W. Kokomoor, H. T. LaBorde, G. B. Lang, J. W. Lasley, Jr., T. H. Lee, R. J. Levit, C. F. Lewis, P. E. Lewis, W. R. Longley, E. L. Mackie, J. S. MacNerney, W. R. Mann, G. W. Medlin, E. P. Miles, Jr., H. C. Miller, Benjamin Ernest Mitchell, J. C. Morelock, C. G. Mumford, H. M. Nahikian, J. D. Novak, F. R. Olson, H. V. Park, W. V. Parker, B. J. Pettis, T. J. Pignani, J. H. Roberts, W. A. Rutledge, A. L. Shields, J. R. Shoenfield, C. B. Smith, W. S. Snyder, W. L. Strother, P. M. Swingle, Olga Taussky, John Todd, R. Z. Vause, J. H. Wahab, M. J. Walsh, H. C. Wang, W. M. Whyburn, L. S. Winton, G. N. Wollan.

By invitation of the Committee to Select Hour Speakers for Southeastern Sectional Meetings, Professor Wallace Givens of the University of Tennessee and Dr. H. H. Goldstine of The Institute for Advanced Study addressed the Society. Dr. Goldstine's address, entitled *Some remarks on numerical analysis*, was delivered at 8:00 P.M. on Friday, with Dr. A. S. Householder presiding. The address by Professor Givens, entitled *Polarities and their signature in von Neumann's continuous geometry*, was given at 10:00 A.M. on Saturday, with Professor Gerald Huff presiding.

Four sessions for the presentation of contributed papers were held on Friday afternoon and Saturday morning. Presiding officers for these sessions were Professors J. H. Roberts, F. W. Kokomoor, W. V. Parker, and B. J. Pettis.

All sessions were held in Room 242 of the Riddick Laboratories Building on the campus of the North Carolina State College. Lounges and conference rooms in the building were also used. A resolution expressing the appreciation of the Society to its hosts for the meeting was offered by Professor W. L. Duren and unanimously adopted.

Abstracts of the papers presented follow below. Abstracts whose numbers are followed by the letter "*t*" were presented by title. Paper number 51 was read by Professor Rutledge. Professor J. W. Ellis was introduced by Dr. Goodner, Mr. McAuley by Professor Jones, Dr. Jaeger by Professor Cairns, Dr. Gordon and Mr. Capel by Professor Wallace, Mr. Pellicciaro and Dr. Shields by Professor Whyburn.

ALGEBRA AND THEORY OF NUMBERS

37. A. T. Brauer: *On the distribution of Jacobian symbols.*

Let p be a prime and $\epsilon_\nu=0$ or ± 1 $(\nu=1, 2)$. N. Aladow (Mat. Sbornik vol. 18 (1896) pp. 61–75) determined the number of integers a in a complete system of residues (mod p) for which the Legendre symbols $(a/p)=\epsilon_1$ and $(a+1/p)=\epsilon_2$ for each given ϵ_1 and ϵ_2. Other proofs for these results were given by E. Jacobsthal (*Anwendung einer Formel aus der Theorie der quadratischen Reste*, Dissertation, Berlin (w. Fr. Kästner, Göttingen, 1906)) and by O. Perron (Math. Zeit. vol. 56 (1952) pp. 122–130). In this paper the corresponding problem for Jacobian symbols for odd squarefree moduli is solved. These results will be used in another paper for the construction of Hadamard determinants of certain orders. (Received November 28, 1952.)

38*t*. Leonard Carlitz: *A note on modular invariants.*

Consider a system of forms $f_1, \cdots, f_r$, with coefficients $\in GF(q)$, which are subjected to the transformation of the full linear group over $GF(q)$; thus the number of *classes* is some finite number k. Let the invariants of the system take on values in an arbitrary field Φ. We show that if Φ contains at least k distinct numbers, then there exists an invariant J such that every invariant can be exhibited as a polynomial in J of degree $\leqq k-1$ and with coefficients in Φ. More generally if $\Phi=GF(q_1)$ and $q_1^{s-1}<k\leqq q_1^s$, then there exist s invariants $J_1, \cdots, J_s$ such that every invariant is a polynomial in the J's. Moreover the number s cannot be diminished. (Received October 6, 1952.)

39*t*. Leonard Carlitz: *A reciprocity formula for weighted quadratic partitions.*

This note is concerned with a reciprocity theorem for the sum $S(\alpha, \lambda, Q)=\sum_{Q(\xi)=\alpha} e(2\lambda_1\xi_1+\cdots+2\lambda_r\xi_r)$, where $\alpha, \lambda_i, \xi_i \in GF(q)$ and Q is a quadratic form with coefficients in $GF(q)$. Some extensions and a rational analogue are also discussed. (Received October 6, 1952.)

40. Leonard Carlitz: *Invariant theory of systems of equations in a finite field.*

The results of the author's paper: *Invariant theory of equations in a finite field* are extended to the general case. (Received October 6, 1952.)

41. B. V. Dean: *Regular isotopes of a near ring.*

A near ring N is a system having two laws of operation, addition and multiplication, where (1) addition is single-valued and has a zero element, (2) multiplication is single-valued and associative, (3) $ao=o=oa$, a in N, and (4) multiplication is left distributive over addition. e is left (right) regular if $ea=eb$ $(ae=be)$ implies $a=b$. The totality of functions on a system satisfying (1), which preserve zero, is a near ring. (A, B, C) is an isotopy of N onto the near ring M, and M is an isotope of N, if A, B, and C are three additive isomorphisms of N onto M and $(ab)^C=a^Ab^B$, a, b in N. An isotopy is a regular isotopy if N has a left regular element e and a right regular element f with $(eN)^A=e^AM$ or $(Nf)^B=Mf^B$. Regular isotopic near rings are isomorphic under a mapping D and if N has a multiplicative identity the isotopy is a multiplication of the isomorphism $AC^{-1}B=D=BC^{-1}A$. Examples show that isotopic subrings of a commutative field may not be isomorphic and for regular isotopes we may have $AC^{-1}B\neq D\neq BC^{-1}A$. (Received October 7, 1952.)

42*t*. Trevor Evans and B. H. Neumann: *On varieties of groupoids and loops.*

An infinite irredundant system of groupoid laws is described; it follows that the set of varieties (in the sense of P. Hall) of groupoids has the cardinal c of the continuum. Also an infinite irredundant system of loop laws is exhibited; loops are here equationally defined in terms of three binary operations: multiplication, right division, and left division (cf. Trevor Evans, J. London Math. Soc. vol. 24 (1949) pp. 254–260). It follows that the set of varieties of loops has cardinal c, and so has the set of varieties of quasigroups. The construction of both infinite irredundant systems of laws utilizes an infinite irredundant system of group relations taken from B. H. Neumann, J. London Math. Soc. vol. 12 (1937) pp. 120–127. It is further shown that power associativity of loops or quasigroups or groupoids is necessarily an infinite set of laws, though no irredundant system of laws equivalent to power associativity in loops is known, and the existence of such a system is highly doubtful. (Received October 20, 1952.)

43. E. H. Hadlock: *Primitive ternary indefinite quadratic genera of more than one class.*

Genera, of properly primitive forms with improperly primitive reciprocals, containing at least two classes are shown to exist. The invariants Ω and Δ associated with the forms are defined by $\Omega=\Omega'\Omega''$, $\Delta=\Delta'\Delta''=\Delta'$, $\Omega'=\Omega_1^2P_{\Omega\Delta}$, $\Delta'=-\Delta_1^2P_{\Omega\Delta}$ where Ω'' and Δ'' are the largest powers of 2 dividing Ω and Δ respectively, and Ω_1^2 and Δ_1^2 are the largest squares dividing Ω' and Δ' respectively. $1<P_{\Omega\Delta}\equiv 1 \pmod 8$. Also $\Omega''\geqq 64$. Each of the characters associated with the forms has the value one. (Received October 14, 1952.)

44*t*. A. J. Hoffman and Olga Taussky: *A characterisation of normal matrices.*

One of the definitions for a matrix A to be normal is that $AA^*=A^*A$ where A^* is the transposed and conjugate matrix of A. Commutativity of two matrices A, B implies that every polynomial $p(A, B)$ has as characteristic roots $p(\alpha_i, \beta_i)$ where α_i are the characteristic roots of A and β_i of B in a special ordering. While this property is weaker than commutativity for general matrices A, B, it implies commutativity if $B=A^*$. It can even be shown that either of the two following facts implies the normality of A: (1) $A+A^*$ has as characteristic roots $\alpha_i+\bar{\alpha}_i$ for some ordering. (2) AA^* has as characteristic roots $\alpha_i\bar{\alpha}_i$ for some ordering. (Received September 22, 1952.)

45. G. B. Huff: *Matrices such that A^t is a polynomial in t and principal idempotent elements.*

Let D be an integral domain (with identity) and of characteristic $p=0$, and D_m be the ring of m by m matrices with elements in D. For E_s, $A \in D_m$, E_s is said to be an identity for A if $E_s^2=E_s$ and $E_sA=AE_s=A$. A matrix A is said to be $E_sF(X)$ if and only if E_s is an identity for A and there exists a matric polynomial $F(X) \in D_m[X]$ such that $F(0)=E_s$ and $F(t)=A^t$ for each natural number t. If n is a natural number and E_s is an identity for A, then the principal result states that A is $E_sF(X)$ with $F(X)$ of degree n if and only if $A-E_s$ is nilpotent of index $n+1$. When D is the field of complex numbers, this is connected with the notion of the principal nilpotent and idempotent elements [Wedderburn, Colloquium lectures, pp. 28–30] of A in the following way. If E_j is the principal idempotent element associated with a nonzero characteristic root x_j of A, then $x_j^{-1}AE_j$ is $E_jF(X)$ where $F(X)$ is of degree one less than the multiplicity of x_j as a root of the minimum function of A. The clarity with which some classical and some new results come out of this connection suggests that the notion of $E_sF(X)$ matrices belongs at the beginning of matrix theory. (Received October 9, 1952.)

46. Arno Jaeger: *Linear differential equations in fields of prime number characteristic.*

An *iterative differentiation* D in a field F of characteristic $p \neq 0$ in the sense of *F. K. Schmidt* (J. Reine Angew. Math. vol. 190 (1952) p. 4) with the property $Da \neq 0$ for at least one $a \in F$ is used to define modified *formal differential equations* in F. A generalization of the basis theorem for these differentiations is the main tool to calculate the solutions of the differential equations by solving algebraic equations only. Hence solutions of a differential equation can be found already in *algebraic* extensions of F provided that solutions exist at all (for instance $Dy=a^{p-1}$ has no solution if $Da=1$ holds). The ring of all differential operators $\sum_{r=0}^{n} a_rD^r$ differs from its classical analogon by the existence of zero divisors caused by the iteration rule $D^iD^j=C_{i+j,i}D^{i+j}$. Thus new methods for finding the inverse of a given differential operator have to be adopted. The solutions of all linear differential equations with constant coefficients $\sum_{r=0}^{n} a_rD^ry=v$ can be given explicitly; if $a_0 \neq 0$ there exists a unique solution. For arbitrary linear differential equations the number of independent parameters entering the general solution may be greater or less than its formal order, but an upper bound can be found. (Received October 14, 1952.)

47. T. H. Lee: *Matrices with generalised quaternions as elements.*

Let α and β be square-free rational integers and r_0, r_1, r_2, and r_3 real numbers. We define i and j by the equations $i^2=-\alpha$, $j^2=-\beta$, and $ij=-ji$. The elements $r=r_0+r_1i+r_2j+r_3ij$ form a real generalized quaternion algebra $Q'(\alpha, \beta)=Q'$. If both α and β are positive, Q' is isomorphic to the ordinary real quaternions. But Q' is not even a division algebra if either α or β is less than zero. Louise A. Wolf (1935), H. C. Lee (1949), and J. L. Brenner (1951) have studied certain properties of matrices whose elements are ordinary real quaternions. In this paper, matrices A whose elements are generalized quaternions are studied. An element λ of Q' is called a right-hand characteristic root of A if a nonzero vector x with components in Q' exists such that $Ax=x\lambda$. It is proved here that such a root exists if at least one of α and β is positive. It remains unsolved whether characteristic roots always exist if α and β are negative. But similar matrices either have no roots or the same roots λ_k. Transforms $\rho\lambda_k\rho^{-1}$ of λ_k are themselves characteristic roots for each ρ in Q' with a nonvanishing norm. Moreover, if A is triangular, a characteristic root λ_k always exists, and for each characteristic root λ_k a nonzero element x of Q' exists such that $a_{mm}x=x\lambda_k$ where a_{mm} is an element of the main diagonal of A. It is shown that one of the a_{ii} is always a characteristic root, but not necessarily all of them are. (Received November 28, 1952.)

48. R. J. Levit: *Division rings in terms of a single operation.*

It was shown by N. Wiener [Trans. Amer. Math. Soc. vol. 21 (1920) pp. 237–246] that a field can be characterized as a system with only one undefined operation $a\nabla b$, which can be expressed in terms of the usual field operations as $1-a/b$. Another characterization in terms of the single class-closing operation $a\Delta b=a(1-b)$ was given by the author [Trans. Amer. Math. Soc. fol. 57 (1945) pp. 426–440]. In the present paper this work is extended to the non-commutative case. Sets of postulates for a division ring are given in terms of each of the operations $a\Delta b$ and $a\nabla b=1-b^{-1}a$. (Received October 14, 1952.)

49. G. W. Medlin: *A note on a theorem of Parker.*

Recently W. V. Parker [Proceedings of the American Mathematical Society vol. 1 (1950)] proved the following theorem. Let A be an $n\times m$ matrix of rank r such that $ACA=kA$ (k a scalar). If B is an $m\times n$ matrix, the characteristic equation of AB is $x^{n-r}\phi(x)=0$ and that of $A(B+C)$ is $x^{n-r}\phi(x-k)=0$. In the same manner that W. T. Reid [Proceedings of the American Mathematical Society vol. 1 (1950)] generalized an earlier theorem of Parker [Bull. Amer. Math. Soc. vol. 55 (1949)], this result will be generalized by proving the following theorem. Let A be an $n\times m$ matrix of rank r and D an $n\times n$ matrix such that $DA=kA$ (k a scalar). If B is an arbitrary $m\times n$ matrix, then the characteristic equation of AB is $x^{n-r}\phi(x)=0$ and that of $AB+D$ is $g(x)\phi(x-k)=0$, where $g(x)$ is a polynomial of degree $n-r$. (Received November 28, 1952.)

50. W. V. Parker: *Characteristic roots of a set of matrices.*

A square matrix A may be uniquely written as $A=H+iK$ where H and K are Hermitian matrices. The characteristic roots of H and K determine a rectangular grid in the complex plane. All characteristic roots of A are within or on this grid. All matrices B given by $B=UHU^*+iVKV^*$, where U and V are unitary matrices, have the same associated grid. It is the purpose of this paper to examine the distribution of the characteristic roots of these matrices as U and V range over the set of all unitary matrices. Two special cases are considered. (Received October 15, 1952.)

51. W. V. Parker and W. A. Rutledge: *Equivalence of matrices over a polynomial domain.*

Let $a=(a_1, a_2, \cdots, a_n)$ be a vector over a field F and $a(x)=a_1+a_2x+\cdots+a_nx^{n-1}$. The vectors e_i are defined by $e_i(x)=x^{i-1}$. A matrix whose rows are e_2, e_3, $\cdots$, e_n, a is the companion matrix of $f(x)=x^n-a(x)$, denoted by $C(f)$. The matrix whose last row is the vector a, and other rows zeros, is the associate of $a(x)$, denoted by $A(a)$. Let $M=(M_{ij})$ be a $k\times k$ block matrix, with M_{ii} the companion matrix $C(f_i)$, and M_{ij}, $i\neq j$, the associate matrix $A(f_{ij})$. It is shown that $M-xI$ is equivalent to diag. $\{I_{n-k}, M(x)\}$ where $M(x)$ is the $k\times k$ matrix (m_{ij}) with $m_{ii}=f_i(x)$ and $m_{ij}=f_{ij}(x)$, $i\neq j$. Several known results are corollaries (e.g. Theorem I in M. F. Smiley, *The rational canonical form of a matrix*, Amer. Math. Monthly vol. 56 (1949) p. 542). A simple proof using matrix theory is given of a theorem of Flanders (H. Flanders, *Elementary divisors of AB and BA*, Proceedings of the American Mathematical Society vol. 2 (1951) p. 872). (Received October 15, 1952.)

52*t*. W. E. Roth: *On the characteristic polynomial of the product of two matrices.*

The following theorem is proved: If A and B are $n\times n$ matrices with elements in the field F, whose characteristic polynomials are $a_0(x^2)-xa_1(x^2)$ and $b_0(x^2)-xb_1(x^2)$ respectively, where $a_0(x^2)$, $a_1(x^2)$, $b_0(x^2)$, and $b_1(x^2)$ are polynomials in x^2 in $F[x]$, the polynomial domain of F; and if the rank of $A-B$ is less than or equal to unity; then the characteristic polynomial of their product, AB, is $a_0(x)b_0(x)-xa_1(x)b_1(x)$. (Received September 23, 1952.)

53*t*. Olga Taussky: *Generalized commutators of matrices.*

It was proved by Shoda that any unimodular $n\times n$ matrix can be expressed as the commutator of two $n\times n$ matrices. A more general result is: if X, Y are any two $n\times n$ matrices with equal nonvanishing determinants, then $n\times n$ matrices C, D can be found such that $Y=C^{-1}D^{-1}XCD$. Applications of this fact are discussed. (Received September 22, 1952.)

54. J. H. Wahab: *New cases of irreducibility for Legendre polynomials.* II.

Let $P_n(x)$ be the Legendre polynomial of degree n. Let p be any prime. If (1) $n=(p-1)(p^{k_1}+p^{k_2}+\cdots+p^{k_s})$, where $k_i\neq k_r$ for $i\neq r$, and if $P_n(x)$ be reducible in the field of rational numbers, then the degree of each factor can be similarly expressed in terms of p and each power of p occurring in (1) will occur once and only once among the representations of the degrees of the factors. This necessary condition for reducibility if $p=2$ was proved in the author's first paper (Duke Math. J. vol. 19 (1952) pp. 165–176) and was used to obtain certain irreducible $P_n(x)$. A proof for arbitrary p is given here and the result is applied to prove additional irreducible cases of $P_n(x)$. (Received October 15, 1952.)

ANALYSIS

55. Seymour Ginsburg: *Some results on fixed points of simply ordered sets.*

Let A be a simply ordered set. The element p of A is called a fixed point of A if $f(p)=p$ for each similarity transformation f of A into A. Theorem: Let p be a fixed point of A, q a fixed point of B, and let $A\times B$ be ordered by first differences. Then (a) either p or q is a fixed point of the ordered sum $A+B$; and (b) there exists a fixed point (r, q) of $A\times B$, where r is some fixed point of A for which both of the sets, $\{x \mid r\leq x\leq p,\ x\in A\}$ and $\{x \mid p\leq x\leq r,\ x\in A\}$, are finite. (Received October 14, 1952.)

56. Ernest Ikenberry: *A note on the Liouville equation* $Df=0$.

By decomposing the distribution function in phase space into its even and odd parts with respect to the momentum coordinates, so that $f=f_e+f_0$ and applying the operator D (J. of Chem. Phys. vol. 19 (1951) pp. 467–470; vol. 20 (1952) pp. 568–570) to f_e and f_0 separately, we find that, in the stationary case in which $\partial f/\partial t=0$, Df_e is an odd function and Df_0 is an even function of the momentum coordinates, unless it is assumed that the intermolecular forces depend on the momenta. Hence we have not only $Df=0$, but also $Df_e=0$ and $Df_0=0$, in the stationary case; the distribution function is a superposition of two functions between which the Liouville equation gives no relation. In this case, Gibbs' principle of "conservation of density-in-phase" is inadequate for obtaining a relation between the heat current and the temperature gradient, or between the components of the strain tensor and the velocity gradients. To obtain such relations by application of Gibbs' principle it is necessary to consider the non-stationary case, or to assume that the force components R_i depend on the momentum components p_i. (Received October 15, 1952.)

57t. W. H. Ingram: *An integral related to the Pollard-Moore-Stieltjes integral.*

Let f be bounded and with simple discontinuities only, g be in the class $\mathfrak{B}$ of functions of bounded variation, $2\bar{f}(x)=f(x+)+f(x)$, then $(M)\int_a^b f\,dg=\lim_\sigma \sum_{(i)} \bar{f}(x_i)[g(x_{i+1})-g(x_i)]$, with $\lim_\sigma$ the Smith-Moore limit, exists. The M-integral has an ordinary integration by parts formula for functions continuous on the left; it is the mean of the N-integral defined in Bull. Amer. Math. Soc. Abstract 55-7-350 and the $LCS\sigma$-integral of Price (Bull. Amer. Math. Soc. vol. 49 (1943) p. 627). Let $\mathbf{f}=(\bar{f}(x), \bar{f}(x+))$, $d\mathbf{g}=(jg, g'(x+)dx)$, $dx>0$, $jg=g(x+)-g(x)$, then the lower (inner) vector product being understood, $(M)\int_a^b \mathbf{f}\,d\mathbf{g}=\lim_\sigma \sum_{(i)} \mathbf{f}(x_i)\,d\mathbf{g}(x_i)$, $dx=\Delta_i x$, for g in that subclass of $\mathfrak{B}$ such that $g'(x+)$ exists and is bounded (i.e., is in $\mathfrak{B}'$) and moreover such that the difference quotient $(g(x+\delta)-g(x+))/\delta$, $\delta>0$, converges uniformly on $[ab]$ (i.e., is in $\mathfrak{B}^*$), also, $d\int_a^x f\,dg=f\,dg$, $a\leqq x\leqq b$, almost everywhere, g in $\mathfrak{B}$, or everywhere, g in $\mathfrak{B}'$. The M-integral has effective application to the problem $dy=dH(x,\mu)y$, $Ly(a)+Ry(b)=0$, H in $\mathfrak{B}^*$ and continuous on the left. (Received November 22, 1952.)

58. J. S. MacNerney: *Essential adjoints and balanced transformations.*

The space S is assumed to be linear and complete, with inner product (x, y)—notation is consistent with J. von Neumann, *Functional operators* (vol. II), Annals of Mathematics Studies, no. 22, Princeton, 1950; "transformation" means single-valued operator. The essential adjoint, T^e, of an operator T is the operator with graph consisting of all ordered pairs of the form $\langle P_B y, P_A y'\rangle$ where $\langle y, y'\rangle$ belongs to the graph of T^*, $A=[D(T)]$, and $B=[R(T)]$; balanced operators are those operators T such that $T^{ee}=T$. It is proved that (1) if T is an operator, then T^e is a closed linear reversible transformation, and (2) every closed linear reversible transformation is balanced. (Received October 14, 1952.)

59. E. P. Miles: *A note on harmonic functions generated by analytic functions of a hypervariable.*

In a preliminary report [Bull. Amer. Math. Soc. Abstract 58-1-61] the author considered a class of 3-dimensional harmonic functions generated by analytic functions of a certain hypervariable. A closely related class of harmonic functions is generated by analytic functions of the hypervariable $t=x+\omega y+\omega^2 z$ where $\omega^3=1$. For analytic $f(t)=U(x, y, z)+\omega V(x, y, z)+\omega^2 W(x, y, z)$ we have $\nabla^2 U=\nabla^2 V=\nabla^2 W$; thus the function $(\omega-1)f(t)$ has harmonic components $W-U$, $U-V$, and $V-W$. Let $H(x, y, z)$ be one such harmonic function, any level surface $H(x, y, z)=C$ is seen to be a cylindrical surface normal to the plane $x+y+z=0$. $H(x, y, z)$ is thus equivalent, under a rigid motion carrying the complex plane into the plane $x+y+z=0$, to an ordinary 2-dimensional harmonic function generated by an analytic function of the complex variable $x+iy$. (Received October 14, 1952.)

60t. C. N. Moore: *On the infinitude of prime triplets.*

A question closely related to the infinitude of prime pairs is that of the infinitude of prime triplets, namely the infinite repetition of sequences of three primes whose successive differences are 2, 4, like 5, 7, 11, or 4, 2, like 7, 11, 13. The method which the author used for the discussion of the prime pair problem can be extended to this other problem by making suitable modifications. Consider three infinite columns of numbers formed by adding successive multiples of 30 to 7, 11, 13. The multiples of 7 in the first column form an arithmetic progression with common difference 210. The numbers in columns two or three opposite the terms of this progression form arithmetic progressions whose first terms and common differences are relatively prime. One then has formulas for the total number of primes in these progressions which are $\leqq$ a given number x. By considering also all composites in the first column which are multiples of 11, 13, $\cdots$, p_λ, where p_λ is the greatest prime $\leqq x^{1/2}$, and also those which are multiples of the product of two or more of these primes, one can get an estimate of the number of primes in columns (2) or (3) which are opposite to composites in column (1). Similarly one can estimate the number of primes in column (3) which are opposite to composites in (2). The totality of primes in (3) $\leqq x^{1/2}$ is found to exceed the totality of primes opposite composites, as above determined. Thus the announced result is obtained. The infinite repetition of prime quadruplets and other similar prime groups can be discussed in analogous fashion. (Received October 14, 1952.)

61. E. J. Pellicciaro: *A study of critical sets of functions.*

A study of critical sets is made by use of their related complementary domains. $f(x)$ is a single-valued, continuous function of the n-real variables $x_1, \cdots, x_n$ throughout $R+C$, where R is a bounded, connected, open subset of the n-dimensional Euclidean space and C is its boundary. On $C, f(x)=K$, a constant, while at all points of $R, f(x)<K$. By a critical point of $f(x)$ is meant a point p of R such that there exists $\epsilon>0$ with the property that $[f(p+mE)-f(p)][f(p+nE)-f(p)]\geqq 0$ for all unit vectors E and all scalars m, n satisfying $0<|m|, |n|<\epsilon$. The critical set $M(H, p)$ that contains p is defined to be that component of H, the set of all critical points of $f(x)$ on which $f(x)$ is constant, which contains p. With the aid of a few theorems concerning $M(H, p)$ and the complementary domains of $G[f=f(p)]$ with respect to R, a classification of critical sets is realized. This classification is exhaustive and mutually exclusive, dividing the collection of critical sets of $f(x)$ into five types, type 0, hitherto not encountered, and types 1, 2, 3, and 4, respectively comparable to the maximal, minimal, min-max, and flex types. (Received November 28, 1952.)

62. A. L. Shields: *On additive properties of real numbers.* Preliminary report.

Let A and B be two sets of real numbers. $A+B$ is the set of all numbers $a+b$, a in A, b in B. Then $m_0(A+B)\geqq m_0(A)+m_0(B)$ where m_0 is Lebesgue outer measure. Result true for inner measure. Proof consists in showing $A+B$ contains a translate of A and a translate of B which are almost disjoint. Let R be reals, and R_1 the reals mod 1. Let A, B be Borel subsets of R_1, and $m(A)+m(B)\leqq 1$. Then $m(A+B)\geqq m(A)+m(B)$ where the addition in $A+B$ is mod 1. Proof consists in reducing the problem to additive number theory by means of ergodic theory. A is said to be rational (integral) basis for R (R_1) if every x in R (R_1) may be represented as a finite linear combination of elements of A with rational (integral) coefficients. Uniqueness of representation is not required. If A is a Borel set and a rational basis for R (R_1) then it is an integral basis. Let A be an integral basis for R_1. It is said to be of finite order if there is an N such that every x in R_1 admits a representation in which the sum of the magnitudes of the coefficients doesn't exceed N. If A is a Borel set and an additive basis for R_1, then it is of finite order. Finally, let A have positive measure and let F be a finite set. Then there exist numbers r, s such that $rF+s\subset A$. (Received October 17, 1952.)

63. G. N. Wollan: *Euler methods of summability for double series.*

Each of the double sequences $A^q_{mn}=(q+1)^{-m-n-2}\sum_{0\leq\mu\leq m+1, 0\leq\nu\leq n+1}C_{m+1,\mu}C_{n+1,\nu}q^{m+n+2-\mu-\nu}A_{\mu-1,\nu-1}$, where $A_{\mu\nu}=0$ whenever μ or ν is negative, and $\mathcal{A}^q_{mn}=(q+1)^{-m-n}\sum_{0\leq\mu\leq m, 0\leq\nu\leq n}C_{m,\mu}C_{n,\nu}q^{m+n-\mu-\nu}A_{\mu\nu}$ may be regarded as an Euler transform of the qth order of the double sequence A_{mn}. In this paper it is shown that for any $q\geqq 0$ if $\lim_{m,n\to\infty}A^q_{mn}=A$, then $\lim_{m,n\to\infty}\mathcal{A}^q_{mn}=A$, but that the converse is not true. It is also shown that if $A_{mn}=\sum_{0\leq\mu\leq m, 0\leq\nu\leq n}a_{\mu\nu}$ is bounded and if $\lim_{m,n\to\infty}\mathcal{A}^1_{mn}=\lim_{m,n\to\infty}2^{-m-n}\sum_{0\leq\mu\leq m, 0\leq\nu\leq n}C_{m,\mu}C_{n,\nu}A_{\mu\nu}=A$ and $\lim_{m,n\to\infty}((m+1)^{1/2}+(n+1)^{1/2})\cdot((m+1)(n+1))^{1/2}\cdot a_{mn}=0$, then $\lim_{m,n\to\infty}A_{mn}=A$. (Received October 14, 1952.)

APPLIED MATHEMATICS

64. A. T. Hind, Jr.: *On the convergence of approximate solutions of the two-dimensional wave equation to the exact solution.*

It is shown that approximate solutions of the two-dimensional wave equation can be obtained which converge to the exact solution for any mesh-ratio r greater than zero, where $r(\Delta x)=\Delta t$ and $r(\Delta y)=\Delta t$. This is an extension of a paper on the one-dimensional wave equation by Leutert and O'Brien in which they extend the values of the mesh-ratio r, for which approximate solutions converge to the exact solution, from $0<r\leqq 1$ to $0<r$. The convergence of approximate solutions of the two-dimensional wave equation is shown by the use of a lemma which treats all $0<r$ in one case instead of in three cases as required by the method of Leutert and O'Brien. As it is to be desired, this lemma contains theirs as a special case. (Received October 15, 1952.)

65. W. R. Mann: *Mean value methods in iteration.*

In order to use the Schauder fixpoint theorem to prove that a given problem has a solution, one must somehow associate with the problem a convex compact set in a Banach space and a continuous transformation which carries the set into itself. Assume that the fixpoint has been proved to be a solution of the original problem and assume furthermore that this solution has been proved to be unique. The problem then is: Given a convex compact set E in a Banach space and a continuous transformation, T, carrying E into itself and having a unique fixpoint, x^*, in E, construct a sequence of elements which converges to x^*. Assuming the ordinary interation process to fail, one considers the following modified iteration scheme. Taking x_1 to be an arbitrary point in E, we define $x_{n+1}=T(v_n)$ where $v_n=(1/n)\sum_{i=1}^{n}x_i$. The following results are obtained. If either of the sequences $\{x_n\}$ and $\{v_n\}$ converge, then the other does and they have the common limit x^*. In the special case where E is a line segment, $\{x_n\}$ always converges to x^*. In more general spaces properties of the sets of accumulation points of $\{v_n\}$ and $\{x_n\}$ are obtained. (Received October 14, 1952.)

66. C. B. Smith: *An infinite orthotropic plate containing a disk of a different orthotropic material.*

A large rectangular orthotropic plate containing a small circular disk of a different orthotropic material subjected to a uniform tension is discussed. The axes of elastic symmetry of the plate and the disk are taken to be parallel and, for simplicity, the uniform tension is assumed to act parallel to one of these common directions. The resulting stress distribution is derived from two stress functions, one stress function giving the stresses in the region exterior to the disk and the other giving the stresses interior to the disk. (Received October 14, 1952.)

GEOMETRY

67. D. O. Ellis: *A modification of the parallelogram law characterization of Hilbert spaces.*

Denote the "parallelogram law" which characterizes Hilbert spaces (real) among Banach spaces (real) by (†). Using methods of vector lattice and metric lattice theory, the classical characterization ((1) is equivalent to (4) below) is extended to: Theorem. *The following four conditions are pairwise equivalent for a Banach space (real) B: (1). B is a Hilbert space. (2) There exists a partial ordering in B so that B is a Banach lattice and (†) holds over the B^+ defined by this ordering. (3) There exists a partial ordering of B so that B is a Banach lattice, the B^+ defined by this ordering is a metric lattice under the norm $\nu(x)=\|x\|^2$, and (†) holds for pairs of comparable elements in this B^+. (4) (†) holds over B.* It is also shown that any Hilbert space which is a Banach lattice (and any Hilbert space can be made one by the main theorem) is a UMB lattice and that if B is a UMB lattice, the Banach space norm is *strictly* monotone increasing on B^+. (Received September 26, 1952.)

68*t*. Jack Levine: *A canonical form for a conformally flat symmetric space.*

It is shown that if a conformally flat space satisfies the conditions $R_{ijkm,p}=0$ (definition of a symmetric space of Cartan), then a coordinate system exists in which the quadratic form may be written as $ds^2=(1/u^2)\sum e_i(dx^i)^2$ $(e_i=\pm 1)$, where $u^2=aR^2+LR+Q$, $R=\sum_i^n e_i(x^i)^2$, $L=\sum b_i e_i x^i$, $Q=c_{ij}x^i x^j+2d_i x^i+f$, with a, b_i, c_{ij}, d_i, and f constants. This canonical form is obtained by the actual solution of the differential equations characterizing the above named space. (Received October 9, 1952.)

TOPOLOGY

69. C. E. Capel: *Inverse limit spaces.* I. (Preliminary report.)

The continuity axiom for a cohomology theory states that if $\{(X_\lambda, A_\lambda), \Pi_{\lambda\mu}, \Lambda\}$ is an inverse mapping system of compact pairs, then the pth cohomology group of the inverse limit is isomorphic, under a natural homomorphism, to the direct limit of the induced direct system $\{H^p(X_\lambda, A_\lambda), \Pi^*_{\lambda\mu}, \Lambda\}$. It is known that the continuity axiom implies the extension and reduction theorems. In the presence of the Eilenberg-Steenrod axioms, it is shown that the extension and reduction theorems imply the continuity axiom. The principal lemma is analogous to a fundamental lemma of E. Spanier (Ann. of Math. vol. 49 (1948) pp. 407–427). (Received October 17, 1952.)

70. J. W. Ellis: *Complementary spaces of cartesian products of groups and linear spaces.*

Let X be called an *S-space* if X is a topological group and S a commutative semigroup whose members act as continuous functions on X and leave the identity fixed. If Z is a second S-space, the *Z-complement* of X is the group of all continuous homomorphisms on X to Z which "commute with the members of S." If, for each α, X_α is an S-space, it is true, under certain conditions on Z, that the Z-complement of PX_α is the weak product of the Z-complements of the X_α. Similarly, the Z-complement of the weak product of the X_α, when this space has a suitable topology, is isomorphic to the full cartesian product of the Z-complements. The conditions on Z are satisfied by the reals as a real-space, and by the reals modulo one as a space over the integers; this theorem is then shown to include Kaplan's result on the character group of a product of groups, and Katětov's theorem on the adjoint of a product of linear spaces. Other applications generalize results by Dixmier, by Šmulian, and by Hewitt and Zuckermann. (Received October 14, 1952.)

71. Mary E. Estill: *An effect of completeness.*

The purpose of this paper is to answer two questions raised by F. B. Jones, the first in a paper entitled *Connected and disconnected plane sets and the functional equation* $f(x)+f(y)=f(x+y)$ which appears in the February, 1942 issue of the Bulletin of the American Mathematical Society, and the second in a paper entitled *Certain consequences of the Jordan curve theorem* which appears in the July, 1941 issue of the American Journal of Mathematics. It is shown that the following theorem settles both questions and its proof is given. There is a connected, linear, complete Moore space in which there exists a totally disconnected domain. (Received October 14, 1952.)

72. W. L. Gordon: *On the coefficient group in cohomology.*

Cohomology groups of the Alexander-Kolmogoroff type may be defined using only p-cocycles which locally are finite-valued. For fully normal Hausdorff (=paracompact) spaces, the groups so defined are naturally isomorphic with the usual Alexander-Kolmogoroff groups and hence with the Čech groups based on arbitrary coverings. In the locally-compact case it is proven that a cocycle with compact support has a finitely valued cocycle in the same cohomology class. For compact Hausdorff-closed pairs X, A the following results are obtained: If the abelian coefficient group G is the direct sum of groups G_α, then the pth cohomology group $H^p(X, A; G)$ of the pair (X, A) with coefficients in G naturally decomposes as the direct sum of the groups $H^p(X, A; G_\alpha)$. An analogous result is obtained in case G is the direct limit of a direct system of groups, reducing the problem of the universal coefficient group to a problem involving only finitely-generated coefficient groups. Applications include an algebraic proof that the use of arbitrary coefficient groups in a cohomological definition of dimension will yield a numerical invariant that cannot exceed the (usual) dimension. The classical equality of the pth Betti number of a finite polyhedron with the dimension of the pth homology group over the rational field is given a natural extension to compact spaces. (Received October 15, 1952.)

73t. J. S. Griffin, Jr.: *A note on transfinite sequences.*

A transfinite sequence is defined to be a net (for definition see J. L. Kelley, Duke Math. J. vol. 17, pp. 277–285) whose domain is a limit ordinal. If X is a topological space and $A \subset X$, then (1) x is in the closure of A iff there is a transfinite sequence in A clustering at x; (2) a function f on X is continuous iff f carries cluster points of transfinite sequences in X into cluster points of their images; (3) X is compact iff each transfinite sequence in X has a cluster point; (4) X is Frechet-compact iff each (ordinary) sequence in X has a cluster point. There may be constructed a topological space with a subset A which has an accumulation point x but such that no transfinite sequence in A converges to x (although there are those in A which cluster at x). (Received October 20, 1952.)

74. J. R. Isbell: *Notes on homogeneous spaces.*

All microhomogeneous linear continua are homogeneous, and either Birkhoff linear homogeneous continua or uncountable unions of such continua. With reference to van Dantzig, *Über das topologische homogene Kontinua*, Fund. Math. vol. 16 (1930) pp. 102–125: the classification of homogeneities and the product theorem are extended and a problem is solved: all locally Euclidean manifolds are two-point homogeneous but one is not involutory homogeneous. (Received November 26, 1952.)

75t. J. R. Isbell: *Unicoherence of meets and joins.*

The union of a chain of locally connected open sets each component of each of which is unicoherent is disconnected or unicoherent. The intersection of a chain of locally connected compact Hausdorff sets each component of each of which is unicoherent is disconnected or unicoherent. The intersection theorem generalizes Borsuk's corresponding result for Peano continua (Fund. Math. vol. 17 (1931) p. 208); it is proved by use of the lemma: unicoherence of normal connected locally connected spaces is equivalent to the property that if A and B are connected open sets covering the space, $\overline{A \cap B}$ is connected. (Received September 15, 1952.)

76. R. J. Koch: *A lemma on mobs.*

Following A. D. Wallace, we define a mob to be a Hausdorff space which admits a continuous, associative multiplication. The purpose of this note is to exhibit the following lemma: Let X be a mob, and A a compact subset of X, let $\{p_\alpha\}$ be a net (generalized sequence) in X, and p_0 any cluster point of $\{p_\alpha\}$; suppose $Ap_\beta \subset Ap_\gamma$ if $\beta > \gamma$. Then $\bigcap_\alpha Ap_\alpha = Ap_0$. Wallace has shown that every compact mob contains at least one idempotent. It follows as a consequence of this result with the present lemma that if X is a compact mob and $a \in X$, then there is an idempotent e such that $\bigcap_n Xa^n = Xe$. (Received November 24, 1952.)

77. L. F. McAuley: *On the aposyndetic decomposition of continua.*

Suppose that M is a continuum and that p is a point of M. Let $M(p)$ denote the set of all points x of M such that there does not exist an uncountable collection $G(p, x)$ of mutually exclusive closed subsets of M such that (1) no element of $G(p, x)$ separates an element of $G(p, x)$ in M and (2) each element g of $G(p, x)$ separates p from x in M. An example is given in the plane of a compact continuum M which contains a point p such that $M(p)$ is not connected. With the aid of several lemmas, the following theorem is proved. If M is a compact metric continuum and H is the collection of all continua C such that, for some point p of M, C is a component of $M(p)$, then H is an upper semi-continuous collection of mutually exclusive continua filling up M; and furthermore, with respect to its elements as points, H is a compact aposyndetic metric continuum. An example is given in the plane to show that H is not necessarily connected im kleinen. (Received October 14, 1952.)

78. W. L. Strother: *Multi-homotopy.*

Definitions are given for (1) homotopy of multi-valued functions, (2) multi-retracts, and (3) multi-homotopy groups. Tietze's extension theorem is generalized and a theorem is proved which enables one to generalize numerous theorems on homotopy groups to multi-homotopy groups. (Received October 14, 1952.)

79. P: M. Swingle: *A note on n-point connected sets.*

A connexe C is n-point connected if there does not exist a subset of power n whose omission disconnects C (Bulletin, 1931, p. 254). A connexe C will be said to be n-densely connected if there exist n, but not a greater cardinal number, of mutually exclusive subconnexes, each dense in C. The main problem of this note deals with the existence of n-densely connected subsets, and related subconnexes, in an (aleph-null)-point connected set. (Received October 14, 1952.)

80. M. J. Walsh: *The paracompactness of the CW-complex.*

J. H. C. Whitehead (Bull. Amer. Math. Soc. vol. 55 (1949) pp. 213–245) has introduced the concepts of the generalized cell complex and the weak topology in complexes. It is shown that a closure finite generalized cell complex with the weak topology (CW-complex) is paracompact. An equivalent condition for the paracompactness of a space is also given. (Received October 14, 1952.)

81. H. C. Wang: *Compact manifolds with homogeneous complex structure.*

All manifolds here are assumed to be simply-connected and closed. Very few examples are known of a manifold which either (a) admits infinitely many inequivalent complex structures, or (b) admits a nonalgebraic complex structure. Hirzebruch exhibited two spaces with property (a), and later Calabi-Eckmann showed that $S^{2p+1} \times S^{2q+1}$ has both properties (a) and (b). In this paper, all homogeneous complex manifolds are determined. As consequences, we find that nonalgebraic complex manifolds are not so limited and most of them in the homogeneous case have property (a). The main results are as follows: (I) Each homogeneous complex manifold is homeomorphic with a fibre decomposition space, with torus as fibre, of a product of certain spaces, called elementary spaces. All elementary spaces are determined. Among them are, in particular, the Stiefel manifolds $V_{n,2k}$, the complex Stiefel manifolds $W_{n,k}$ (odd spheres are included as special cases), and the space of compact simple Lie groups. (II) The product of elementary spaces is nonalgebraic, and has a homogeneous complex structure when its dimension is even. (III) The number of differentiably inequivalent homogeneous complex structures over a manifold is either noncountable or finite according as the Euler characteristic vanishes or not. (Received October 14, 1952.)

W. M. WHYBURN,
Associate Secretary

THE NOVEMBER MEETING IN LAFAYETTE

The four hundred eighty-sixth meeting of the American Mathematical Society was held at Purdue University, West Lafayette, Indiana, on November 28–29, 1952. There were approximately 130 registrations, including the following 86 members of the Society:

D. N. Arden, Nachman Aronszajn, W. L. Ayres, P. T. Bateman, P. O. Bell, R. L. Blair, Leonard Bristow, E. F. Buck, R. C. Buck, I. W. Burr, J. W. Carr, Lamberto Cesari, Y. W. Chen, J. A. Dieudonné, C. L. Dolph, John Dyer-Bennet, W. E. Edington, B. J. Eisenstadt, J. G. Elliott, Paul Erdős, Chester Feldman, D. A. Flanders, J. S. Frame, C. G. Fry, R. E. Fullerton, Murray Gerstenhaber, Casper Goffman, Michael Golomb, A. W. Goodman, S. H. Gould, G. H. Graves, L. M. Graves, R. L. Graves, H. E. H. Greenleaf, J. K. Hale, P. C. Hammer, Sister M. Agnes Hatke, C. T. Hazard, L. L. Helms, Melvin Henriksen, J. G. Hocking, H. K. Hughes, Ralph Hull, P. E. Irick, W. E. Jenner, Meyer Jerison, Samuel Kaplan, M. W. Keller, J. H. B. Kemperman, C. F. Kossack, J. R. Lee, Y. L. Luke, J. D. McKnight, Jr., J. M. Mitchell, M. A. Moore, R. H. Niemann, Rufus Oldenburger, D. B. Owen, Sam Perlis, J. C. Polley, L. E. Pursell, C. R. Putnam, Gustave Rabson, Tibor Radó, W. T. Reid, B. V. Ritchie, P. C. Rosenbloom, Arthur Rosenthal, E. H. Rothe, W. C. Sangren, Lowell Schoenfeld, W. T. Scott, M. E. Shanks, M. F. Smiley, A. H. Smith, R. B. Stone, E. W. Titt, E. A. Trabant, N. H. Vaughan, G. L. Walker, M. S. Webster, C. P. Wells, George Whaples, Oswald Wyler, L. C. Young, J. W. T. Youngs.

By invitation of the Committee to select Hour Speakers for Western Sectional Meetings, Professor Nachman Aronszajn of the University of Kansas addressed the Society on *Approximation methods in eigenvalue problems* at 2:00 P.M. on Friday. The speaker was introduced by Professor Tibor Radó, and the interest in the topic is attested to by the large number of contributed papers on the program which followed Professor Aronszajn's address and dealt with the same field.

There were sessions for contributed papers at 3:30 P.M. on Friday and at 9:30 A.M. and 11:00 A.M. on Saturday. Presiding officers for these sessions were Professors Arthur Rosenthal, W. T. Reid, Ralph Hull, and L. C. Young.

There was a dinner for members of the Society and their guests at the Purdue Memorial Union on Friday night, followed by open house in the homes of the various members of the local department. The Society takes this opportunity to record its pleasure and appreciation.

Abstracts of papers presented at the meeting are recorded below. Papers with abstract numbers followed by "*t*" were presented by title. Paper number 88 was read by Dr. Sangren, number 95 by Professor Young.

ALGEBRA AND THEORY OF NUMBERS

82. Melvin Henriksen: *On the prime ideals of the ring of entire functions.*

The existence of nonmaximal ideals of the ring R of entire functions was pointed out to the author by Professor I. Kaplansky. If we let $0_n(f)$ be the multiplicity of the nth zero of f, these ideals are characterized as the prime ideals P such that $0_n(f)$ is unbounded, for all $f \in P$. Any nonmaximal prime ideal of R is free (for definitions, see Pacific Journal of Mathematics vol. 2, pp. 179–184). Each prime ideal is contained in a unique maximal (free) ideal M. The class P of all prime ideals contained in M is linearly ordered under set inclusion, and $P^* = \bigcap_{k=1}^{\infty} M^k$ is the largest nonmaximal ideal of P. The residue class ring R/P is a valuation ring and is Noetherian if and only if $P = P^*$. The residue class ring R/P^* is isomorphic to the ring of all formal power series over the complex field. The structure theory of Cohen (Trans. Amer. Math. Soc. vol. 59, pp. 54–106) of complete local rings is used. (Received December 4, 1952.)

83. George Whaples: *Additive polynomials.*

Additive polynomials are those of form $f(x) = \sum a_\nu x^{p^\nu}$ over a field k of characteristic p. They are characterized by the identity $f(x+y) = f(x) + f(y)$, form a ring under addition and composition, and have been studied by O. Ore (Trans. Amer. Math. Soc. vol. 35 (1933) pp. 559–584 and vol. 36 (1934) pp. 243–274) for arbitrary k and for k a Galois field. This paper assumes k has characteristic p, has no inseparable extensions, and has for each n at most one extension of degree n (exactly one for $n = p$) in its algebraic closure. There are many such fields not absolutely algebraic. Main results (k^+ = additive group of k, $f(x)$ = an a.p.): 1. (Index of $f(k^+)$ in k^+) = number of zeros of $f(x)$ in $k \leq$ degree $f(x)$. 2. Every group between $f(k^+)$ and k^+ is set of values of some a.p. 3. k^+ can be topologized by taking the sets of values of the a.p. as neighborhoods of 0. Under the induced uniform topology on the ring C of continuous endomorphisms of k^+, the a.p. are everywhere dense in C. (Received October 15, 1952.)

ANALYSIS

84. R. C. Buck: *Generalised asymptotic density.*

Let μ_n be a sequence of measures on X, and $K_1 \subset K_2 \subset \cdots$ be a sequence of sets of finite measure exhausting X such that (i) $\lim_n \mu_n(K_j) = 0$ for each j, (ii) $\mu_n(X) = 1$ for all n, (iii) the support of each μ_n lies in some K_j. Write $A \overset{\cdot}{\subset} B$ when there is a j such that $A - K_j \subset B - K_j$. A measurable set S has density $D(S) = d$ if $d = \lim \mu_n(S)$. Theorem: if $\{A_n\}$ is a sequence of mutually disjoint sets having density, there is a set S, unique up to sets of zero density, such that $D(S) = \sum D(A_n)$, and such that for each n, $S \overset{\cdot}{\supset} \bigcup_1^n A_k$. Application: let $f(x)$ be real-valued, measurable, and bounded below on X. For any $\epsilon > 0$, let $f(x) > -\epsilon$ on a set of unit density, and suppose that $\lim_{n\to\infty} \int f d\mu_n = 0$. Then, there is a set E of zero density such that as $x \to \infty$ in $X - E$, $f(x) \to 0$. The topology at ∞ in X is that defined by $X - K_n$. (Received October 16, 1952.)

85*t*. R. C. Buck: *On limit points of complex sequences.*

Let $\{s_n\}$ be a bounded complex sequence, and let C be its core, the union of the closed convex hulls of the sets $\{s_n, s_{n+1}, \cdots\}$. Any extreme point of C will be called an outer limit point of $\{s_n\}$, generalizing the upper and lower limits of a real sequence. Theorem: If $\{s_n\}$ is Cesaro summable to one of its outer limit points p, then except for a subsequence of zero density, $\{s_n\}$ converges to p. The theorem remains true if $\{s_n\}$ is allowed to be unbounded, provided that p is a regular extreme point of C for which there is a supporting line touching C only at p. (Received October 16, 1952.)

86. Lamberto Cesari: *On a geometrical proof of the equality $V(S)=L(S)$ for continuous parametric surfaces.*

Let S: $p=p(w)$, $w=(u, v)\in Q$, $p=(x, y, z)\in E_3$ denote any continuous parametric surface in E_3, i.e., any continuous mapping from a set Q of the w-plane into E_3. Then the projections of S on the coordinate planes are plane mappings T_i, $i=1, 2, 3$. Let $L(S)$ and $V(S)$ be the Lebesgue area and the so-called Geocze area of S. The equality $V(S)=L(S)$ is generally proved for all surfaces S of the type of the 2-cell, i.e., for mappings from a 2-cell Q, by making use first of the standard inequality (*) $V(S)\leq L(S)\leq \sum_i V(T_i)$, and then of the concepts of F-equivalence, Dirichlet integral, generalized conformal representation, and topological structure properties of continuous mappings from a 2-cell. All these important concepts, however, are far from the simple geometrical meaning of the equality $V=L$. A direct proof of the equality $V(S)=L(S)$ for all continuous mappings from any arbitrary open set Q is now given. The proof, which is independent of the concepts mentioned above, is based on the existence of everywhere dense rectifiable contours on each surface of finite Lebesgue area (L. Cesari, Bull. Amer. Math. Soc. vol. 57 (1951) p. 168; R. E. Fullerton, ibid. vol. 58 (1952) pp. 57 and 180) and makes systematic use of the inequality (*). (Received October 15, 1952.)

87. Y. W. Chen: *Degenerate solutions of quasi-linear partial differential equations in n independent variables.*

Consider equation $A_{ij}(p_1, \cdots, p_n)\partial^2u/\partial x_i\partial x_j=0$ with $A_{ij}=A_{ji}$ and $p_i=\partial u/\partial x_i$. A degenerate solution of the equation is called an s-tuple wave if it satisfies $n-s$ relations $p_\alpha=F^{(\alpha)}(p_1, \cdots, p_s)$, $\alpha=s+1, \cdots, n$. An s-tuple wave will be represented by $u=\sum_{s+1}^{n}x_\alpha F^{(\alpha)}+\phi+\sum_1^s p_k x_k$ where the $n-s+1$ unknown functions $F^{(\alpha)}$ and ϕ of s variables $p_1, \cdots, p_s$ satisfy $(n-1)!/(n-s)!\,(s-1)!$ partial differential equations in s independent variables. The problem of finding simple waves $(s=1)$ is under-determined (not enough equations) unless $n=2$, and that of finding s-tuple waves with $s>2$ is over-determined (too many equations). Only when $s=2$ are there as many equations as unknown functions. In this case of double waves the system of $n-1$ equations has the form $C_{11}F_{11}^{(\alpha)}+2C_{12}F_{12}^{(\alpha)}+C_{22}F_{22}^{(\alpha)}=0$ where $F_{ij}^{(\alpha)}=\partial^2F^{(\alpha)}/\partial p_i\partial p_j$, and the C_{ij} depend on p_1, p_2, $F^{(3)}\cdots F^{(n)}$ and their first partial derivatives with respect to p_1 and p_2. When $F^{(\alpha)}$ are obtained, one finds ϕ by solving $C_{11}\phi_{11}+2C_{12}\phi_{12}+C_{22}\phi_{22}=0$, $\phi_{ij}=\partial^2\phi/\partial p_i\partial p_j$. Each coefficient C_{ij} is the same in all $n-1$ equations. Such system, if hyperbolic, admits solutions for initial value problems, as is well known. The method used here to derive the differential equations for F^α is an application of contact transformations together with suitable separation of variables. Special examples of degenerate solutions are Meyer-Prandtl flows in the plane, conical flows in the space, and others. (Received October 14, 1952.)

88. S. D. Conte and W. C. Sangren: *An eigenfunction expansion associated with a pair of first order differential equations.*

Consider the equations $u'(x)-[\lambda+a(x)]v(x)=0$ and $v'(x)+[\lambda+b(x)]u(x)=0$ where $a(x)$, $b(x)$, and their derivatives are continuous and $L(0, \infty)$. Further consider those solutions (u, v) which satisfy the boundary condition $u(0)\cos\alpha+v(0)\sin\alpha=0$ and the condition $\int_0^\infty(|u|^2+|v|^2)dx<\infty$. Then the values of λ for which such solutions exist form a continuous spectrum extending over the entire real axis. Two arbitrary functions $[f(x), g(x)]$ which satisfy the boundary condition $f(0)\cos\alpha+v(0)\sin\alpha=0$ may under appropriate restrictions be represented by the generalized Fourier integrals $f(x)=\int_{-\infty}^{\infty}h(\lambda)u(x, \lambda)d\lambda$ and $g(x)=\int_{-\infty}^{\infty}h(\lambda)v(x, \lambda)d\lambda$. The methods used are patterned after those of E. C. Titchmarsh (*Eigenfunction expansions associated with second order differential equations*, Oxford, 1946). (Received September 18, 1952.)

89. R. E. Fullerton: *The structure of L cones and a geometric characterization of L-spaces.*

Let X be a linear topological space and let C be a cone with vertex at the origin which is the closed convex hull of its set of extreme rays. C is a C-cone if for every $x\in X$ there exists a $y\in X$ with $C\cap(x+C)=y+C$. C is an L-cone if there exists some $x\in X$, $x\notin C\cup(-C)$, and a $y\in X$ such that $C\cap(x+C)=y+C$. It is shown that an L-cone is either a C-cone or the union of two C-cones with a common face and that a necessary and sufficient condition that a Banach space X whose unit sphere S is the closed convex hull of its extreme points be a space of integrable functions is that for any extreme point $v\in S$ there exists an L cone C with $S=(v+C)\cap(-v-C)$. (Received October 15, 1952.)

90. Michael Golomb: *A function-theoretic method for computing eigenvalues and eigenfunctions.* Preliminary report.

Suppose $f(s)$ is regular analytic for $|s|\leq r$ except for the simple poles $\lambda_1, \cdots, \lambda_n$, $0<|\lambda_i|<r$. It is well known how the product $\prod\lambda_i$ can be obtained as the limit of a quotient of determinants involving the coefficients $a_\nu=f^{(\nu)}(0)/\nu!$. Similarly a limit involving the symmetric functions of $\lambda_1, \cdots, \lambda_n$ beside the coefficients a_ν is found for the sum of the residues of $f(s)$. These results are used to compute the eigenvalues and eigenprojectors of a bounded normal linear operator A in a Hilbert space. Let $\mathcal{A}$ be the Banach algebra over the complex field generated by A and $\tau(X)$ a complex-valued bounded linear functional on $\mathcal{A}$. Put $R(\lambda)=A(1-\lambda A)^{-1}$ and assume the part of the spectrum of A for which $|s|\geq r^{-1}$ consists of the eigenvalues $\lambda_1^{-1}, \cdots, \lambda_n^{-1}$, the corresponding eigenprojectors being $P_1, \cdots, P_n$. Then $f(s)=\tau(R(s))$ has simple poles at $\lambda_1, \cdots, \lambda_n$ if $\tau(P_i)\neq 0$, and $f^{(\nu)}(0)/\nu!=\tau(A^{\nu+1})$. Hence, $\prod\lambda_i$ can be expressed as a limit involving the numbers $\tau(A^\nu)$ and $\sum P_i$ can be expressed as a limit involving the symmetric functions of $\lambda_1, \cdots, \lambda_n$ beside the operators A^ν. Specialization of the function τ leads to various iteration formulae for the eigenvalues, some known, some new. Application is made to eigenvalues and eigenfunctions of integral kernels. (Received October 14, 1952.)

91*t*. Michael Golomb: *A general Fredholm theory for linear equations.* Preliminary report.

Suppose A is a bounded linear operator in a complex Banach space, $\mathcal{A}$ is the Banach algebra generated by A, and $\tau(X)$ is a complex-valued bounded linear functional on $\mathcal{A}$. Define the numbers $n!d_n=\sum(-1)^r\tau(A^{p_1})\cdots\tau(A^{p_r})$ where the sum extends over all permutations of $1, 2, \cdots, n$, and $p_1, \cdots, p_r$ are the orders of the cycles constituting the permutation. Likewise define the operators $n!D_n=\sum(-1)^r$

$\cdot\tau(A^{p_1})\cdots\tau(A^{p_r})A^p$ with the sum extending over all permutations of $0, 1, \cdots, n$; p being the order of the cycle containing 0. The functions $d(\lambda)=\sum d_n\lambda^n$ and $D(\lambda)=\sum\lambda^nD_n$ satisfy the identities $D(\lambda)(1-\lambda A)=(1-\lambda A)D(\lambda)=d(\lambda)A$, hence function like the fundamental determinant and minors of the Fredholm theory of integral equations. Conditions on the functional τ are investigated under which $d(\lambda)$, $D(\lambda)$ are entire functions and under which the poles and residues of $D(\lambda)/d(\lambda)$ are the eigenvalues and eigenprojectors of A. (Received October 14, 1952.)

92. J. H. B. Kemperman: *Exact solutions for a certain class of integral equations.*

Let $a=a(x)$ and $F(x, y)$ be defined for $x, y\in E$, with $F(x, y)=\sum_1^p d_k(x)F_k(y)$ if $a(x)\neq 0$ and $a(y)\neq 1$ (restricted degeneration). Let $Ug=\int_E g(y)dF(x, y)$ and $Tg=aUg$. Let $(1-\lambda U)g_i=h_i$ ($i=1, \cdots, q$; $q\geq p$), where λ is a scalar and where h_i is linearly dependent on the set of p functions $d_k=d_k(x)$ ($k=1, \cdots, p$). Finally, let $(1-\lambda U)g_0=h_0$. Then, given a certain independence condition, the constants $\alpha_1, \cdots, \alpha_q$ can be found such that the function $g=a(g_0+\alpha_1g_1+\cdots+\alpha_qg_q)$ satisfies $(1-\lambda T)g=ah_0$. This principle sometimes enables us to construct a solution of the latter equation. An important application is that where E is the real axis, $a(x)=1$ for $0<x<c$, $a(x)=0$ otherwise (eventually, $c=+\infty$). Assume that $F(x, y)=F(y-x)$, where, for $x<0$, $F(x)$ is a finite sum of terms $c_jx^{m_j}e^{\lambda_jx}$ (c_j, λ_j = complex numbers, m_j = integer ≥ 0). Only if $c<\infty$ we require a similar condition for $x>0$. Further, suppose that $\phi(t)=\int_{-\infty}^{\infty}e^{tx}dF(x)$ converges for Re $(t)=t_0$ and, moreover, that $\phi(t)$ can be extended as an analytic and single-valued function in a certain open and connected region D (if $c<\infty$, $\phi(t)$ is a rational function). If $g_t=e^{tx}$, $t\in D$, $\phi(t)=1/\lambda$, and Tg_t exists, it turns out that g_t serves the above purpose even if Ug_t does not exist (if λUg_t exists it is equal to g_t, thus, $h_t=0$). Example: if $F(x)=e^x$ for $x<0$ and $\phi(t_1)=\phi(t_2)=1/\lambda$, the function $g(x)=(1+t_2)e^{t_1x}-(1+t_1)e^{t_2x}$ satisfies $g(x)=\int_0^\infty g(y)dF(y-x)$, provided that the latter integral exists. (Received November 12, 1952.)

93*t*. A. J. Lohwater and George Piranian: *On the derivative of a univalent function.*

It is known that, if $f(z)$ is analytic and univalent in $|z|<1$, then (*) $\lim_{r\to 1}(1-r)^{1/2}|f'(re^{i\theta})|=0$ for all $e^{i\theta}$ on $|z|=1$ except for a possible set of linear measure zero (Seidel and Walsh, Trans. Amer. Math. Soc. vol. 52 (1942) p. 141). In view of Beurling's success in cutting down the size of certain exceptional sets associated with a univalent function and its derivative, it is natural to ask whether the exceptional set associated with (*) can be described in terms of logarithmic capacity. It is proved that there exists in $|z|<1$ a univalent function $f(z)$ such that $\lim_{r\to 1}(1-r)^{\beta}|f'(re^{i\theta})|$, $0\leq\beta<1$, is infinite on a set of positive capacity. This result depends on the following lemma: If $\mu(t)$ is of bounded variation in $(-\pi, \pi)$, and if $t_0(-\pi<t_0<\pi)$ is such that $\lim_{h\to 0}(\mu(t_0+h)-\mu(t_0-h))/h^{\alpha}=\infty\,(0<\alpha\leq 1;\ h>0)$, then the harmonic function $u(r, \theta)=(1/2\pi)\int_{-\pi}^{\pi}K(r, \theta-t)d\mu(t)$, where $K(r, \theta)$ is the Poisson kernel $(1-r^2)/(1+r^2-2r\cos\theta)$, has the property that $\lim_{r\to 1}(1-r)^{1-\alpha}u(r, t_0)=\infty$. (Received September 17, 1952.)

94. Arthur Rosenthal: *On functions with infinitely many derivatives.*

For real functions $f(x)$, analytic at $x=x_0$, the derivatives at $x=x_0$ cannot be chosen quite arbitrarily. It is shown that in the case of infinitely often differentiable functions $f(x)$ the values of the successive derivatives at $x=x_0$ can be assigned in an entirely arbitrary manner. The proof uses polynomials which are closely related to the Hermite polynomials. (Received October 13, 1952.)

95. C. J. Titus and G. S. Young: *The maximum modulus principle for certain differential systems.*

Generalizing the work of G. Welsky, A. Krowade, and others, the authors prove the following. Let $u_i = u_i(x_1, \cdots, x_n)$, $i = 1, \cdots, n$, be a mapping of class C^1 defined on a domain D in E^n. Suppose that the Jacobian of $\{u_i\}$ does not change sign in D and that the function is constant on every connected open set on which the Jacobean vanishes. Then each $|u_i|$ takes its maximum on the boundary of every open set with a compact closure in D. Methods are those of the authors' paper (Michigan Mathematical Journal vol. 1 (1952) 89–93). (Received October 15, 1952.)

APPLIED MATHEMATICS

96. C. L. Dolph: *On the Timoshenko theory of the transverse vibration of beams.*

The Timoshenko beam equations (London Philosophical Magazine (1921)), which include correction terms for shear and rotary inertia, are investigated by means of solutions sinusoidal in time. A two term relation, valid independent of the uniformity of the beam, is derived and shown to imply a type of orthogonality of the spatial modes under all the simplest boundary conditions. For the special case of the uniform hinged-hinged beam the normal modes are determined explicitly and a general boundary and initial value problem solved by a superposition of them. The coefficients in these series are determined by the two term orthogonality relation and shown to be equal to Fourier coefficients. Such a solution exists although the given system is not separable without the additional assumption of a sinusoidal time behavior and although this assumption does not appear a priori to yield sufficiently many such modes to satisfy the four initial conditions. However, each spatial part is in fact paired with two functions sinusoidal in time, with different frequencies so that the resulting eigenvectors are orthogonal in the above sense. Universal curves for this case are compared with those computed electronically for the uniform free-free beam where it is not known that the sinusoidal assumption is sufficient. (Received March 24, 1952.)

97. Y. L. Luke: *Mechanical quadrature near a singularity.*

Purpose of paper is to present coefficients to facilitate computation of $\int_0^{nh} x^{-1/2} f(x)\,dx$. Coefficients are derived on basis of writing the Lagrangian polynomial which precisely fits $f(x)$ at the points $f(rh)$, $r = 0, 1, 2, \cdots, n$. Exact coefficients are given for $n = 1(1)10$. Integration formula is exact if $f(x)$ is a polynomial of degree n. The remainder term for the $(n+1)$ point formula is derived after the manner of W. E. Milne (*The remainder in linear methods of approximation*, Journal of Research, National Bureau of Standards, vol. 43 (1949) pp. 501–511). He writes $R_n = \int_0^{nh} f^{(n+1)}(s) G(s)\,ds$. Mean value theorem is applicable if $G(s)$ is of constant sign in the interval of integration. This is so for the even point formulae. For the odd point formulae, $G(s)$ vanishes at one point in the open interval of integration. Let ξ be the point. For each n exact values of $\int_0^{nh} G(s)\,ds$ are tabulated. For the odd point formulae, values of ξ and $\int_0^{\xi} G(s)\,ds$ are also presented, mostly to 6*d*. (Received October 10, 1952.)

98*t*. K. O. May: *Note on the complete independence of the conditions for simple majority decision.*

A method of group decision may be characterized by specifying a function giving group choice in terms of the choices of the individuals making up the group. In a previous paper (*A set of independent necessary and sufficient conditions for simple majority decision*, Econometrica, October 1952) the author showed that the familiar method of deciding between two alternatives by simple majority vote is characterized categorically by four conditions on the group decision function. These conditions are here shown to be *completely* independent by exhibiting functions satisfying every possible combination of their truth values. (Received October 14, 1952.)

99t. L. E. Payne: *Axially symmetric problems in elasticity.* II. *The crack problem.*

This paper gives a new approach to the problem of determining the distribution of stress in an infinite elastic medium when a normal pressure is applied to the faces of a disk-shaped crack (see, I. N. Sneddon, *Fourier Transforms*, 1951). Let the plane of the crack be the plane $x=0$ and let y be the radial coordinate. The problem then amounts to the determination of the stress distribution throughout the medium with the conditions that on the plane $x=0$ all shearing stresses vanish, for $y>b$ no displacement normal to this plane exists, and for $y<b$ either the normal stress or normal displacement be prescribed. In this paper the problem is reduced to that of determining the potential of a magnetic disk. By use of curvilinear coordinates the customary tedious procedure of solving a pair of dual integral equations can be eliminated. This new method of formulation, which is the same as that employed in treating the punch problem (L. E. Payne, *Axially symmetric problems in elasticity.* I. *The punch problem*, Bull. Amer. Math. Soc. Abstract 58-6-681) shows that the crack and punch problems are actually equivalent to certain half space torsion problems. (Received October 3, 1952.)

GEOMETRY

100. P. O. Bell: *On certain limit points determined by means of a prescribed metric at a generic point of a curve.*

Let $x(-\sigma)$, $x(0)$, $x(\sigma)$ denote neighboring points of a space curve C in ordinary projective space. Let π_μ denote an arbitrary plane passing through the tangent to C at $x(0)$. Let $P(\sigma, \mu)$ denote the point of intersection of the line joining $x(-\sigma)$, $x(\sigma)$ with the plane π_μ. As σ tends to zero the point $P(\sigma, \mu)$ approaches a limit point $P(0, \mu)$ which is a point of the tangent to C at $x(0)$, and it is distinct from $x(0)$ if, and only if, π_μ osculates C at $x(0)$. If π_μ osculates C at $x(0)$, the limit of the line joining $x(0)$ and $P(\sigma, \mu)$ is a line distinct from the tangent to C at x. Use is made of these concepts to characterize geometrically: (1) A covariant reference frame for C associated with an element of projective arc length, (2) the projective normal of an analytic surface, and (3) the curvature and torsion of a curve C. The last of these characterizations is as follows: Let ϕ denote the angle in the positive sense from the osculating plane to the plane π_μ, and let Δ denote the directed distance from $x(0)$ to $P(0, \mu)$. Tan ϕ is proportional to Δ if, and only if, $k^{1/3}ds/d\sigma=$const., s denotes arc length, $k=$curvature. In this case $\tan\phi=T\Delta/3$, $T=$torsion. (Received October 14, 1952.)

101. Oswald Wyler: *Incidence geometries.*

The lattice L of flats of a geometry satisfying Hilbert's axioms of incidence is an exchange lattice (MacLane, Duke Math. J. vol. 4 (1938) pp. 455–468), and the quotient lattice L/P is modular for every point P of L. These two properties furnish a lattice-theoretic characterization of incidence geometries of any dimension. Two systems of axioms for such geometries of dimension $\geqq 3$ are given. The first one, with points and planes as primitive notions, is valid for any dimension; the second one, with points and hyperplanes as primitive notions, is valid for finite-dimensional geometries. An incidence geometry is called irreducible if every pencil of lines contains at least three lines. It is shown that any irreducible incidence geometry of dimension $\geqq 4$ can be embedded into a projective geometry of the same dimension by the construction of bundles. (Received October 7, 1952.)

LOGIC AND FOUNDATIONS

102*t*. J. C. E. Dekker: *Productive sets.*

A collection of non-negative integers is called a set. Let $\Phi(n, x)$ be the partial recursive function discussed by Kleene [Trans. Amer. Math. Soc. vol. 53 (1943), p. 58] which generates all partial recursive functions of one variable. Following Rice [*Classes of recursively enumerable sets and their decision problems*, to be published] we use $\Phi(n, x)$ to characterize recursively enumerable (r.e.) sets. Let ω_n denote the range of $\Phi(n, x)$, then $\{\omega_n\}$ is a sequence of r.e. sets in which every r.e. set occurs at least once. The set α is called productive if there exists a partial recursive function $p(n)$ such that $\omega_n \subset \alpha$ implies: (1) $p(n)$ is defined, (2) $p(n) \in \alpha - \omega_n$. Every such function $p(n)$ is called a productive function of α. Let Dom α denote the set of all n such that $\omega_n \subset \alpha$. The set π is called a productive center of the productive set α if $\pi = p(\text{Dom } \alpha)$ for some productive function $p(n)$ of α. *Theorems.* A productive set is not r.e., but has an infinite r.e. subset. There exists a productive set among whose productive centers denumerably many are mutually disjoint. If π is a productive center of the productive set α and $\pi \subset \beta \subset \alpha$, then β is a productive set with a productive center included in π. (Received October 9, 1952.)

103*t*. J. C. E. Dekker: *Two notes on recursively enumerable sets.*

A collection of sets of non-negative integers is called a class. Q denotes the class of all sets which are finite or empty, P the class of all sets which have a finite or empty complement, E the class of all recursive sets, F the class of all recursively enumerable (r.e.) sets. Post introduced the following concepts: simple set, hypersimple set, one-one reducible, many-one reducible [Bull. Amer. Math. Soc. vol. 50 (1944) pp. 284–316]. The complete set defined by Post [loc. cit. p. 295] is denoted by κ, the simple set defined by Post [loc. cit. p. 298] by ζ. H denotes the class of all r.e. sets whose complement is productive [see the preceding abstract], Z the class of all simple sets, Z_0 the class of all hypersimple sets, J the class $(F-E)-(H+Z)$. *Theorem of the first note.* Both $Z+P$ and Z_0+P are dual ideals in F. *Theorem of the second note.* There exists a set ν belonging to J whose degree of unsolvability relative to one-one reducibility lies between that of ζ and that of κ, while its degree of unsolvability relative to many-one reducibility is the same as that of ζ. (Received October 9, 1952.)

104*t*. J. C. E. Dekker: *Productive classes.*

For notations see the two preceding abstracts. Let Ω_n denote the class of all r.e. sets over which ω_k ranges if k ranges over ω_n. The subclass A of F is called r.e. if $A = \Omega_n$ for some n. The author proved in an earlier (not yet published) paper that the classes P, Q, E, F are r.e. The subclass A of F is called productive if there exists a partial recursive function $p(n)$ such that $\Omega_n \subset A$ implies: (1) $p(n)$ is defined, (2) $\omega_{p(n)} \in A - \Omega_n$. Every such function $p(n)$ is called a productive function of A. Let

Dom A denote the set of all n such that $\Omega_n \subset A$. The class T is called a productive center of the productive class A if there exists a productive function $p(n)$ of A such that $\omega_{p(n)}$ ranges over T, when n ranges over Dom A. *Theorem.* If T is a productive center of the productive class A and $T \subset B \subset A$, then B is a productive class with a productive center included in T. Let D denote the class $E-(P+Q)$. A subclass A of F is called creative if A is r.e. and $F-A$ productive. *Theorem.* The class $F-Q$ is productive; among its productive centers at least one is included in D, at least one in H and at least one in J. *Corollaries.* D, H, J are productive. Q and E are creative. (Received October 9, 1952.)

105. P. C. Hammer: *Generalized extreme points.*

Let M be a set, $\mathfrak{S}_1$ and $\mathfrak{S}_2$ classes of subsets of M, and f a function with domain $\mathfrak{S}_1$, range $\mathfrak{S}_2$, where $\mathfrak{S}_1$ does not contain the null set, N. A subset X of M is said to be *f-closed* if $X \supset Y \in \mathfrak{S}_1$ implies $X \supset f(Y)$. f is *canonical* if $Y \cap f(Y) = N$ for every $Y \in \mathfrak{S}_1$. Every f has an equivalent canonical function. Consider only canonical functions. x is an *extreme point* of an f-closed set X if $X \supset Y \in \mathfrak{S}_1$ implies $x \notin f(Y)$. A set Z is *f-discrete* if $Z \supset Y \in \mathfrak{S}_1$ implies $Z \cap f(Y) = N$. The set of extreme points of an f-closed set is f-discrete. Every subset of an f-discrete set is f-discrete. Every f-discrete set contains the extreme points of its f-closure. An f-closed set minus its extreme points is f-closed. (Received September 29, 1952.)

TOPOLOGY

106. J. G. Hocking: *Homeomorphic approximations to monotone maps on noncompact 2-manifolds.* Preliminary report.

Using a suitable definition of a 1-monotone map of a (noncompact) locally compact (separable metric) space, the following theorem is established: Let f be a uniformly continuous map of a (not necessarily compact) ulc[1], 2-manifold M^2 (with or without ulc boundary) onto itself. Then f can be approximated uniformly by homeomorphisms of M^2 onto itself if and only if the map f is 1-monotone. The methods of proof are largely those of G. T. Whyburn and G. S. Young. (Received October 10, 1952.)

107. J. D. McKnight, Jr.: *Characterizations of function-rings.* Preliminary report.

Let $C(X)$ be the ring of all continuous real-valued functions over a compact Hausdorff space X. Among the characterizations of $C(X)$ obtained by using the Stone topology for special classes of ideals, one is found by noting that the following conditions are fulfilled: (1) $C(X)$ is semi-simple; (2) $C(X)$ is an algebra with unity over the real field; (3) there is an isomorphism of $C(X)-I$ into the real field for every primitive ideal I; (4) the space of maximal (primitive) ideals of $C(X)$ is a Hausdorff space, i.e., if I_1 and I_2 are different maximal ideals, there exist ideals J_1 and J_2 such that $J_i \subset I_i$, $J_i \not\subset I_j$ for $i \neq j$, and $J_1 \cap J_2 = 0$; (5) the elements of $C(X)$ satisfy a boundedness property: there is a real number a such that for any $b \in C(X)$ there is another element $c \in C(X)$ such that $n(a1) - b = c^2$ for some natural number n; (6) $C(X)$ has no proper extension satisfying (1)–(5) which preserves the topology on the space of maximal ideals, i.e., if $C(X) \cong C \subset B$, B satisfying (1)–(5), and if the correspondence $I \rightarrow I \cap C$ is one-one from the set of maximal ideals of B to the set of maximal ideals of C and $J \cap C \subset I \cap C$ implies $J \subset I$ for ideal J and maximal ideal I of B, then $B = C$. (Received October 14, 1952.)

108. P. S. Mostert: *Fiberable spaces.* I. Preliminary report.

Seven classes, $F_0, \cdots, F_6$, of fiberable spaces are defined, where F_0 is the class of fibre bundles, and an element of F_6 is a triple $\{X, B, \pi\}$ where X and B are topological spaces, and π is a continuous, open map of X on B. A *partial cross section* w.r.t. a class K of subsets of B is a mapping f of some set A of K into X such that $\pi f(a) = a$. It is the object of this and subsequent papers to study the interrelations of these concepts. In particular, criteria are discussed for the existence of partial cross sections when the class K consists of the arcs of B. (Received October 16, 1952.)

109. M. E. Shanks: *Homology and rings of continuous functions.* Preliminary report.

Homology groups of an associative ring $\mathfrak{R}$ have been defined by S. T. Hu in his paper, *Homology groups of a ring*, Bull. Calcutta Math. Soc. vol. 42 (1950) pp. 123–130, where he shows that if $\mathfrak{R} = C(X) =$ the ring of all real continuous functions over a compactum X, then his homology groups are the Alexander-Kolmogoroff groups of X. In this paper the above result is shown to be valid for: (1) a larger class of spaces, (2) certain proper subrings of $C(X)$, (3) much smaller subrings if X is a polytope. In addition the whole theory is recast in a form depending on the support of $f \in C(X)$ instead of on the zero set of f as was done by Hu. Finally, a comparison is made between the above theory and the theory of differential forms on manifolds. (Received October 17, 1952.)

J. W. T. YOUNGS,
Associate Secretary

THE NOVEMBER MEETING IN LOS ANGELES

The four hundred eighty-seventh meeting of the American Mathematical Society was held at the University of Southern California, Los Angeles, California, on Saturday, November 29, 1952. Attendance was approximately 100, including the following 81 members of the Society:

M. G. Arsove, E. W. Barankin, E. F. Beckenbach, M. M. Beenken, Clifford Bell, R. E. Bellman, L. D. Berkovitz, H. F. Bohnenblust, J. V. Breakwell, L. H. Chin, E. A. Coddington, L. M. Coffin, J. M. Danskin, G. B. Dantzig, P. H. Daus, A. C. Davis, A. H. Diamond, R. P. Dilworth, Milton Drandell, Roy Dubisch, L. K. Durst, H. A. Dye, Arthur Erdélyi, Harley Flanders, G. E. Forsythe, I. L. Glicksberg, J. W. Green, L. D. Gregory, C. J. Halberg, Jr., L. A. Henkin, M. R. Hestenes, P. G. Hodge, Jr., P. G. Hoel, D. G. Humm, Rufus Isaacs, P. B. Johnson, Jan Kalicki, L. D. Kovach, D. H. Lehmer, R. B. Leipnik, H. A. Linstone, M. M. Loève, Charles Loewner, N. M. Martin, H. A. Osborn, T. K. Pan, Emanuel Parzen, R. S. Phillips, D. H. Potts, W. T. Puckett, Jr., R. M. Redheffer, Edgar Reich, R. E. Roberson, J. B. Robinson, Raphael M. Robinson, G. F. Rose, Peter Scherk, Raymond Sedney, Bernard Sherman, Seymour Sherman, I. M. Singer, James C. Smith, Ernst Snapper, R. H. Sorgenfrey, D. V. Steed, Robert Steinberg, E. G. Straus, A. C. Sugar, J. D. Swift, Alfred Tarski, A. E. Taylor, F. B. Thompson, R. A. Wagner, L. F. Walton, W. R. Wasow, M. A. Weber, P. A. White, G. T. Whyburn, B. R. Wicker, F. L. Wolfe, P. S. Wolfe.

The two sections at 10:00 A.M. for contributed papers were presided over by Dr. G. E. Forsythe and Professor Ernst Snapper. At 2:00 P.M., Professor M. M. Loève of the University of California, Berkeley, delivered the invited address, *Ergodic theorems in probability theory.* Professor Loève was introduced by Professor P. G. Hoel. At 3:15 there was a section for late papers, at which Professor D. V. Steed presided.

After the meetings, those attending were the guests of the Mathematics Department of the University of Southern California at tea at the Town and Gown.

Abstracts of papers presented at the meeting follow. Those preceded by the letter "*t*" were presented by title. Paper 114 was presented by Professor Tarski, paper 116 by Professor Kalicki, and paper 137 by Professor Scherk. Mr. Mack was introduced by Professor G. C. Evans, Mr. Scott by Professor Kalicki, and Mr. Adams by Professor J. C. C. McKinsey.

ALGEBRA AND THEORY OF NUMBERS

110*t*. Anne C. Davis: *Square roots of some denumerable order types.*

All solutions of $\xi^2=\alpha$ are found, when α is an order type of the form $\alpha=\tau\cdot\eta$, τ denumerable and scattered (Bull. Amer. Math. Soc. Abstract 58-4-330; Davis and Sierpinski, C. R. Acad. Sci. Paris vol. 235 (1952) pp. 850–852). Every order type μ is (essentially) uniquely representable as an ordered sum $\mu = \sum_{x\in M}\phi(x)$ where $\phi(x)\in R = \{\omega, \omega^*, \omega^*+\omega, 1, \cdots, n, n+1 \cdots\}$, and, if x immediately precedes x' in M, then $\phi(x)+\phi(x')\notin R$. For every type μ let $R(\mu)$ be the range of ϕ in this representation. Let A, B respectively denote the set of nonzero denumerable scattered types τ of the form $\tau=1+\tau'$, $\tau=\tau''+1$. Let $S(\tau)$ be the set of all solutions of $\xi^2=\tau\cdot\eta$. Results: (1) If $\tau\in A\cap B$, $S(\tau)=\{\tau\cdot\eta\}$. (2) If $\tau\in A-B$, or if $\tau\in B-A$, $S(\tau)=\{\tau\cdot\eta, \tau\cdot\eta+\tau\}$, or $S(\tau)=\{\tau\cdot\eta, \tau+\tau\cdot\eta\}$. (3) If $\tau\notin A$, $\tau\notin B$ then; (3a) If $R(\tau)=\{\omega^*+\omega\}$, $S(\tau)$ consists of $\tau\cdot\eta$ and of all $\xi=\sigma+\tau\cdot\eta+x$, where $x+\sigma=\tau$, $\sigma\notin A$, $x\notin B$, and $\xi\leqq S(\tau)\leqq\aleph_0$; (3b) If $\{\omega\}\subseteq R(\tau)\leqq\{\omega, \omega^*+\omega\}$, $S(\tau)$ consists of $\tau\cdot\eta$, $\tau\cdot\eta+\tau$, and of all $\zeta=\tau+\tau\cdot\eta+x$, where $x+\tau=\tau$, $x\notin B$, and $S(\tau)\in\{\zeta\cdot\aleph_0\}$; (3c) The case when $\{\omega^*\}\subseteq R\leqq\{\omega^*, \omega^*+\omega\}$ is symmetric with (3b); (3) If $n\in R(\tau)$, n finite, or if $\{\omega, \omega^*\}\subseteq R(\tau)$, $S(\tau)=\{\tau\cdot\eta, \tau+\tau\cdot\eta, \tau\cdot\eta+\tau\}$. (Received October 20, 1952.)

111. Harley Flanders: *The norm function of an algebraic field extension.*

Let K be an extension of finite degree n of an algebraic field k. With respect to a basis $\omega_1, \cdots, \omega_n$ of K over k, one may define the general norm function $N(X)$. It is a polynomial over k, homogeneous of degree n in the n variables $X=(X_1, \cdots, X_n)$. It is proved that $N(X)$ is equal to an irreducible polynomial raised to the power $s=n/m$, where m is the maximum of the field degrees $(k(\theta):k)$ for all θ in K. This result is applied to the problem of characterizing the field norm $N_{K/k}$ by means of its algebraic properties. (Received October 17, 1952.)

112*t*. Alfred Horn: *On the eigenvalues of a matrix with prescribed singular values.*

H. Weyl [Proc. Nat. Acad. Sci. U. S. A. vol. 35 (1949) pp. 408–411] has shown that if A is a matrix with eigenvalues $\lambda_1, \cdots, \lambda_n$ and singular values $\alpha_1, \cdots, \alpha_n$ (the singular values are the non-negative square roots of the eigenvalues of A^*A), and if (1) $\alpha_1\geqq \cdots \geqq\alpha_n\geqq 0$, then (2) $|\lambda_{i_1}, \cdots, \lambda_{i_k}|\leqq\alpha_1\cdots\alpha_k$ whenever $1\leqq i_1 < \cdots < i_k\leqq n$, $1\leqq k\leqq n$, and (3) $|\lambda_1\cdots\lambda_n| = \alpha_1\cdots\alpha_n$. In this paper it is shown that the simultaneous relations (2), (3) are best possible. In other words, given any pair of n-tuples $(\lambda_1, \cdots, \lambda_n)$, $(\alpha_1, \cdots, \alpha_n)$ satisfying (1), (2), and (3), then there exists a matrix with the former as eigenvalues and the latter as singular values. This solves a question raised by Seymour Sherman. (Received October 10, 1952.)

113*t*. J. R. Jackson: *Extension of relations on finite sets to partial orderings.*

A *strict partial ordering* on set A is a transitive relation $\prec$ on A such that $a\prec a$ is false for $a\in A$. The result given here is easily translated into the language of "ordinary" partial orderings. Let R be a relation on finite set A, represented by a segment of integers: $A=\{1, 2, \cdots, N\}$. Define $N\times N$ matrix $P=(p_{mn})$ by setting $p_{mn}=1$ if mRn, $p_{mn}=0$ otherwise. Let $P+P^2+\cdots+P^N=\bar{P}=(\bar{p}_{mn})$. Define relation $\bar{R}$ on A by including $m\bar{R}n$ if and only if $\bar{p}_{mn}>0$ (this definition can be made more elegant by appropriately choosing the p_{mn} from the Boolean algebra with two elements). Then the following six conditions are equivalent: (i) $P^N=0$; (ii) tr $\bar{P}=\bar{p}_{11}+\bar{p}_{22}+\cdots+\bar{p}_{NN}=0$; (iii) there exists an $N\times N$ permutation matrix G such that all the nonzero entries of $G'PG$ are below the main diagonal; (iv) R can be extended to a strict partial ordering on A (and hence, in fact, to a complete ordering of A); (v) $\bar{R}$ is a strict partial ordering on A; (vi) $\bar{R}$ is the (unique) minimal extension of R

to a strict partial ordering on A. (Prepared under Contract Nonr 233(02), Office of Naval Research.) (Received October 14, 1952.)

114. Bjarni Jónsson and Alfred Tarski: *Factor relations over algebras*. Preliminary report.

Let $\mathfrak{A}=\langle A, f_0, f_1, \cdots\rangle$ be any algebra. (For notations see Jónsson-Tarski, Amer. J. Math. vol. 73, pp. 894 ff.) $S\perp S'$ means that S, S' are *complementary* congruence relations over $\mathfrak{A}$, i.e., $S|S'=A\times A$ and $S\cap S'$ is the identity relation on A. S is called a *factor relation*, $S\in FR$, if $S\perp S'$ for some S'. The following conditions are equivalent: (α) FR is a Boolean algebra under $S|T$ and $S\cap T$; (β) $S|T=T|S$ and $(S|T)\cap S'\subseteq T$ whenever $S\perp S'$, $T\in FR$; (γ) $(S|T|S)\cap S'\subseteq T$ whenever $S\perp S'$, $T\in FR$. As is known, every algebra satisfying (α) has at most one direct decomposition into indecomposable factors. In particular, (α) holds for every $\mathfrak{A}=\langle A, +\rangle$ such that $x+x=x$ for every $x\in A$ and for any $x_1, x_2, x_3\in A$ there is a $z\in A$ with $x_i+z=z+x_i=z$ [or $x_i+z=z+x_i=x_i$], $i=1, 2, 3$. Under a modified definition of $S\perp S'$, (α)-(γ) remain equivalent for algebras in the wider sense; (α) holds for every $\mathfrak{A}=\langle A, R\rangle$ where R is a reflexive, antisymmetric relation and for any $x_1, x_2, x_3\in A$ there is a $z\in A$ with x_iRz [or zRx_i], $i=1, 2, 3$. This improves the known factorization theorems for lattices, doubly directed systems, etc. (Birkhoff, *Lattice theory*, 1948, pp. 25–27; Nakayama, Math. Japonicae vol. 1, p. 49; Tarski, *Cardinal algebras*, pp. 277, 308). (Received October 21, 1952.)

115*t*. Bjarni Jónsson and Alfred Tarski: *Decomposition functions on algebras*.

For notations see preceding abstract. Let $\mathfrak{A}=\langle A, +\rangle$. A function f on $A\times A$ to A is called a *decomposition function* (on $\mathfrak{A}$), $f\in DF$, if $f(x, x)=x$, $f(x, f(y, z))=f(f(x, y), z)=f(x, z)$, and $f(x+z, y+u)=f(x, y)+f(z, u)$ for all $x, y, z, u\in A$. If $S\perp S'$ and, for all $x, y\in A$, $f(x, y)$ is defined as the only $z\in A$ with $xSzS'y$, then $f\in DF$; conversely, if $f\in DF$, and S and S' are defined by $xSy\leftrightarrow f(x, y)=y$, $xS'y\leftrightarrow f(x, y)=x$, then $S\perp S'$. Thus, the study of direct decompositions reduces to that of DF. Given $u\in A$ and $f\in DF$, let $f_u(x)=f(x, u)$ and $f'_u(x)=f(u, x)$ for $x\in A$. Then $f_u(x+y)=f_u(x'+y')$ whenever $f_u(x)=f_u(x')$ and $f_u(y)=f_u(y')$, and hence f_u is a homomorphism on $\mathfrak{A}$; if $u+u=u$, then $f_u(x+y)=f_u(x)+f_u(y)$, and f_u is an endomorphism; similarly for f'_u. Condition (α) of preceding abstract is equivalent to each of the following conditions: (ϵ) $f_ug_u=g_uf_u$ for all f, $g\in DF$ and every $u\in A$; (ζ) $f_ug_u=g_uf_u$ for all f, $g\in DF$ and some $u\in A$; (η) for all f, $g\in DF$ and $x_1, x_2, x_3, x_4\in A$ there is a $u\in A$ such that $f_u(g_u(x_i))=g_u(f_u(x_i))$ for $i=1, 2, 3, 4$. (This theorem proves helpful in deriving some results of preceding abstract.) With appropriate modifications the results extend to arbitrary algebras in the wider sense. (Received October 16, 1952.)

116. Jan Kalicki and Dana Scott: *Some equationally complete algebras*. Preliminary report.

Continuing the investigations of Bull. Amer. Math. Soc. Abstract 58-6-582 it was proved that (1) $\mathfrak{A}=\langle A, '\rangle$ with one unary operation is equationally complete if and only if either $x'=x$ whenever $x\in A$ or $x=y'$ whenever $x, y\in A$; (2) a lattice is equationally complete if and only if it is distributive; (3) a group is equationally complete if and only if it is Abelian and each of its elements is of the same prime order. This applies to groups considered as systems $\langle G, \cdot\rangle$, $\langle G, \cdot, {}^{-1}\rangle$, or $\langle G, \cdot, {}^{-1}, 1\rangle$; (4) a ring $\mathfrak{R}=\langle R, \cdot, +, -, 0\rangle$ with an element $z\neq 0$ for which $z\cdot z=0$ is equationally complete if and only if, whenever $x, y\in R$, $x\cdot y=0$ and $px=0$ ($1x=x$, $(k+1)x=kx+x$, and p

is a prime); (5) every ring for which there is a prime p such that for every element x, $px=0$ and $x^p=x$ ($1x=x1=x$, $(k+1)x=kx+x$, $x^{k+1}=x^k\cdot x$) is equationally complete; (6) every Boolean algebra is equationally complete (this result is essentially known from the literature); (7) a Brouwerian algebra is equationally complete if and only if it is Boolean; (8) a closure algebra $\langle A, \cup, \cap, -, C\rangle$ is equationally complete if and only if $Cx=x$ whenever $x\in A$; (9) a relation algebra is equationally complete if and only if it is a Boolean relation algebra. (Received October 14, 1952.)

117. M. S. Klamkin: *On transcendental numbers generated by the method of Liouville.*

If the number $\sum_1^\infty 10^{-a_n}$ can be shown to be transcendental by application of Liouville's theorem on algebraic numbers, it is then shown that integral sequences $\{b_n\}$ of lower order than $\{a_n\}$ $[b_n=o(a_n)]$ can be chosen such that the number $\sum_1^\infty 10^{-b_n}$ is also transcendental. (Received October 14, 1952.)

118. L. D. Kovach: *On elementary nilpotent algebras.*

An elementary nilpotent algebra of degree n over a field $\mathcal{J}$, denoted by $\mathfrak{C}_l^{(n)}$, is defined to be any isomorphic copy of the algebra of $n\times n$ matrices over $\mathcal{J}$ with zeros on and above the main diagonal and beyond the lth column. Interest in this class of algebras stems from the fact that $\mathfrak{C}_l^{(n)}$ is a characteristic ideal of the total nilpotent algebra $\mathfrak{C}_{n-1}^{(n)}$. The latter is significant by virtue of a theorem proved by Dubisch and Perlis (*On total nilpotent algebras*, Amer. J. Math. vol. 73 (1951) pp. 439–452), namely, that the total nilpotent algebras together with their subalgebras constitute the totality of associative nilpotent algebras. In the present paper the results of Dubisch and Perlis are generalized to elementary nilpotent algebras. All automorphisms and all ideals of $\mathfrak{C}_l^{(n)}$ are explicitly determined and it is determined which ideals are characteristic. (Received July 2, 1952.)

119. Dana Scott: *Algebraic characterization of equational completeness.* Preliminary report.

For terminology see Bull. Amer. Math. Soc. Abstract 58-6-582. Let $P_n[\mathfrak{A}]$ be the algebra of polynomial functions of n arguments over the algebra $\mathfrak{A}$, i.e. the subalgebra of the infinite direct power of $\mathfrak{A}$ generated by the n identity functions $f_i(x_1, \cdots, x_n)=x_i$ for $i=1, \cdots, n$. $\mathfrak{A}$ is a *P_n-algebra* if $\mathfrak{A}$ is isomorphic to $P_n[\mathfrak{A}]$. $\mathfrak{A}$ is *P_n-simple* if there are no proper homomorphs of $\mathfrak{A}$ that are P_n-algebras. S is an *EC-extension* of $\mathfrak{A}$ if there is an algebra $\mathfrak{B}$ in the same similarity class as $\mathfrak{A}$ such that $\mathfrak{B}\in EC$, $Id(\mathfrak{A})\subseteq Id(\mathfrak{B})$, and $S=Id(\mathfrak{B})$. Results: (1) if $\mathfrak{A}\in EC$, then for every $n>1$, $P_n[\mathfrak{A}]$ is P_n-simple and $Id(\mathfrak{A})=Id(P_n[\mathfrak{A}])$; (2) if $\mathfrak{A}$ is a P_n-simple P_n-algebra with at least two elements and $n>1$, then $\mathfrak{A}\in EC$; (3) if $\mathfrak{A}$ is a finite algebra, then there are only finitely many EC-extensions of $\mathfrak{A}$. (Received October 14, 1952.)

ANALYSIS

120. M. G. Arsove: *A convergence theorem for δ-subharmonic functions.*

Let $\{w_n\}$ be a sequence of (quasi) δ-subharmonic functions on a region Ω, having mass distributions of uniformly bounded total variation. If the functions w_n all lie between two δ-subharmonic functions and converge point-wise (quasi everywhere) on Ω to a function w, then there exists a δ-subharmonic function w^* differing from w

at most on a set of capacity zero. Moreover, the mass distributions for $\{w_n\}$ converge vaguely to that for w^*. This theorem, which contains as a particular case the theorem of Brelot-Cartan for increasing sequences of subharmonic functions, is proved by introducing a special mode of convergence in the Banach space of functions of potential type and employing the following results: (i) every set of positive interior capacity supports a distribution of the unit mass having a continuous potential; (ii) (Choquet) all Borel sets are capacitable. (Without the Choquet theorem, the exceptional set can only be shown to be of zero *interior* capacity.) The derivation does not employ the Hilbert space technique of Cartan (or the intermediary of functions of finite energy), and thus affords an independent proof of the Brelot-Cartan theorem. (Received October 14, 1952.)

121. E. F. Beckenbach: *Some results in geometric function theory.*

Concerning functions $f(z)$ which are analytic in the unit circle $|z|=r<1$, certain inequalities are known to hold only for restricted values of r. By means of properties of convex functions, bounds valid for all $r<1$ are now obtained in some of these situations. Thus it is known that if on each concentric circle of radius $r<1$ the mean of $|f(z)|$ is $\leqq 1$, then the mean of $|f(z)|^2$ is $\leqq 1$ for $r^2 \leqq 1/2$; it is now shown that the latter mean is $\leqq 1/[2(1-r^2)]$ for $1/2 \leqq r^2 < 1$. (Received November 26, 1952.)

122*t*. Ben Bernholtz: *Analytic functions in locally convex spaces.*

The theory of analytic functions with arguments and values in a Banach space is generalized to the case of functions with arguments and values in complete locally convex complex linear topological spaces (l.t.s.) L and L' respectively. A function $f(x)$ on L to L' is analytic in the domain $D \supset L$ if it is single-valued, continuous, and G-differentiable in D. An analytic function is F-differentiable and can be expanded in an infinite series of F-differentials. This theory rests heavily on that for analytic functions $x(\zeta)$ of a complex variable ζ with values in a complete locally convex complex l.t.s. L. In the latter case, $x(\zeta)$ analytic means $x^*[x(\zeta)]$ is analytic in the classical sense for all x^* in L^*, the space of continuous linear functionals on L. Series expansions for these analytic functions are obtained and from them expansions for G-differentiable functions are deduced. (Received September 16, 1952.)

123. Rufus Isaacs: *Pursuit games.*

The two players each control certain aspects of the motion of two points; P, the pursuer, and E, the evader. Either motion may suffer restrictions as to velocity, acceleration, curvature, etc., and each player may enjoy full or partial knowledge of the other's position. A variety of games may be formulated with such pay-off's as time of capture. A strategy consists in specifying a player's navigation for every possible position. Some such games have been solved; others embody formidable difficulties. (Received October 15, 1952.)

124. R. B. Leipnik: *A generalisation of the Hahn-Banach theorem.*

Let S be an abelian group, S' a boundedly complete lattice-ordered abelian group. A pair (f, g) is called an outside gauge if $f(0)=0$, $f(x-y) \leqq f(x)-g(y)$, and an inside gauge if $g(0)=0$, $g(x-y) \geqq g(x)-f(y)$, where f and g are functions on S to S'. In case (f, g) is both an inside and an outside gauge, it is called a dual guage. If h satisfies $f(s) \geqq h(s) \geqq g(s)$ for each s in domain h, then h is said to be gauged by (f, g). Theorem: If a convex (concave) or linear h is gauged by an outside (inside) gauge (f, g), then h can be extended to a function H on domain $f \cap$ domain g which is convex (concave) or linear and is gauged by (f, g). If S and S' are real vector spaces, then sub (super)-homogeneity of k is retained in H. If (f, g) is taken to be a dual gauge, then this reduces to the Hahn-Banach Theorem. (Received October 16, 1952.)

125. S. F. Mack: *The discontinuity of the second normal derivative of the potential.* Preliminary report.

O. D. Kellogg (*Derivatives of harmonic functions*, Trans. Amer. Math. Soc. vol. 33 (1931) pp. 486–510) gave sufficient conditions with respect to smoothness of surface boundary and boundary values in order that derivatives of harmonic functions take on limiting values on the surface. The following statement applies to discontinuities across the surface boundary even when the derivatives do not exist at the boundary. Let the surface S in the neighborhood of a point q be of class C_3 with reference to the tangent plane at q, and s, $-s$ be equal distances along the normal on either side of the tangent plane; and let $u(p)$ be the potential of a density distribution σ which is continuous at q. Then on the normal at q, $\lim_{s\to 0}\{(d^2u/dn^2)_s-(d^2u/dn^2)_{-s}\}=-8\pi k\sigma$, where k is the mean curvature of S at q. The proof is obtained by referring S to a surface tangent to S at q, the normal sections of which are circles osculating the normal sections of S. For a closed surface conductor the above theorem is equivalent to $d^2u/dn_+^2=2k(q)du/dn_+$. (Received October 14, 1952.)

126. R. S. Phillips: *An inversion formula for Laplace transforms and semi-groups of linear operators.*

This paper is primarily concerned with the problem of determining necessary and sufficient conditions that a closed linear operator be the infinitesimal generator of a semi-group of bounded linear transformations. The principal tool is an inversion formula for Laplace transforms defined as follows: Suppose $\int_0^\infty \exp(-\omega\sigma)|f(\sigma)|d\sigma<\infty$. Set $F(\lambda)=\int_0^\infty \exp(-\lambda\sigma)f(\sigma)d\sigma$ and $f_\lambda(s)=\exp(-\lambda s)\sum(-1)^n(\lambda^2 s)^{n+1}[n!(n+1)!]^{-1}F^{(n)}(\lambda)$. Then $\lim_{\lambda\to\infty}f_\lambda(s)=f(s)$ a.e. Now let $T(s)$ be a semi-group of transformations such that $\int_0^1\|T(\sigma)\|d\sigma<\infty$. Then $T(s)$ is said to belong to class $(1, A)$ or $(1, C)$ according as it is strongly Abel or Cesaro summable to the identity at $s=0$. For $T(s)$ of class $(1, A)$, the infinitesimal generator A is not in general closed. However the smallest closed extension $\overline{A}$ (called c.i.g.) exists and the Laplace transform of $T(s)$ is the resolvent of $\overline{A}$. Necessary and sufficient conditions that a closed linear operator be the c.i.g. of a semi-group of class $(1, A)$ (or $(1, C)$) are found. The foregoing results are then applied to a perturbation theory for semi-groups. It is shown that if $\overline{A}$ is the c.i.g. of a semi-group of class $(1, A)$ (or $(1, C)$) and if B is a linear bounded operator, then $\overline{A}+B$ is likewise the c.i.g. of a semi-group of class $(1, A)$ (or $(1, C)$). (Received August 26, 1952.)

127. R. M. Redheffer: *An inequality for entire functions of finite type.*

Let $F(z)$ be an entire function such that $\limsup\,[\log|F(z)|]/|z|\leq 1$, as $|z|\to\infty$; and suppose $F(x)$ is real, $-\infty<x<\infty$. Then the inequality $|F(x)|^2\leq A$ implies $|F(x)|^2+|F'(x)|^2\leq A$, $|F'(x)|^2+|F''(x)|^2\leq A$, and so on. If any of these relations becomes an equality at a point where the higher of the two derivatives is not zero, then $F(z)=\pm A\cos(\pm z+B)$. (Received November 21, 1952.)

128*t*. R. M. Redheffer: *Completeness of* $\{e^{\lambda_n z}\}$.

Let $\{\lambda_n\}$ be a set of positive numbers, $\lambda_{n+1}\geq\lambda_n$, let $\Lambda(u)$ be the number of λ_n's $\leq u$, and assume $\Lambda(u)/u\leq M$. If $\limsup_{x\to\infty}(\Lambda[x+x(\log x)^{a}]-\Lambda(x))/x(\log x)^{a}\leq d$

for $a < -1$, then the set $\{e^{\pm i\lambda_n x}\}$ cannot be complete L^2 on an interval of length greater than $2\pi d$. This improves a previous result of the author, and verifies a conjecture of Beurling. When $\lambda_{n+1} - \lambda_n \geqq c$ and no further assumption is made, the set $\{e^{i\lambda_n x}\}$ cannot be complete on an interval of length $> 2\pi/c$. If a set $\{e^{i\lambda_n x}\}$ is complete on a closed interval of length d, and remains complete when N terms but not when $N+1$ terms are removed, Paley and Wiener call N the *excess* of the set. If $|\Lambda(u) - du| \leqq A$, then the excess does not exceed $[4A - 3/2]$; and there is a sequence with $|\Lambda(u) - du| \leqq A$ for which the excess is as large as $[4A - 2]$. These statements improve and complete a result of Paley and Wiener. (Received November 21, 1952.)

129*t*. R. M. Redheffer: *Entire functions with zeros having a density.*

Let $F(x) = \Pi(1 - x^2/\lambda_n^2)$, where $\lambda_{n+1} \geqq \lambda_n$ are positive numbers, let $\Lambda(u)$ represent the number of λ_n which are $\leqq u$, and assume $\Lambda(u)/u \leqq M$. If $\lim_{x\to\infty}\{\Lambda(vx)/vx - \Lambda(x/v)/x/v\} = 0$ whenever $0 < v < 1$, then $\lim_{x\to\infty} \log^+|F(x)|/x = 0$; and therefore, by the Phragmen-Lindelöf theorem, $|F(re^{i\theta})| = O(e^{r\pi d|\sin\theta| + \epsilon r})$, $\epsilon > 0$, $d \leqq M$. In particular the result holds whenever $\lim \Lambda(u)/u = d$ exists. This generalizes a theorem of Carlson, which gives the same conclusion under the hypothesis $\lambda_{n+1} - \lambda_n \geqq c > 0$, $\lim \Lambda(u)/u = d$. The converse is false; in fact if a and b are any numbers with $0 \leqq a \leqq b$, there exists a sequence $\{\lambda_n\}$ such that $\Lambda(u)/u \leqq M$, $\lim\sup\{\Lambda(vx)/vx - \Lambda(x/v)/(x/v)\} = b$, $\lim\inf = a$, for a fixed v less than a preassigned δ, yet such that $\log^+|F(x)|/x \to 0$. It is possible to have arbitrarily large gaps free of zeros; if $M > 0$, there is a sequence $\{\lambda_n\}$ such that $\Lambda(u)/u < M$, $\lambda_{n+1} - \lambda_n \geqq c$ for some $c > 0$, $\Lambda(x_i) = \Lambda(y_i)$ for an infinite sequence $x_i > y_i \to \infty$, $x_i/y_i \to \infty$, and yet $\log^+|F(x)|/x \to 0$. (Received November 21, 1952.)

130*t*. R. M. Redheffer: *Entire functions with zeros uniformly distributed.*

In the notation of the previous abstract, if $|\Lambda(u) - du| \leqq A$ for $A \geqq 1/2$, $d > 0$, then there is a constant C such that $|F(x)| < Cx^{4A-2}$. This improves a result of Paley and Wiener, who obtained the exponent $4A - 1$. Moreover, there exists a set $\{\lambda_n\}$ such that $\lambda_{n+1} - \lambda_n \geqq 1/d$, $|\Lambda(u) - du| \leqq A$, and $|F(x)| > C_1 x^{4A-2}$ for an infinite sequence $x = x_i \to \infty$. One can give estimates for the constants C in terms of λ_0, d, and A. If $a(u)$ is a positive function such that $a'(u)$ tends monotonically to 0 as $u \to \infty$, then the condition $|\Lambda(u) - du| < a(u)$ implies $\log^+|F(x)| < g(x)$, where $g(x) \sim [2a(x) - 1]\log[x^2(d/\lambda_0)a(x)] + 2a(x)$. Similar results allow comparison of two functions $F_1(x)$, $F_2(x)$; for example if $|\Lambda_1(u) - \Lambda_2(u)| \leqq A$, then $|F_1(x) - F_2(x)| \leqq Cx^{4A-2}$. Let K denote the class of entire functions which are $o(e^{|z|})$ as $|z| \to \infty$, and which belong to L^2 on $(-\infty, \infty)$. Let a be fixed and satisfy $0 < a < 2\pi$. Then the class of functions $G'(z)$, where G ranges over K, is identical with the class of functions $H(z+a) - H(z)$, where H ranges over K; but if $a \geqq 2\pi$, the latter class is a proper subset of the former. (Received November 21, 1952.)

131. R. E. Roberson: *The polynomial transform.*

The definition of a "polynomial transform" of a number sequence given by F. W. Bubb is generalized by the introduction of a multiplicative function $\phi(x)$ whose form can be chosen with some latitude. It is shown that such a transform exists if and only if the terms of the sequence can be bounded by a function which is transformable in the bilateral Laplace sense. Properties are given for the generalized form of the transform. Provided certain restrictions be placed on the choice of $\phi(x)$ (which are implicit in the particular choice of Bubb's definition), the polynomial transform is shown to be precisely the bilateral Laplace transform of a function constructed from the number sequence by means of an interpolating function $\Phi(t)=\mathfrak{L}_{\mathrm{II}}^{-1}\{\phi(e^{-s})\}$. (Received October 9, 1952.)

APPLIED MATHEMATICS

132*t*. G. W. Evans, II. *Applications of the Mauro Picone theorem for heat conduction.*

Use is made of a restricted statement of a theorem due to M. Picone (Math. Ann. vol. 101 (1929) pp. 701–712), which essentially asserts that a nonconstant solution of the heat conduction equation must have its maximum and minimum values along the boundary of the half-open xt-region of description, to prove the uniqueness of the solution $u(x, t)$ of the differential equation $\alpha u_{xx}=u_t-Q(t)$ for $0\leqq x<a$, and the differential equation $\alpha u_{xx}=u_t$ for $a<x\leqq L$. The initial condition is $u(x, 0)=f(x)$ and the boundary conditions are $u_x(0, t)=0$, $u_x(a+, t)=u_x(a-, t)=u_x(a, t)$, $u(a+, t)=u(a-, t)=u(a, t)$, and $ku_x(L, t)=h(t)[u(L, t)-g(t)]$, where $Q(t)$, $h(t)$, and $g(t)$ are known functions. The Picone theorem is also used to analyze the action of a thermal controlling unit, which consists of two thermocouples, for an oven. (Received September 22, 1952.)

133. P. G. Hodge: *On plastic strains in slabs with cutouts.*

The complete elastic-plastic problem is set up for a circular slab with a central circular cutout, subjected to uniform external tension. The analysis is carried out under the assumption of generalized plane stress but with possibly finite deformations. The material is assumed to be isotropic, homogeneous and incompressible, to yield according to the Tresca yield condition, and to satisfy Hooke's law in the elastic domain and the plastic potential law in the plastic domain. The equations are solved by a perturbation method based on the ratio of the maximum shear stress to the shear modulus, in which each of the significant quantities, stress, displacement, and slab thickness, is expanded in a power series in this ratio. A similar technique is employed to solve the fully plastic flow problem. It is shown that for cutout radii greater than one percent of the radius of the slab, the "first approximations" obtained by neglecting all but the leading term in each series are satisfactory, up to loads at which the slab becomes wholly plastic. The ratio of the maximum strain in the just fully plastic slab to that in the completely elastic slab is computed. It is found that this ratio is less than about 6 if the cutout radius is at least one percent of the radius of the slab. Some observations are advanced on the case where the cutout is very small compared to the slab. (The results presented in this paper were obtained in the course of research conducted under Contract N7onr-35810 between the Office of Naval Research and Brown University.) (Received September 2, 1952.)

134. Edgar Reich: *A random walk related to the capacitance of the circular-plate condenser.*

Let C be the capacitance of a circular parallel-plate condenser of radius a, and plate separation ta. It follows from the integral equation of E. R. Love that $C=a/\pi$ times the mean number of steps of the following random walk: The walk starts at a random point in $[-1/t, 1/t]$; each step is chosen from a Cauchy distribution with unit semi-interquartile range; the walk ends when a step crosses a boundary point of $[-1/t, 1/t]$. Since this discrete random walk can be looked upon as a continuous Cauchy "Brownian" motion (such as studied by M. Kac and H. Pollard), *observed* at discrete times, a simple interpretation for the value of "stray" capacitance is possible. (Received October 10, 1952.)

135. W. R. Wasow: *On the differential equation for the stability of plane Couette flow.*

The differential equation $u^{(4)}-2\alpha^2u''+\alpha^4u+\lambda^2z(u''-\alpha^2u)=0$, where α is a real constant, is solved asymptotically for large values of the complex parameter λ, when z ranges over a full complex neighborhood of the origin. The results are used to reduce the problem whether plane Couette flow is stable at large Reynolds numbers to the question whether all roots of the equation $\int_{-\infty}^{z}Ai(-t)dt=0$ lie in the sector $|\arg z|<\pi/6$. Here $Ai(z)$ denotes Airy's integral. It is also proved that small disturbances of Couette flow do not possess the inner viscous layer occurring for other flow profiles. This is shown to be a rather special property of the differential equation under consideration, not shared, e.g., by the otherwise very similar differential equation $u^{(4)}+\lambda^2x(u''+u)=0$. (Received September 12, 1952.)

GEOMETRY

136. J. W. Green: *Curves encircling a cylinder.*

Let C be a curve encircling a right cylinder of unit radius in such a manner that each element of the cylinder contains a single point of C. It is further required that the maximum distance between points of C not exceed 2. Under these conditions bounds are found for the length L and the height H of C, where by H is meant the minimum distance between two planes normal to the cylinder and containing C between them. These bounds are respectively: $2\pi\leq L\leq 2^{3/2}\pi$ and $0\leq H\leq 2^{1/2}$. The bounds are the best possible and are all attainable, with the possible exception of the upper bound for length. (Received October 15, 1952.)

137. N. D. Lane and Peter Scherk: *Differentiable points in conformal plane.*

The authors define the differentiability of a point on an arc in conformal plane by means of two conditions involving pencils of circles. An analysis of these conditions yields a classification of the differentiable points. (Received October 14, 1952.)

138*t*. T. K. Pan: *A proof of a sufficient condition that two surfaces be applicable.*

On page 176 of his book *Differential geometry* (Macmillan, 1947), W. C. Graustein stated that two surfaces are applicable if and only if they can be mapped geodesically so that the total curvatures in corresponding points are equal. That the condition is necessary is obvious. The proof of the sufficiency of the condition was divided into two cases. Graustein proved one case and omitted the other, saying that it can be shown though not without difficulty. This note offers a simple complete proof with the use of the tensor calculus. (Received October 14, 1952.)

139. T. K. Pan: *Variation of congruences of curves of an orthogonal ennuple in a Riemannian space.*

Consider any three congruences of an orthogonal ennuple at a point of a Riemannian space. When one congruence is moved by local and a second congruence is moved by parallel displacement in the direction of the third congruence, the rate of change of cosine of the angle between the first two congruences is well known as Ricci's coefficient of rotation and has been considerably studied. This paper investigates the corresponding rate of change when the third congruence is replaced by an arbitrary one. Variation of one congruence with respect to another is defined and in terms of which equidistance and parallelism of congruences are expressed. (Received October 14, 1952.)

LOGIC AND FOUNDATIONS

140. Ernest Adams: *A representation theorem for classical rigid body mechanics.*

A representation theorem is proved for systems of classical rigid body mechanics, as axiomatized jointly by Herman Rubin and the author. This theorem establishes that for every system of rigid body mechanics there exists an isomorphic system of pseudo-rigid bodies: where by a psuedo-rigid body is meant a sub-system of a system of classical particle mechanics (Bull. Amer. Math. Soc. Abstract 57-6-524 of McKinsey, Sugar, and Suppes) such that the distance between any two particles of the sub-system remains constant over time. (Received October 15, 1952.)

141. L. A. Henkin: *On certain subsystems of the second-order functional calculus.*

One formulation of the second-order calculus employs two axiom schemata of the following form: $(\alpha)A \supset B$. In schema (i), α is any individual variable and B is formed from the wff A by replacing all free occurrences of α by some other symbol of the same type. In (ii), α is any functional variable and B is obtained from A by the rather complicated rule of substitution for such variables. Following an old suggestion of Leśniewski, it is shown that if (i) is strengthened to (i′) by allowing "α" to range over variables of any type, then (ii) may be replaced by the simpler schema (iii): $(\exists\gamma)(\beta_1)\cdots(\beta_n)(\gamma(\beta_1, \cdots, \beta_n) \equiv A)$, where A is any wff not containing the n-ary function symbol γ. Let Σ be the formal system obtained from this formulation of the second-order calculus by omitting (iii). Then the formal theorems of Σ are precisely those wffs which are valid with respect to all models in which the range of n-ary functional variables may be an arbitrary set of n-ary relations of individuals. Interesting subsystems of second-order calculus are thus obtained by adding to Σ a form of (iii) in which "A" is restricted to some special class of wffs; in particular, if the class of formulas of first order calculus is chosen, the resulting system is closely connected to the Gödel set-theory. (Received October 16, 1952.)

STATISTICS AND PROBABILITY

142*t*. Aryeh Dvoretzky, Jack Kiefer, and Jacob Wolfowitz: *Sequential decision problems for processes with continuous time parameter. Estimation of parameters.*

The problem considered is that of sequential point and interval estimation (through continuous observation in time) for continuous time parameter stochastic processes. For example, for certain weight functions and a variety of cost functions, minimax sequential procedures are obtained for estimating the unknown parameter of the Wiener, Poisson, negative binomial, and gamma processes. The results for the last three processes contain as special cases new results on sequential estimation with discrete time parameter. (Received September 12, 1952.)

143t. Aryeh Dvoretzky, Jack Kiefer, and Jacob Wolfowitz: *Sequential decision problems for processes with continuous time parameter. Testing hypotheses.*

The problem considered is that of testing sequentially (through continuous observation in time) which of two laws represents the true distribution of a stochastic process with continuous time parameter. Results on Bayes' solutions, the optimum property of the sequential probability ratio test, Wald's equation, and fundamental identity, as well as other results from the discrete time case, carry over to the present case. In many examples, several of which are treated in detail, the *exact* operating characteristic function and the average sample number function can be found. The theory of general sequential decision problems may also be extended to the continuous time parameter case. (Received September 12, 1952.)

144t. Emanuel Parzen: *Invariance principle for certain uniform probability limit theorems.*

Erdös, Kac, and Donsker (Memoirs of the American Mathematical Society, no. 6, 1951) have established an "invariance principle" for certain probability limit theorems. The author has recently obtained a uniform continuity theorem, determining conditions in terms of characteristic functions that a sequence of distribution functions converge uniformly in a parameter, and has applied this theorem to determine conditions that a sequence of independent random variables X_n whose distribution functions $G_n(x, \theta)$ contain an unknown parameter θ satisfy the central limit theorem uniformly in θ, that the maximum likelihood estimate be uniformly asymptotically normally distributed, etc. These results are used to adapt Donsker's proof of the invariance principle to show that if the X_n satisfy the central limit theorem uniformly in θ then the distributions of various functions of the consecutive sums of the sequence also approach their limiting distributions uniformly in θ. (Received October 16, 1952.)

145t. Jacob Wolfowitz: *Consistent estimators of the parameters of a linear structural relation.*

Let $X=\xi+u$, $Y=\alpha+\beta\xi+v$, where ξ, u, v are chance variables such that u and v are jointly normally distributed with zero means and unknown covariance matrix, ξ is distributed independently of (u, v), and the distribution of ξ is not normal. If $F(x)$ and $G(x)$ are two-distribution functions, we define the distance $\delta(F, G)$ between them as $\sup_x |F(x)-G(x)|$. Let N^* be the class of all normal distributions with mean zero, and $\delta(F, N^*)=\inf_{N\in N^*} \delta(F, N)$. Let (x_i, y_i), $i=1, \cdots, n$, be a sequence of independent observations on (X, Y). Let $A(x|c_1, c_2)$ be the empiric distribution function of $\{y_i-c_1-c_2x_i\}$, $i=1, \cdots, n$. Let $a_n(x_1, \cdots, x_n, y_1, \cdots, y_n)$ and $b_n(x_1, \cdots, x_n, y_1, \cdots, y_n)$ be Borel measurable functions of the arguments exhibited such that $\delta(A(x|a_n, b_n), N^*)<1/n+\inf_{c_1,c_2} \delta(A(x|c_1, c_2), N^*)$. Then a_n and b_n converge stochastically to α and β, respectively. The method can be generalized to more than two variables, to non-normal variables, and to other problems in estimation and testing hypotheses. Other definitions of distance can be used. (Received March 24, 1952.)

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BOOK REVIEWS

Tensor analysis. Theory and applications. By I. S. Sokolnikoff. New York, Wiley, 1951. 10+335 pp. $6.00.

This volume is an outgrowth of a course of lectures given by the author over a period of years at the University of Wisconsin, Brown University, and the University of California. It consists of a general introduction to vector analysis and tensor calculus with special emphasis on the various applications. A short bibliography of some of the texts of the subject is given on p. 327 and an index is on pp. 329–335.

The book consists of six chapters. The first two deal with linear vector spaces, matrices, and the calculus of tensors. The last four chapters consist of applications to geometry, analytical mechanics, relativistic mechanics, and mechanics of continuous media. Throughout the book there are exercises to illustrate the already established concepts and formulas.

In the first chapter, the student is introduced to the concept of vector by intuitive and geometrical means. Thereafter, linear vector spaces over the fields of real and complex numbers, Euclidean spaces of n dimensions, linear transformations and matrices, reductions of quadratic forms, orthogonal and Hermitean matrices are discussed. In the second chapter, the usual concepts of the tensor calculus are introduced with a short section on relative tensors. After a short treatment of the Riemannian metric tensor and the Christoffel symbols, covariant differentiation is defined by writing the well known formulas. By means of Ricci's commutation rule for the covariant differentiation of tensors, the Riemann-Christoffel tensor is obtained. After treating the Ricci tensor and the Riemannian (including the Euclidean) spaces, this second chapter closes with a discussion of the generalized epsilons and Kronecker deltas together with applications to determinants.

The third chapter, on geometry, begins with a short exposition of the development of non-Euclidean geometries. The author defines the metric of a general Finsler space of which Riemannian geometry is a special case. The remainder of the chapter is devoted to the study of classical differential geometry of curves and surfaces in Euclidean three-dimensional space, as is done in such books as ***An introduction to differential geometry***, by L. P. Eisenhart, Princeton University Press, and *Applications of the absolute differential calculus*, by A. J.

McConnell, Blackie. After showing how the sphere and pseudosphere are physical representations of Riemannian elliptic geometry and Lobachevskean hyperbolic geometry, the author obtains the Serret-Frenet formulas, curvature, and torsion of a space curve. The differential equations of the geodesics of a Riemannian manifold are obtained by means of the Euler-Lagrange differential equations in the calculus of variations. In the theory of surfaces, it is shown how the Gaussian curvature is related to the Riemann-Christoffel tensor. Among the topics discussed on surface theory may be mentioned the first and second fundamental forms, the equations of Gauss, Mainardi-Codazzi, and Weingarten, Meusnier's theorem, principal curvatures, and the lines of curvature. This third chapter concludes with a short excursion into n-dimensional space.

In the fourth chapter, on analytical mechanics, the three general laws of motion of Newton are stated on p. 199. The motion of a particle in three-dimensional Euclidean space is studied by means of general curvilinear coordinates. Among the subjects covered are such topics as work, energy, the Lagrangian equations of motion, the energy equation in a conservative field of force, Hamilton's Principle of Least Action, the Hamiltonian equations, holonomic and nonholonomic systems, and virtual work. After showing how this applies to a constrained motion on a surface, the author extends the preceding concepts to general holonomic dynamical systems with n degrees of freedom. In the concluding part of this fourth chapter, there is a discussion of potential theory including the theorem of Gauss and the equation of Poisson. Elsewhere the notions concerning dynamical systems have been extended so as to include not only dynamical trajectories, but also brachistochrones, catenaries, velocity systems, and other families of curves in a given field of force of a Riemannian space of n dimensions. Also there is a development of potential theory in a Euclidean universe of n dimensions in other works.

The fifth chapter, on relativistic mechanics, is mainly concerned with the special theory of relativity. The two Einstein postulates of 1905 are stated on p. 265. From these, the Lorentz-Einstein transformation is deduced. After a discussion of the Minkowski velocity and acceleration, the energy equation and the gravitational equations of Einstein are obtained. Next it is shown how to solve Einstein's equations for the gravitational field produced by a spherically symmetric mass particle, which situation corresponds to the gravitational field of the sun. This is followed by a discussion of the planetary orbits in such a field. This chapter concludes with some remarks concerning the advance of the perihelion of Mercury.

The sixth and final chapter, on mechanics of continuous media, contains a general formulation of the basic concepts of mechanics of continua and a derivation of the fundamental equations governing the behavior of continuous media. The treatment of this chapter follows the ideas of F. D. Murnaghan. (See the paper, *Finite deformations of an elastic solid*, Amer. J. Math. vol. 59 (1937) pp. 235–260.) For the deformation of a continuum of identifiable material particles from its undeformed state into its deformed state, the Lagrangian and Eulerian strain tensors are obtained. After discussing the physical significance of the components of these tensors, the author finds the compatibility relations on the Eulerian strain tensor, which for the linearized case reduce to the conditions first obtained by B. Saint Venant in 1860. The deformation of a medium at any point is characterized by a certain quadratic form, which is represented geometrically as a quadric surface, called the strain quadric. From this is deduced the fact that the principal directions are those orthogonal directions in the undeformed state which remain orthogonal after deformation. Next it is shown that the state of stress at a point of a body, in equilibrium under prescribed surface and body forces, is characterized by a certain symmetric tensor, called the stress tensor. For this state of stress, the stress quadric of Cauchy is derived. The differential equations of a body in a state of equilibrium under the action of prescribed body and surface forces are obtained. It is proved that if the virtual displacement consists of translations and rotations, then Killing's equations hold, that is, the virtual work in an arbitrary rigid virtual displacement is zero. In the study of stress-strain relations, the author makes use of the first law of thermodynamics and deduces a law that includes the stress-strain relations of the classical theory of elasticity as a special case. This final chapter is concluded with the derivation of the equations of elasticity and also of fluid motion.

The material of this book is given in a straightforward and elementary fashion. The ideas and concepts are presented in such a manner that it is possible for a student with only a course in advanced calculus to have no difficulty in comprehending the subject matter. The last few chapters on the various applications would perhaps influence the more advanced student of tensor analysis to desire its possession. This book is a good introduction to the subject of tensor analysis, its usefulness, and its various applications.

JOHN DECICCO

The Lebesgue integral. By J. C. Burkill. Cambridge University Press, 1951. 8+87 pp. $2.50.

This tract is an exposition of classical Lebesgue theory on the real line, written for "those who have no wish to plumb the depths of the theory of real functions." The treatment throughout is fairly standard. The chapters are as follows: I. Sets of points; II. Measure (after de la Vallée Poussin); III. The Lebesgue integral (using ordinate sets); IV. Differentiation and integration (based on the Vitali covering theorem); V. Further properties of the integral (change of variable, the Fubini theorem, and the L^p-spaces); VI. The Lesbesgue-Stieltjes integral.

The limitations which the author imposes on the scope of his treatment are severe. For example, general measure theory is not even mentioned, which automatically excludes such timely topics as the Radon-Nikodým theorem and the theory of Cartesian product measures. Nor does the concept of set function occur, even for the real line; monotone functions, functions of bounded variation, and absolutely continuous functions are treated as point functions only. It follows that the book will be of little direct use to students interested in such fields of modern analysis as the general theory of Hilbert space, integration on locally compact groups and its consequences, or the fundamentals of probability theory, to name a few.

Of course, it can always be argued that the study of general measure and integration theory can with benefit be preceded by the study of a short treatment of the classical case, such as is presented here. And the present tract is a very clear and concise exposition of those topics with which it is concerned.

L. H. Loomis

Measure theory. By P. R. Halmos. New York, Van Nostrand, 1950. 11+304 pp. $5.90.

In this book Professor Halmos presents an account of the modern theory of measure and integration in the generality required for the study of measure in groups. Thus finiteness conditions are imposed only where necessary, and algebraic and topological aspects are appropriately stressed. Although written primarily for the student, the many novel ideas in the book and its store of interesting examples and counter examples have already made it an indispensable reference for the specialist. The clarity of expression and the sprightly style which are characteristic of the author make the exposition a pleasure to follow.

In the later chapters a knowledge of general topology and topological groups is presupposed, but the first part is intended to assume only undergraduate algebra and analysis. However, some previous acquaintance with the Lebesgue integral in the real domain is almost indispensable for an appreciation of the more abstract theory set forth here. The book does not aim to be exhaustive, but many alternative developments and points of contact with other mathematical theories are indicated especially in the exercises at the end of each section, which constitute an extensive and important part of the book. Some of these are examples, corollaries, and variant formulations intended to sharpen the student's understanding, while others are really supplementary sections of the text, in condensed form. Many are interrelated, for instance those concerned with measure in semirings. With the aid of brief but well chosen hints a large amount of additional material is thus brought within the scope of the book. An adequately prepared reader who takes the trouble to work through many of these exercises can hardly fail to acquire a feeling for the subject and find it exciting.

The first half of the book contains the standard theory, but in a somewhat more general formulation than is usual. After discussing extension theorems and various types of convergence the integral is defined by a completion process, starting with simple functions. The integral of a general function f is defined as the limit of the integrals of a mean fundamental sequence of simple functions that converges in measure to f. This definition is technically convenient, and has the advantage of achieving full generality in one step, without the need for successive extension to unbounded functions, domains of infinite measure, or functions not of constant sign. A chapter on additive set functions, their decompositions and Radon-Nikodým derivatives, and one on product measure, giving a very simple treatment of Fubini's theorem, complete this part of the book. A chapter is then devoted to an introductory treatment of measurable transformations, function spaces, and isomorphisms of measure algebras, but there is no attempt to develop ergodic theory proper. A chapter on probability sets forth the basic theorems and definitions in measure-theoretic terminology, and includes an heuristic introduction along the lines of the author's well-known expository article on the foundations of probability [Amer. Math. Monthly vol. 51 (1944) pp. 493–510].

The last three chapters, about a quarter of the book, contain an elegant account of the theory of measure in locally compact Hausdorff spaces, Haar measure in groups, and Weil's duality theorem. For the most part the development here follows Kodaira and Kakutani and culminates in a proof of their theorem on the completion regularity of Haar measure. Much of this material appears here for the first time in book form. These chapters constitute the focal point of the whole development, a fact which explains to some extent the arrangement and choice of material in the earlier chapters. For instance, properties of Carathéodory outer measure, and most results pertaining specifically to measure in metric spaces, appear only among the exercises. For the same reason, many important results that belong properly to the theory of real functions are omitted altogether. There is no discussion of integration as inverse to differentiation, and the density theorem appears only in a generalized form applicable to Haar measure. The surprising thing, however, is that so much material is included without undue condensation. It seems likely that this book will come to be recognized as one of the few really good text books at its level. It can hardly fail to exert a stimulating influence on the development of measure theory.

J. C. OXTOBY

A decision method for elementary algebra and geometry. By Alfred Tarski. Prepared for publication with the assistance of J. C. C. McKinsey. Berkeley and Los Angeles, University of California Press, 1951. 3+63 pp. $2.75.

The results of this monograph were obtained by the author in 1930. The material in its full development was published privately (and hence not reviewed in the Bulletin) by the Rand Corporation in 1948. The present edition is simply a reprint of that edition with corrections and some supplementary notes.

By elementary algebra Tarski means that part of the theory of real numbers which can be expressed in the formal language described as follows: (1) Variables of this language stand for real numbers. (2) There are three constants, "0", "1", and "-1". (3) A term is defined (recursively) as a constant or variable, or else of the form $\alpha+\beta$ or $\alpha\cdot\beta$ where α and β are any terms. (4) An atomic formula is either of the form $\alpha=\beta$ or of the form $\alpha>\beta$ where α and β are terms. (5) A formula is either an atomic formula or made up from atomic formulas by means of truth functions and quantifiers in the usual manner of the first order function calculus. In this language we can discuss any integer and any polynomial we choose; but, although we can talk of all real numbers having a certain property, we cannot talk of *all* integers having a property or of *all* polynomials having a property, since we have no variables to stand for these. In this monograph Tarski presents a decision procedure for elementary algebra, based on Sturm's theorem.

Sturm's theorem itself can be regarded as a decision procedure to determine the truth or falsity of any sentence of the form, "The polynomial p has exactly k roots." Tarski generalizes Sturm's theorem to provide (as a lemma for his main result) a decision procedure for any statement of the following form: "There are exactly k real numbers satisfying σ," where σ is a given system of a finite number of equations in ξ of the form $\alpha_{i,m_i}\xi^{m_i}+ \cdots +\alpha_{i,0}=0$ and inequalities in ξ of the form $\beta_{j,n_j}\xi^{n_j}+ \cdots +\beta_{j,0}>0$. (Cf. Definition 28 and Theorem 29, pp. 31–34.) Then, using this result, Tarski shows that there is a constructive method for determining from any given formula Φ, a formula $U(\Phi)$ which is equivalent to Φ, has no quantifiers, and contains no variables except those appearing free in Φ (Theorem 31, p. 39). From this last theorem the main result follows immediately. For only statements without free variables are true or false, and, for a statement Φ without free variables, $U(\Phi)$ is a truth function of equations and inequalities without variables. It is easy to determine the truth or falsity of such equations and inequalities, whence the truth or falsity of $U(\Phi)$ can be determined by the familiar truth table method.

A formula without free variables is true if it is a true statement in the intuitive sense, concerning the real numbers. Tarski's result then is that there is a decision procedure which determines, for the formal language of elementary algebra, the truth or falsity (in this sense) of every formula with no free variables. Alternatively, this theory could be presented as an axiomatic system; a formula is true in a second sense if it is a theorem of the system. Tarski suggests (in note 9, pp. 48–50) possible axiom systems. The same decision method holds for truth in this second sense, so that the two senses of "truth" are equivalent. It follows from this that such axiom systems are complete and consistent (note 15, pp. 53, 54). Moreover, since these axioms are all satisfied by every real closed field, it follows that any formula of elementary algebra true in one real closed field (as defined e.g. in van der Waerden, Section 70, Chapter IX) is true in every real closed field. Finally, this is still true if we extend elementary algebra by introducing special variables for integers allowing quantification over them. (See supplementary note 7, pp. 62, 63 where the further significance for the theory of real closed fields is explained.)

It may seem paradoxical that the elementary theory of real numbers is decidable (has a decision procedure) while the seemingly simpler elementary theory of natural numbers (i.e. the non-negative integers) is undecidable and in fact incompletable (i.e. has no axiomatic system yielding all and only truths as theorems), as proved by the famous Gödel theorem. The reason for this is probably the fact that the language of elementary natural number theory more fully represents the complexity involved in that theory than the language of elementary algebra represents the complexity of the theory of real numbers. As Tarski points out (in note 13, p. 53), it is impossible to define the notion of a rational number or integer or natural number in elementary algebra; apparently this failure deprives elementary algebra of much of the complexity we ordinarily associate with the theory of real numbers. (Indeed, if the predicate "$NN(x)$" were added to elementary algebra meaning that x is a natural number, then there could no longer be a decision procedure for this enriched language, since the Gödel undecidable statement could be constructed in it.)

In section 3, Tarski gives a language for n-dimensional elementary geometry for a positive integer n, and shows how formulas of this language can be translated into statements of analytic geometry, thereby becoming statements of algebra. Hence elementary geometry is decidable. Certain geometric statements, however, are not regarded as statements of elementary geometry, e.g. statements involving the phrase "all polygons" or the notion of constructibility by means of an arbitrary number of operations of ruler and compass (p. 3).

The Introduction and the many notes are rich with informative and suggestive comments. The Introduction contains a well written, nontechnical summary of the significance of the main result. The proof of the main result should be easy to follow for anyone familiar with symbolic logic, although the reader who has never before seen a proof of Sturm's theorem may wish for a more intuitive account of $G_{\xi}^{*}(\alpha,\beta)$ and its rôle. There are a few errata and comments on small points. On page 10 the blank space after line 26 should read "$((\Theta\rightarrow\Phi)\wedge(\Phi\rightarrow\Theta))$." On page 23, line 18, the statement that "division . . . is not available in our system" may be misleading; certainly "x/y" is definable as "the real number z such that $y\cdot z=x$," to which Russell's technique for eliminating descriptions can be applied. On page 23, line 28, insert "and $m\geqq n$" after "β_{n} of β is 0." (If $m<n$ and $\beta_{n}=0$ there may be a different kind of undesirable consequence for the notion of a negative remainder.) On page 27, line 13, the word "computed" is badly chosen, since the process indicated in the remainder of the paragraph is not effective.

ROBERT MCNAUGHTON

Lezioni sulla teoria moderna dell'integrazione. By M. Picone and T. Viola. Edizioni Scientifiche Einaudi, 1952. 404 pp. 5000 lire.

The word "modern" in the title is to be taken with a grain of salt; the book could just as well have been written fifteen or twenty years ago. The notation and the point of view are old-fashioned; the exposition is verbose and occasionally vague. Integration, for instance, is defined in terms of limits on directed sets, but the definition of directed set is couched in the language of "variables" taking "well determined" numerical values. The bibliography contains no reference to Radon or to Nikodým. As a matter of fact, except for a half dozen references to relatively recent papers by Italian mathematicians, no reference is made to any part of the mathematical literature of the last fifteen years.

The material itself is essentially standard: it includes interval functions, Riemann-Stieltjes and Lebesgue-Stieltjes integration, Egoroff's theorem, theorems on term-by-term integration, the Riesz-Fischer theorem, Fubini's theorem, bounded variation in the sense of Vitali, integration by parts, absolute continuity, differentiation, and change of variables. All the sets considered are subsets of finite-dimensional Euclidean spaces. Since the authors took great care to make the proof rigorous, the book might be useful for a beginning student who is anxious to have all the epsilons spelled out for him.

PAUL R. HALMOS

Numerische Behandlung von Differentialgleichungen. By L. Collatz. (Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen mit besonderer Berücksichtigung der Anwendungsgebiete, vol. 40.) Berlin, Göttingen, Heidelberg, Springer, 1951. 13+458 pp., 1 plate. 45 DM; Bound, 48 DM.

Few are the books devoted exclusively to the numerical treatment of differential equations. One may recall Levy and Baggott in English, Panov in Russian, von Sanden in German and perhaps a few others—say Runge and Willers' Encyclopedia article. These are brief, limited in scope, handbooks rather than treatises. Here now is a more ambitious undertaking, namely to make a comprehensive study of numerical methods for both ordinary and partial differential equations. Of course such an ambition cannot be fully realized in a book of reasonable size, and inevitably some topics are touched only lightly. For example one notes that scant attention is given to those methods for ordinary equations which use open type quadrature formulas to predict and corresponding closed type formulas to correct, and where the correction provides a measure of the truncation error. For partial differential equations even greater limitations are encountered, so that instead of giving a general theory the author is perforce content to exhibit typical methods illustrated by selected practical problems in physics and engineering.

Nevertheless this text contains the most thorough and comprehensive presentation of numerical methods for differential equations that has yet been published. The term "numerical methods" is liberally interpreted, and far from being content with a mere description of procedures the author gives a detailed theoretical analysis of each process described. The crucial question of error control has received the careful attention that its importance merits.

A quick glance through the 458 pages discovers the major headings: (I) initial-value problems for ordinary equations; (II) boundary-value (i.e., two end-point) problems for ordinary equations; (III) initial-value problems and initial plus boundary-value problems for partial equations; (IV) boundary-value problems for partial equations; and (V) integral and functional equations.

A second and longer look reveals the wealth of material covered. Under I we find first the necessary background material of difference formulas and quadrature formulas; then come simple methods of summation, error estimates, the Runge-Kutta methods, methods using finite differences, Taylor's series, quadrature formulas, the "extrapolation" process, the "interpolation" process, and methods of iteration. Each of these is applied to both first and higher order equations and the error for each is examined. Under II appear linear boundary value problems, use of finite differences, relaxation, iteration, the minimum principle, the Ritz method, least square methods, Galerkin's method, step-by-step methods, and methods for calculating eigenvalues.

Chapter II takes up those partial differential equations, principally of parabolic or hyperbolic type, where the initial values are known at $t=0$, where the space boundary values are given, and for which the solution is computed step-by-step for successive values of t. The method of difference equations is applied to the values of the unknown at lattice points of a rectangular net and the dangers due to incorrect formulation of the difference equation are vividly exhibited. Further topics include choice of the mesh size, propagation of errors, graphical constructions for solving the difference equations, and the use of characteristics.

In IV the author takes up those partial differential equations, mainly of elliptic type, where the solution is not obtainable by a step-by-step process with respect to one variable, but is found as the solution of a set of simultaneous equations. The method of difference equations for a square lattice is explained and applied to such examples as the Dirichlet problem. Further topics include proof of existence and uniqueness of a solution, estimate of error, convergence of the iteration process for solving the equations, relaxation, triangular and hexagonal nets, method of least squares, the Ritz method, Trefftz' method, use of series of known functions, eigenfunctions, and eigenvalues.

Chapter V deals mainly with integral equations, and explains methods where the integrals are replaced by sums, methods of iteration, use of series, cases of singular integral equations, Volterra's equation, and closes with some examples of functional equations.

The text is enlivened by interesting numerical examples, some short biographical notes, and appropriately has as its frontispiece a photograph of C. Runge.

W. E. MILNE

Linear algebra and matrix theory. By R. R. Stoll. New York, McGraw-Hill, 1952. 15+272 pages. $6.00.

In this treatment of the algebra of matrices with elements in a field, the author adopts the modern viewpoint that the linear transformations are the basic objects for study, rather than their carriers, the matrices. Abstract algebraic concepts are introduced as they arise. (It should be observed that this "background of modern algebra" consists of really elementary concepts, and does not include, for example, such a powerful tool as the Krull-Schmidt theorem.) A number of the usual proofs have been replaced by fresh ones. For example, the reduction of symmetric matrices is done so carefully that Kronecker's reduction is a very easy consequence. The important similarity problem is solved by adapting the arguments developed for division rings to the present case. (Cf., for example, N. Jacobson, *The theory of rings*, Chap. 3.) The discussion of normal transformations in (finite-dimensional) unitary and Euclidean spaces is more thorough than usual.

The author has naturally left to the reader the duty of supplying some of the details. Besides these "natural" exercises, there are ample lists of more formal exercises and many illustrative examples. There is an index and a useful list of the special symbols employed.

This text is heartily recommended to the teacher of matric theory who feels obliged to introduce his students at the same time to basic concepts of modern algebra.

M. F. SMILEY

A university algebra. By D. E. Littlewood. London, Heineman, 1950. 8+292 pp. 25s.

Something less than the first half of this book is concerned with the same kind of subject matter as in the text by Stoll reviewed above. However, Littlewood's matrix theory goes beyond Stoll's (and beyond most standard treatments) by including a great deal of material on the theory of determinants, which in turn motivates a smattering of permutation group theory and a large amount of detail on symmetric and alternating polynomials (all this presumably with an eye to applications to group representations later in the book). The next third of the book contains a more or less standard account of the theory of equations, some elementary theory of numbers, a little field theory in connection with the subject of polynomials, and a short chapter on the Galois theory of equations (by resolvents). The remainder of the book is devoted to the more advanced topics of invariants and group representations, the latter attacked via the Wedderburn theory of algebras, ending with a discussion of the irreducible characters and representations of the symmetric, general linear, and orthogonal groups.

The text is pitched at the advanced undergraduate or beginning graduate level. The calculational sections and the examples are worked out in detail, and the book is well supplied with exercises. On the other hand, the theoretical sections of the text tend to be condensed or slurred over, making them quite obscure at times, and of dubious accessibility to the uninitiated.

The fundamental approach in this volume is almost the opposite of Stoll's. In matrix theory, for example, the center of the stage is occupied by n-tuples and matrices, with their attendant canonical bases, rather than by abstract vector spaces and linear mappings thereon. This lack of what one might call a geometrical approach results, of course, in a computational matrix theory, in contrast with the more conceptual linear transformation theory where no basis is a preferred one. However, algebra and geometry, which had been divorced by decree in the very first section, are reconciled at last in one of the late chapters, on the theory of invariants, which theory "eliminates all the heavy and irrelevant work which the superfluous frame of reference carries with it, and restores the simplicity and elegance which are chacteristics of pure geometry."

Like Stoll, Littlewood introduces the concepts of group, ring, integral domain, and field in short parenthetical sections when the concepts arise naturally. However, this leaning toward "abstract algebra" is perfunctory. For example no definitions are given for "vector space" (although "algebra" is defined) or "quotient group" (although the phrase and the concept are ultimately used, without warning); definitions of abstract concepts that are given are almost never linked to one another (e.g., the underlying additive group of a ring or field is not mentioned); in fact, the only groups that are treated as such are groups of permutations or matrices.

Lastly, we should mention a third contrast between the present text and the one above. A fatal carelessness pervades the whole book, even when abstract concepts are only in the background. To give two examples, the reader is hard put to it to discover in any particular section whether "number" means real number or complex number (see especially the section on orthogonal matrices); linear dependence of a set of vectors is defined incorrectly (the phrase "scalars not all zero" is omitted) and so is linear dependence of one vector on a given set of vectors (the phrase "scalars not all zero" is included). This kind of carelessness finally leads the author into at least two real errors: (1) A gap in the proof of Wedderburn's theorems, where the author uses without proof the existence of a unit element in a semisimple algebra, the "justification" presumably being this blanket statement a few sections back: "If an algebra does not possess a modulus [unit element] then a modulus may be *adjoined*. . . . It will be assumed henceforth that if necessary a modulus has been adjoined." (2) Theorem II, Chapter X, p. 156 is false. It reads: "If $F(\xi_1, \xi_2)$ is the extension field obtained by adjoining to F two roots ξ_1, ξ_2 of an irreducible equation in F, then the correspondence $\xi_1 \rightarrow \xi_2$, $\xi_2 \rightarrow \xi_1$ constitutes an automorphism of the extension field [over F]." A counter-example is provided by any cyclic extension field of degree >2.

DANIEL ZELINSKY

Linear transformations in n-dimensional vector space. An introduction to the theory of Hilbert space. By H. L. Hamburger and M. E. Grimshaw. Cambridge University Press, 1951. 10+195 pp. $4.50.

Since the purpose of this book is clearly indicated by its title and subtitle, this review can consider first the order (quite reasonable) and clarity (good) with which the authors present their material.

The vector space V_n considered is always the space of sequences of n complex numbers. The scalar product, linear dependence, linear subspaces, orthogonal complements, and the algebra of linear transformations are discussed in Chapter I. Chapter II describes the elementary properties of Hermitian, normal, unitary, and projection operators and their eigenmanifolds. Chapter III derives the spectral representation of a Hermitean H, locating the successive eigenvalues as maxima of (Hx,x) on sections of the unit sphere; this argument has a Hilbert space analogue only for completely continuous operators. After some study of resolvents and other functions of Hermitian operators, the chapter closes with extension of the preceding results to unitary and normal operators.

Chapter IV discusses principal manifolds, the minimal polynomial, and elementary divisors, gives the Jordan canonical form and Segre characteristic for a general linear transformation, and closes with a study of commutativity. Chapter V discusses the effect of introducing a new scalar product (Gx,y), where G is positive definite. This study is needed to make up for the original concrete choice of V_n; it is then possible to characterize the transformations with simple elementary divisors as those which are normal relative to some such scalar product.

Interesting historical notes, some referring to work of a century ago, show the authors' knowledge of the deep roots of their subject in the structure of classical mathematics. That their presentation is faithful to that same classical tradition may make the book easier for a student to begin, but it seems to this reviewer to make the secondary goal, the introduction of the reader to Hilbert space, so much the more difficult to reach in a small book.

This reviewer can (as the pre-publication reviewer for Mathematical Reviews could not), and therefore should, attempt to compare this book with ***Finite dimensional vector spaces*** by P. R. Halmos. The books overlap much more in subject matter than in attitude; Halmos acknowledges great indebtedness to von Neumann, whose name does not appear in bibliography or index of the book under review. A student unaccustomed, as so many of our undergraduates are, to axiomatic methods might profit more from this concrete and detailed study than from a surfeit of abstractions. On the other hand, Halmos's book, with its racy style and its steady slant toward Hilbert space, when contrasted with this formal, workmanlike, and detailed discussion, seems to offer one of the few examples of a paper-bound book suited better than its slick-paper competitor to the education of any student who has been bent to the appropriate axiomatic attitude.

MAHLON M. DAY

BRIEF MENTION

Mecanique des milieux continus et deformables. By M. Roy. Paris, Gauthier-Villars, 1950. Vol. 1, 22+198+166 pp.; vol. 2, 212+126 +12 pp.

The two volumes represent an introduction to the theory of elasticity and fluid dynamics for the author's students at the École Polytechnique. The common starting point for the treatment of solids and fluids includes the kinematics and thermodynamics of continuous media. Fluid dynamics is developed appreciably more fully than elasticity. In the latter field attention is largely restricted to the linear theory and aside from general developments some consideration is given to problems of plane stress and St. Venant torsion and to the theory of the elastica. In fluid dynamics one finds the customary material on incompressible potential and vortex flow of frictionless fluids. In compressible flow theory considerable space is devoted to shock waves and their applications. In viscous flow theory consideration is given to boundary layer theory and turbulence. A concluding chapter deals with applications of hydrodynamics to hydraulic machinery. The mathematical level of the book is such that the reader must be well grounded in advanced calculus.

E. REISSNER

The theory of lattices. By B. C. Rennie. Cambridge, England, Foister and Jagg, 1951. 51 pp. Paper cover, $1.00; stiff cover, $1.50.

This little pamphlet which consists essentially of the author's Ph.D. dissertation could more properly have been entitled *Selected topics in the theory of topologies on lattices.* It carries on a program initiated by Birkhoff and Frink of investigating the relations which exist between the various topologies which can be defined on lattices. Some applications to special cases are also given.

The results presented in this work once again point up the fact that, except for special cases, none of the topologies which have been defined on lattices has yet proved useful in the study of lattice structure.

R. P. DILWORTH

Mathematische Maschinen und Instrumente. By F. A. Willers. Berlin, Akademie, 1952. 12+318+2 pp. $8.16.

This book describes a fascinating collection of machines and instruments with appropriate emphasis on the modern developments in the various types. These include slide rules, desk machines, automatic sequence calculators, differential analyzers, harmonic analyzers, planimeters, including the Stieltjes type, and instruments for drawing curves and curve measurements. The desk machines described are the continental types, Brunsviga, Walther, Facit, Rheinmetal, Archimedes, Millionaire, Haman, and a new small device, the Curta. The range of instruments involving graphical methods is quite remarkable and most of these seem to be current. The discussion of large scale digital computers gives a list of machines, including ENIAC, EDSAC, UNIVAC, SWAC, and SEAC. Various possibilities for the memory of such devices are described and the logical diagrams for serial addition, subtraction, and multiplication in these machines are given. There are brief descriptions of programming, checking, and procedures on these machines. The differential analyzers described include the machines built by Rosseland, Walther-Ott, Sauer-Pösch, and the latest Bush machine. The photographs and drawings are clear and the explanations easy to follow. The printing is excellent. There is a bibliography of 871 references at the end.

F. J. MURRAY

NOTES

The editors of the Bulletin and the Proceedings wish to make grateful acknowledgment of the services rendered by the following persons who have refereed papers: A. A. Albert, T. M. Apostol, Richard Arens, Nachman Aronszajn, R. G. Ayoub, Reinhold Baer, E. W. Barankin, E. G. Begle, R. E. Bellman, Stefan Bergman, Lipman Bers, R. H. Bing, L. M. Blumenthal, Salomon Bochner, T. A. Botts, Fred Brafman, A. T. Brauer, Richard Brauer, R. H. Breusch, J. P. Brewster, R. H. Bruck, R. C. Buck, Herbert Busemann, R. H. Cameron, Lamberto Cesari, S. S. Chern, Herman Chernoff, C. C. Chevalley, Sarvadaman Chowla, K. L. Chung, R. V. Churchill, A. B. Coble, E. A. Coddington, Eckford Cohen, I. S. Cohen, L. W. Cohen, A. H. Copeland, V. F. Cowling, H. S. M. Coxeter, A. T. Craig, Philip Davis, M. M. Day, J. A. Dieudonné, R. P. Dilworth, J. L. Doob, Arnold Dresden, F. G. Dressel, G. E. D. Duff, R. J. Duffin, Nelson Dunford, Ben Dushnik, Aryeh Dvoretzky, W. F. Eberlein, Samuel Eilenberg, Arthur Erdélyi, Paul Erdös, Herbert Federer, William Feller, Werner Fenchel, Harley Flanders, E. E. Floyd, R. H. Fox, Evelyn Frank, Philip Franklin, Bernard Friedman, W. H. Fuchs, P. R. Garabedian, J. J. Gergen, A. M. Gleason, Oscar Goldman, Michael Golomb, W. H. Gottschalk, L. M. Graves, L. W. Green, William Gustin, Franklin Haimo, Marshall Hall, Jr., P. R. Halmos, Philip Hartman, A. E. Heins, M. H. Heins, Henry Helson, J. G. Herriott, M. R. Hestenes, T. H. Hildebrandt, J. D. Hill, Einar Hille, I. I. Hirschman, Jr., Vaclav Hlavatý, T. R. Hollcroft, Eberhard Hopf, Alfred Horn, G. A. Hunt, Eugene Isaacson, V. G. Iyer, J. A. Jenkins, R. E. Johnson, F. B. Jones, Mark Kac, Shizuo Kakutani, L. H. Kantor, Irving Kaplansky, P. W. Ketchum, Fred Kiokemeister, V. L. Klee, S. C. Kleene, Kunihiko Kodiara, B. O. Koopman, Harold W. Kuhn, Cornelius Lanczos, B. Kuttner, R. E. Lane, Marguerite Lehr, Norman Levinson, B. W. Lindgren, W. G. Lister, Charles Loewner, A. J. Lohwater, L. H. Loomis, E. R. Lorch, E. R. Love, N. H. McCoy, E. J. McShane, A. M. MacBeath, G. R. MacLane, Wilhelm Magnus, Szolem Mandelbrojt, Morris Marden, G. Maruyama, K. S. Miller, W. H. Mills, E. E. Moise, Deane Montgomery, A. P. Morse, F. J. Murray, S. B. Myers, Zeev Nehari, H. W. Oliver, Oystein Ore, O. G. Owens, Oscar Perron, R. S. Phillips, Edmund Pinney, George Piranian, Harry Pollard, George Pólya, Tibor Radó, E. D. Rainville, J. F. Randolph, R. M. Redheffer, P. V. Reichelderfer, W. T. Reid, Eric Reissner, C. E. Rickart, M. S. Robertson, Robin Robinson, P. C. Rosenbloom, M. A.

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The editors of the Transactions wish to acknowledge the services of the following persons, not members of the Editorial Committee, who have been consulted regarding papers submitted for publication: R. P. Agnew, A. A. Albert, Richard Arens, Nachman Aronszajn, Frederick Bagemihl, Valentine Bargmann, Lipman Bers, R. H. Bing, R. P. Boas, R. H. Breusch, F. E. Browder, K. L. Chung, Alonzo Church, Harold Davenport, M.D. Davis, M. M. Day, J. A. Dieudonné, E. E. Floyd, M. K. Fort, Jr., Hilda Geiringer, P. R. Halmos, L. A. Henkin, M. R. Hestenes, I. I. Hirschmann, Jr., T. H. Hildebrandt, Einar Hille, J. A. Jenkins, Bjarni Jónsson, Shizuo Kakutani, Wilfred Kaplan, P. W. Ketchum, V. L. Klee, E. R. Kolchin, J. L. Koszul, Philip LeCorbeiller, D. H. Lehmer, L. H. Loomis, R. C. Lyndon, H. B. Mann, Morris Marden, D. C. Murdock, F. J. Murray, S. B. Myers, David Nelson, Oystein Ore, Sam Perlis, R. S. Phillips, Hans Rademacher, W. T. Reid, C. E. Rickart, H. L. Royden, Arthur Sard, A. C. Schaeffer, O. F. G. Schilling, Lowell Schoenfeld, I. E. Segal, Abraham Seidenberg, Max Shiffman, J. L. Snell, George Springer, H. S. Vandiver, G. S. Young.

The editors of the Bulletin plan to supplement the Book Review section by reviews of new mathematical journals. These will be brief descriptive and critical notices of a journal as a whole (detailed analyses of the contents being the function of Mathematical Reviews). Editors and publishers of new journals are invited to send review copies to the office of the Society.

The School of Mathematics of the Institute for Advanced Study will allocate a small number of grants-in-aid to gifted young mathematicians and mathematical physicists to enable them to study and to do research work at Princeton during the academic year 1953–1954. Candidates must have given evidence of ability in research comparable at least with that expected for the degree of Doctor of Philosophy. Blanks for application may be obtained from the School of Mathematics. Institute for Advanced Study, Princeton, New Jersey.

In honor of the memory of Guido Fubini, the Unione Matematica Italiana announces a Guido Fubini international prize in mathematics. The prize, which will not be divided, consists of a sum in Italian lire equivalent to approximately 550 grams of gold (equivalent to approximately 557 United States Gold dollars). The award will be based on contributions to the advancement of differential geometry published between January 1, 1946 and December 31, 1953.

The award will be made by a Committee consisting of Professors Salomon Bochner of Princeton University, Charles Ehresmann of the University of Strasbourg, and Alessandro Terracini of the University of Turin. The decision of the Committee shall be final. In case circumstances prevent one member of the Committee from serving, the remaining two members shall appoint a successor.

Contestants can submit their own work to the Ufficio di Presidenza, Unione Matematica Italiana, Istituto Matematico, Università di Bologna, Bologna, Italy, until the thirty-first of March, 1954. In addition to the works submitted, the Committee can take into consideration works on differential geometry published within the aforementioned period by mathematicians who have not declared themselves to be contestants for the prize.

In case the Committee should find itself unable to award the prize to a specialist in differential geometry, it can grant the award, on a similar basis, to an outstanding specialist in the theory of automorphic functions or in related theories.

The Electrical Engineering Department of Michigan State College, in cooperation with the National Science Foundation, the National Bureau of Standards, the Instrument Society of America, and the American Society for Engineering Education, announces an invitational National Collegiate-Industry-Government Conference on Instrumentation to be held on its campus, March 19–20. Additional information may be obtained from Professor R. J. Jeffries, Department of Electrical Engineering, Michigan State College, East Lansing, Michigan.

The third Midwestern Conference on Fluid Mechanics will be held at the University of Minnesota on March 23–25. The subject matter will be all features of theoretical and applied fluid mechanics. The program chairman for the conference is Professor E. Silberman, St. Anthony Falls Hydraulics Laboratory, Minneapolis 14, Minnesota.

Professor P. O. Bell of the University of Kansas has been awarded a National Science Foundation Postdoctoral Fellowship for 1952–1953 and is spending this year at the University of California, Berkeley.

Professor Josephine Mitchell of the University of Illinois has been awarded a National Science Foundation grant.

Professor Everett Pitcher of Lehigh University has been awarded a Guggenheim Fellowship for 1952–1953. He has been appointed to a visiting professorship at the University of Oxford for the first term, and will spend the second term at Princeton University.

Associate Professor A. L. Blakers of Lehigh University has been appointed to a professorship at the University of Western Australia.

Dr. C. Y. Pauc of the University of Cape Town has been appointed a maître de conférences at the École Nationale Supérieure de Mécanique, Nantes, France.

Dean Samuel Beatty of the University of Toronto has retired with the title dean emeritus.

Professor Salomon Bochner of Princeton University is on leave of absence and has been appointed to a visiting professorship at the University of California, Berkeley.

Professor Thomas Buck of the University of California, Berkeley, has retired.

Mr. T. Y. Chow of Cornell University has been appointed to an assistant professorship at Rensselaer Polytechnic Institute.

Dr. G. F. Cramer of the Radio Corporation of America has accepted a position as staff scientist with the Engineering Research Associates, Arlington, Virginia.

Miss Maria Davidson of Columbia University has accepted a position as a statistician with the American Heart Association, New York, New York.

Mrs. Anne C. Davis has accepted a position as Junior research mathematician with the Office of Ordnance Research Project, University of California, Berkeley.

Dr. Jim Douglas, Jr., has accepted a position as assistant research engineer with the Humble Oil and Refining Company, Houston, Texas.

Dr. S. E. Dyer, Jr., of the University of Texas has been appointed to an assistant professorship at the University of Georgia.

Dr. J. W. Gaddum of the National Bureau of Standards, Los Angeles, has accepted a position as mathematician with the U.S. Air Force, Washington, D.C.

Dr. L. D. Gates, Jr., of the Iowa State College of Agriculture and Mechanic Arts has accepted a position as mathematician with the Defense Department.

Dr. H. D. Huskey of the National Bureau of Standards, Los Angeles, is on leave of absence and has been appointed technical director of the Computation Laboratory, Wayne University.

Professor D. H. Hyers of the University of Southern California is on sabbatical leave during the present year at New York University.

Professor Jacob Korevaar of the Institute of Technology, Delft, Netherlands, has been appointed to an assistant professorship at the University of Wisconsin.

Mr. F. A. Kros has accepted a position as mathematician with the White Sands Proving Ground, Las Cruces, New Mexico.

Associate Professor K. W. Lamson of Lehigh University has retired with the title associate professor emeritus.

Professor Joseph Lehner of the University of Pennsylvania is on leave of absence and has accepted a position as staff member with the Los Alamos Scientific Laboratory.

Assistant Professor Benjamin Evans Mitchell of the Alabama Polytechnic Institute has been appointed to an assistant professorship at the Louisiana State University.

Mr. A. H. Moore of Pratt Institute has accepted a position as operations analyst with the Operations Research Office, Johns Hopkins University.

Dr. R. B. Murphy of the Carnegie Institute of Technology has accepted a position as a member of the technical staff, Bell Telephone Laboratories, Inc.

Dr. S. F. Neustadter of Harvard University has accepted a position as staff member with the Project Lincoln, Massachusetts Institute of Technology.

Professor Jerzy Neyman of the University of California, Berkeley, is on leave of absence to assist in organizing a sampling demonstration center for the government of Thailand under the auspices of the Food and Agriculture Organization of the United Nations.

Mr. P. S. Null of West Virginia University has accepted a position as stress analyst with the Douglas Aircraft Company, Long Beach, California.

Professor Margaret Owchar of Southwest Missouri State College has accepted a position as researcher and statistician with the Manitoba Cancer Relief and Research Institute, Winnipeg, Manitoba, Canada.

Dr. Anne F. O'Neill of Smith College has been appointed to an assistant professorship at Wheaton College, Norton, Massachusetts.

Dr. J. D. Riley of the University of Kansas has accepted a position as mathematician with the Naval Ordnance Laboratory, White Oak, Silver Spring, Maryland.

Mr. R. E. Thomas of the State University of Iowa has accepted a position as mathematician with the Battelle Memorial Institute, Columbus, Ohio.

Mr. M. W. Tolman has accepted a position as junior geophysicist with the Shell Oil Company, Tulsa, Oklahoma.

Professor E. G. Volterra of the Illinois Institute of Technology has been appointed to a professorship at Rensselaer Polytechnic Institute.

Mr. Jack Warga of the Reeves Instrument Corporation has accepted a position as principal engineer with the Republic Aviation Corporation, Farmingdale, New York.

Professor W. J. Wells of the University of Minnesota has accepted a position as mathematician with the United Aircraft Corporation, East Hartford, Connecticut.

Dr. J. G. Wendel of the Rand Corporation has been appointed to an assistant professorship at the Louisiana State University.

Mr. W. D. Wood of Purdue University has accepted a position as staff member with the Sandia Corporation.

Dr. Arthur Wouk of Johns Hopkins University has accepted a position as mathematician with the Reeves Instrument Corporation, New York, New York.

Dr. J. L. Yarnell of the University of Minnesota has accepted a position as staff member with the Los Alamos Scientific Laboratory.

The following promotions are announced:

R. G. Ayoub, Pennsylvania State College, to an assistant professorship.

E. W. Barankin, University of California, Berkeley, to an associate professorship.

M. A. Bassam, Huston-Tillotson College, to a professorship.

Kurt Bing, University of California, Berkeley, to an associate professorship.

Joseph Hilsenrath, National Bureau of Standards, to a physical science administrator.

E. V. Lewis, University of Delaware, to an associate professorship.

Edmund Pinney, University of California, Berkeley, to an associate professorship.

R. G. Stanton, University of Toronto, to an assistant professorship.

R. G. Tutte, University of Toronto, to an assistant professorship.

F. E. Ulrich, Rice Institute, to a professorship.

The following appointments to instructorships are announced: University of California, Berkeley: Dr. Lucien LeCam; Lehigh University: Dr. S. I. Goldberg, Dr. Felix Haas; University of Michigan: Dr. E. L. Griffin; Northwestern University: Dr. J. C. E. Dekker; University of Rochester: Mr. Hewitt Kenyon; Swarthmore College: Mr. E. R. Mullins, Jr.; Syracuse University: Dr. G. F. Leger, Jr.; Virginia Polytechnic Institute: Mr. V. R. Hancock; Wake Forest College: Mr. G. W. Medlin; Yale University: Dr. W. G. Bade.

Dr. R. H. Kyle of Trinity College, Dublin, Ireland, died on October 1, 1952 at the age of twenty-six years.

Professor W. J. Rusk of Grinnell College died on September 10, 1952, at the age of eighty-three years. He had been a member of the Society for forty-nine years.

Professor Marie J. Weiss of Newcomb Memorial College, Tulane University of Louisiana, died on August 19, 1952, at the age of forty-nine years. She had been a member of the Society for twenty-four years.

Dr. Emily C. Williams died on August 8, 1952, at the age of eighty-seven years. She had been a member of the Society for forty-seven years.

NEW PUBLICATIONS

Abetti, B. B. See Abetti, G.

Abetti, G. *The history of astronomy.* Trans. by B. B. Abetti. New York, Schuman, 1952. 12+338 pp. $6.00.

Azumaya, G. *Tanjun-kan no daisûteki riron.* [*Algebraic theory of simple rings.*] Tokyo, Kawaide Shobô, 1951. 2+2+182 pp. 300 Yen.

Becker, O., and Hofmann, J. E. *Geschichte der Mathematik.* Bonn, Athenäum, 1951. 340 pp. $10.00.

Bickley, W. G., Comrie, L. J., Miller, J. C. P., Sadler, D. H., and Thompson, A. J. *Bessel functions.* Part II. *Functions of positive integer order.* (British Association for the Advancement of Science, Mathematical Tables, vol. 10.) Cambridge University Press, 1952. 40+255 pp. $11.00.

Bondi, H. *Cosmology.* Cambridge University Press, 1952. 6+179 pp. $4.50.

Bourbaki, N. *Éléments de mathématique.* XIII. Part I. *Les structures fondamentales de l'analyse.* Book VI. *Intégration.* (Actualités Scientifiques et Industrielles, no. 1175.) Paris, Hermann, 1952. 2+237+5 pp. 2500 fr.

Bourbaki, N. *Éléments de mathématique.* XIV. Part I. *Les structures fondamentales de l'analyse.* Book II. *Algèbre.* (Actualités Scientifiques et Industrielles, no. 1179.) Paris, Hermann, 1952. 2+159+3 pp. 1800 fr.

Busbridge, I. W. See Kourganoff, V.

Comrie, L. J. See Bickley, W. G.

Dijksterhuis, E. J. *Christiaan Huygens (Bij de voltooiing van zijn Oeuvres Complètes).* Haarlem, Bohn, 1951. 29 pp.

Eichler, M. *Quadratische Formen und orthogonale Gruppen.* Berlin, Springer, 1952. 12+220 pp. 27.60 DM.

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Fuks, B. A., and Levin, V. I. *Funkcii kompleksnogo peremennogo i ih priloženiya.* Moscow, Gosudarstvennoe Izdatel'stvo Tehniko-Teoretičeskoi Literatury, 1951. 307 pp. 11 rubles.

Gnyegyenko, B. V., and Kolmogorov, A. N. *Független valószínűségi változók összegeinek határeloszlásai.* [Limit distributions for sums of independent random variables.] Budapest, Akadémiai Kiadó, 1951. 256 pp. 32.00 florints.

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Hofmann, J. E. See Becker, O.

Kawada, Y. *Bibunshiki ron—Grassmann daisû to Lie gun.* [*Theory of differential forms—Grassmann algebras and Lie groups.*] Tokyo, Kawaide Shobô, 1951. 6+194 pp. 300 Yen.

Kolmogorov, A. N. See Gnyegyenko, B. V.

Kourganoff, V. *Basic methods in transfer problems. Radiative equilibrium and neutron diffusion.* With the collaboration of I. W. Busbridge. Oxford, Clarenden, 1952. 15+281 pp. $7.00.

Krieger, C. C. See Sierpinski, W.

Kuratowski, K., and Mostowski, A. *Teoria Mnogości.* [*Theory of sets.*] (Monografie Matematyczne, vol. 27.) Warsaw, Polskie Towarzystwo Matematyczne, 1952. 9+311 pp.

Larmor, J. See Poincaré, H.

Levin, V. I. See Fuks, B. A.

Lohr, E. *Mechanik der Festkörper*. Berlin, de Gruyter, 1952. 8+483 pp. 39.60 DM.

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Maitland, F. See Poincaré, H.

Miller, J. C. P. See Bickley, W. G.

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Poincaré, H. *Science and hypothesis*. With a preface by J. Larmor. New York, Dover, 1952. 28+244 pp., 1 plate. $2.50.

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Russell, B. See Poincaré, H.

Sadler, D. H. See Bickley, W. G.

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Sauer, R. *Anfangswertprobleme bei partiellen Differentialgleichungen*. Berlin, Springer, 1952. 14+229 pp. 29 DM.

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Solodovnikov, V. V. *Vvedenie v statističeskuyu dinamiku sistem avtomatičeskogo upravleniya*. Moscow, Gosudarstvennoe Izdatel'stvo Tehniko-Teoretičeskoi Literatury, 1952. 1+367 pp. 11.45 rubles.

Stewart, B. M. *Theory of numbers*. New York, Macmillan, 1952. 13+261 pp. $5.50.

Tables of the Bessel functions $Y_0(x)$, $Y_1(x)$, $K_0(x)$, $K_1(x)$, $0 \leq x \leq 1$. (National Bureau of Standards Applied Mathematics Series, no. 25.) Washington, Government Printing Office, 1952. 10+60 pp. $.40.

Thompson, A. J. See Bickley, W. G.

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INTEGRATION IN ABSTRACT SPACES

T. H. HILDEBRANDT

In thinking about possible subjects for this address, I could not help thinking back to the first "invited address" which I gave before members of the Society. The address together with one given by the late G. A. Bliss was the basis of the first Symposium of what was then called the Chicago Section of the Society, the subject Lebesgue integration, the time April 1917. These symposia were initiated because of the desire of the Chicago Section to make their meetings more interesting. It was felt that if an occasional session could be devoted to a single topic, with one or two speakers, and an outline of and information concerning the material could be sent out in connection with the program of the meeting, it would make effective audience participation possible and prove more profitable and stimulating to members of the Society. The idea was taken up by other groups and now takes the form of an "invited address," where unfortunately the audience does not have advance preparation, and stimulating discussion is avoided. I note that in recent years we have had committees to consider ways and means of making the sessions of the Society more effective, and we shall probably have this problem always with us. I do feel that the system of invited addresses has proven itself in that it has been worth while for him who gives and him who takes, particularly when the address results in published summaries of research covering a vital field.

For me that first symposium was very much worth while; it aroused and established in me an interest in integration. So a first suggestion for this address was to give a summary of what has happened in integration since that time. But the immensity of such a project is obvious, and the impossibility of including such within the scope of an address no less so. I have, consequently, narrowed my field, and expect to speak on two rather narrow lines of generalization of integration, with emphasis on Lebesgue integration, one in the direction of linear spaces, the other using the order character of some linear spaces.

1. The Lebesgue integration postulates. It might be interesting to recall briefly the axiomatic program which Lebesgue set up, and which led him to the now well known definition of measurable sets

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Retiring presidential address delivered at the Athens, Georgia, meeting of the Society on December 31, 1947; received by the editors June 26, 1952.

and integration as given in his book *Sur l'intégration* (first edition (1904) pp. 98 ff., 2nd edition, pp. 105 ff.). He proposes to define an integration process yielding a real number for any of the class $\mathfrak{F}$ of bounded real valued functions f, and any finite interval $a \leqq s \leqq b$, subject to the following conditions.

(1) For all a, b, and h: $\int_a^b f(s)ds = \int_{a+h}^{b+h} f(s-h)ds$.
(2) For all a, b, c: $\int_a^b f + \int_b^c f + \int_c^a f = 0$.
(3) For all f_1 and f_2: $\int_a^b (f_1 + f_2) = \int_a^b f_1 + \int_a^b f_2$.
(4) If $f \geqq 0$ and $b > a$, then $\int_a^b f \geqq 0$.
(5) $\int_0^1 1 ds = 1$.
(6) If $f_n(s) \leqq f_{n+1}(s)$ for all n and s and $\lim_n f_n(s) = f(s)$ for all s, then $\lim_n \int_a^b f_n = \int_a^b f$.

His analysis of these properties yields among other things that if $b > a$, then (3) and (4) imply (a) $\left|\int_a^b f\right| \leqq \int_a^b |f|$; (b) $\int_a^b cf = c\int_a^b f$ for all real c, so that the integral is linear on the class $\mathfrak{F}$; (c) if f_n approaches f uniformly on (a, b), then $\lim_n \int_a^b f_n = \int_a^b f$. Consequently, the properties (3) and (4) require that $\int_a^b f$ with $b > a$ be a positive linear continuous functional or form on the space $\mathfrak{F}$ of all bounded functions, the least upper bound of the absolute value being the norm of the function. By dividing up the range of values of $f(s)$, Lebesgue notes that his problem would be solved if he knew the value of the integral for the characteristic functions of all subsets of the linear interval. It is remarkable how closely Lebesgue approached the solution of the problem of obtaining the most general linear continuous functional or form of the space of bounded functions on a finite or infinite interval. A solution of this latter problem was published much later, first by the speaker [30] and shortly thereafter by Fichtenholz and Kantorovic [18], viz., that such a form is completely determined by a bounded additive function of all subsets of the interval. It can be expressed in the form $\int f(s)d\alpha(E)$ where the integration is definable either by the Lebesgue process of subdividing the range of $f(s)$ or by successive subdivisions or partitions of the interval into a finite number of disjoint sets. The positive property (4) requires that $\alpha(E) \geqq 0$ for all E, i.e., $\alpha(E)$ is a positive finitely additive measure function on all subsets of the interval. Turning to the other conditions imposed by Lebesgue, the additional continuity restriction involved in condition (6) when applied to $\int f d\alpha$ requires that if E_n is a monotonic nonincreasing sequence of sets without common element, then $\lim_n \alpha(E_n) = 0$, which in turn is equivalent to the complete additivity of the measure function $\alpha(E)$ on all sets. Now condition (1) applied to a measure function requires that it be invariant under translation, and we have here the precursor of the Haar-measure problem, to determine a completely additive measure function on a completely additive class of sets, invariant under translation. Conditions (2) and (3) demand that, for intervals, this measure should reduce to the length of the interval. At this point Lebesgue abandons the project of determining an integral satisfying conditions (1) to (6) for all bounded functions, and sets up a process for determining what is now well known as the class of measurable subsets of the straight line, a measure function on these sets, and a class of measurable functions. Banach ([2] and [3, pp. 30–32]) has given a procedure for finding an integral for all bounded functions on $0 \leqq s \leqq 1$, satisfying conditions (1) to (5) but not (6). This would correspond to a bounded finitely additive invariant measure on all subsets of $0 \leqq s \leqq 1$. So far as I am aware, no one has settled the question whether the set of Lebesgue measurable bounded functions is the largest subclass of these functions, with an integral satisfying all of Lebesgue's postulates.

One might consider a corresponding set of postulates for Stieltjes integrals and their generalization in which the emphasis would be on the bilinear character of these integrals. Thus one might ask for a number corresponding to every pair of bounded functions f and g and interval $a \leqq s \leqq b$, denoted by $S(f, g; a, b)$ reducing to say the Riemann-Stieltjes integral when the latter exists. There would be no postulate corresponding to (1). Condition (2) would go over unchanged. Condition (3) would require bi-additivity in f and g. Condition (4) would take the form: if $f \geqq 0$, g is monotonic non-decreasing, and $b > a$, then $S(f, g; a, b) \geqq 0$. Condition (5) would be $S(1, g; a, b) = g(b) - g(a)$. Condition (6) might involve two convergence properties: (a) if $f_n(s) \leqq f_{n+1}(s)$ for all n and s, and $\lim_n f_n(s) = f(s)$ for all s, then for every monotone function g, $\lim_n S(f_n, g; a, b) = S(f, g; a, b)$; (b) if $g_n(s)$ are monotone nondecreasing, and $\lim_n g_n(s) = g(s)$ for all s, then $\lim_n S(f, g_n; a, b) = S(f, g; a, b)$.

We might recall in passing that Lebesgue's definition of an integral is essentially a Stieltjes integral. If f is measurable and we set $\mu(y) = \text{meas } E(s \mid f(s) < y)$, then $L\int_a^b f(s)ds = S\int_{-\infty}^{\infty} y\,d\mu(y)$, convergence of the right-hand side being necessary and sufficient for the integrability of f, when f is not bounded. This form emphasizes the distributional character of integration. As is well known, there are other approaches to Lebesgue integration, some of which will be mentioned later, the amazing, or perhaps gratifying, fact is that so many different approaches lead to the same thing in the end, justifying Lebesgue's solution.

2. **The Fréchet general integral.** The extension of Lebesgue integration in which the functions are defined on an n-dimensional rectangle instead of a linear interval was an easy one. The Radon suggestion of replacing n-dimensional measure by a positive completely additive set function was the basis of the natural generalization made by Fréchet [20]. In this generalization the linear interval is replaced by a general set $\mathfrak{S}$, measurable subsets are postulated to be a σ-field $\mathfrak{E}$ of subsets E (satisfying the conditions (a) if E_n, $n=1, 2, \cdots$, belong to $\mathfrak{E}$, so does the set sum or join $\sum_n E_n$; (b) if E_1 and E_2 belong to $\mathfrak{E}$, so does E_1-E_2; (c) $\mathfrak{S}$ belongs to $\mathfrak{E}$). Measure is replaced by a set function, $\alpha(E)$, on $\mathfrak{E}$ to real numbers, completely additive on $\mathfrak{E}$, i.e., if $E=\sum_n E_n$ in $\mathfrak{E}$, and E_n are disjoint, then $\alpha(\sum_n E_n)=\sum_n \alpha(E_n)$. Then α is bounded on $\mathfrak{E}$, has a total variation $\int|d\alpha|$ on all subsets of $\mathfrak{E}$, and can be written as the difference of two positive-valued completely additive set functions. A function is measurable relative to $\mathfrak{E}$ if the set $E(s|f(s)<a)$ belongs to $\mathfrak{E}$ for all real numbers a. Then the Lebesgue procedure can be applied relative to each of the positive parts of α, yielding $\int f d\alpha$. Fréchet also called attention to the applicability of a method of defining an integral following the ideas suggested by W. H. Young, by introducing upper and lower integrals. For this it is necessary to assume that there exists a subdivision of $\mathfrak{S}$ into a denumerably infinite number of sets of $\mathfrak{E}$, so that $\sum_n M_n\tau(E_n)$ converges, where $\tau(E)$ is the total variation of α on E, and M_n is the least upper bound of $|f|$ on E_n. Then finite-valued upper and lower integrals based on subdivisions of $\mathfrak{S}$ into sets of $\mathfrak{E}$ exist, equality giving rise to an integral. An alternative procedure is possible by utilizing the Moore-Smith type of limit [45]. For this, subdivisions π of $\mathfrak{S}$ are ordered by inclusion, $\pi_1\geqq\pi_2$ meaning that every subset of π_1 is part of a subset of π_2. Then $f(s)$ has an integral with respect to $\alpha(E)$ if $\lim_\pi \sum_n f(s_n)\alpha(E_n)$ exists, s_n being in E_n. Since a subdivision does not define any order in the sets E_n composing it, the infinite series involved must converge absolutely. The methods of Lebesgue and Young have the same set of integrable functions if the class of measurable sets is complete relative to α, i.e., if $\int_E|d\alpha|=0$, and E_0 is contained in E, then E_0 is in $\mathfrak{E}$ and $\int_{E_0}|d\alpha|=0$.

I. INTEGRALS IN LINEAR NORMED SPACES

3. **Riemann integrals.** The next most obvious generalization of the integration process consists in extending the value space of the functions to be integrated and the integral values to the simplest generalizations of the space of real numbers, viz., a linear normed com-

plete or LNC or Banach space.[1] A space $\mathfrak{X}$ of elements x is linear if addition as a commutative group and multiplication by real (or complex) numbers is defined. A linear space is normed if every x has a norm $\|x\|$ satisfying the conditions: $\|x\| \geqq 0$ for all x; $\|x\| = 0$ is equivalent to $x = 0$ (the null element of the space); and $\|a_1x_1 + a_2x_2\| \leqq |a_1| \|x_1\| + |a_2| \|x_2\|$ for all numbers a_1, a_2 and x_1, x_2 of $\mathfrak{X}$. A normed space is complete if for a sequence x_n such that $\lim_{m,n} \|x_m - x_n\| = 0$ there exists an x such that $\lim_n \|x_n - x\| = 0$.

The simplest integration notion in such an LNC space $\mathfrak{X}$ is that of the Riemann integral of a function $x(s)$ on the linear interval $a \leqq s \leqq b$ to $\mathfrak{X}$, as the limit in the space, of sum $\sum_{i=0}^{n-1} x(s_i')(s_{i+1} - s_i)$, where $a = s_0 < s_1 < \cdots < s_n = b$ is a subdivision of (a, b), $s_i \leqq s_i' \leqq s_{i+1}$, and the limit is taken as the norm of the subdivision, the maximum of $s_{i+1} - s_i$ approaches zero. This generalization is due to L. M. Graves [27]. The proof of the integrability of a function $x(s)$ continuous on (a, b) follows the usual pattern, but the boundedness of the function $\|x(s)\|$ and its Riemann integrability, while sufficient, are not necessary for the integrability of a function $x(s)$. Simple counter examples are available in the space of bounded sequences.

The extension of these notions to Riemann Stieltjes integrals is conceptually simple. There are two possibilities: (a) the integration of a function $x(s)$ on $a \leqq s \leqq b$ to $\mathfrak{X}$ with respect to a real-valued function $g(s)$; (b) the integration of a real-valued function $f(s)$ with respect to a function $x(s)$ to $\mathfrak{X}$ (for the latter see VanderLijn, G. [62]). The limit of the approximating sums $\sum_{i=0}^{n-1} x(s_i')(g(s_{i+1}) - g(s_i))$ or $\sum_{i=0}^{n-1} f(s_i')(x(s_{i+1}) - x(s_i))$ is possible either as the $|\sigma| = \max_i (s_{i+1} - s_i)$ approaches zero, or by successive subdivisions, $\sigma_1 \geqq \sigma_2$ if σ_1 contains all of the points of σ_2. Obviously an integration by parts theorem between the two types of integrals holds, i.e., $\int_a^b x(s)dg(s)$ and the corresponding $\int_a^b g(s)dx(s)$ exist simultaneously and their sum is $x(b)g(b) - x(a)g(a)$. The integral (a) exists if $x(s)$ is continuous and $g(s)$ is of bounded variation, and the integral (b) exists if $f(s)$ is continuous and $x(s)$ is strongly of bounded variation in the sense that $\sum_{i=0}^{n-1} \|x(s_{i+1}) - x(s_i)\|$ is bounded as a function of the subdivision σ of (a, b). Dunford [17, p. 312] has shown that if $f(s)$ is continuous and $x(s)$ is weakly of bounded variation in the sense that if σ consists of the finite disjoint subintervals (s_1', s_1''), $\cdots$, (s_n', s_n''), then $\|\sum_{i=1}^{n}(x(s_i')-x(s_i''))\|$ is bounded in σ, then $\int_a^b f(s)dx(s)$ exists, and this condition is necessary if we want the integral to exist for all continuous functions. Stieltjes integrals of type (a) occur in the resolution of the identity of self-adjoint transformations in Hilbert spaces.

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[1] We shall use the terminology linear normed complete or LNC space in preference to Banach space for two reasons: (a) the concept of such spaces predated Banach's formulation (see F. Riesz, Acta Math. vol. 41 (1916) pp. 72, 73; A. A. Bennett, Proc. Nat. Acad. Sci. U.S.A. vol. 21 (1916) p. 595; K. W. Lamson, Amer. J. Math. vol. 42 (1920) p. 245), (b) there are linear spaces in which the topology is different, and it seems better to include in the name of the space its mathematical character.

4. **The Bochner integral.** The definition of measurability and integrability for $x(s)$ on an n-dimensional interval $\mathfrak{S}$, to an LNC space due to Bochner [6], has proved very effective. A function $x(s)$ is measurable if it is the limit almost everywhere of a sequence of finite-valued measurable functions, a function $x(s)$ being a finite-valued measurable function if there exists a subdivision of $\mathfrak{S}$ into a finite number of disjoint measurable subsets, $E_1, \cdots, E_n$, on each of which $x(s)$ is constant. If $\mathfrak{X}$ is the real number system, the definition of measurability yields the usual set of measurable functions. The function $x(s)$ is integrable on $\mathfrak{S}$ if $x(s)$ is measurable and $\|x(s)\|$, which is therefore also measurable, has a finite Lebesgue integral on $\mathfrak{S}$. It is shown that it is then possible to assign a unique value to $\int x(s)ds$ as the limit of $\int x_n(s)ds$, where the $x_n(s)$ are finite-valued functions approaching $x(s)$ almost everywhere with $\|x_n(s)\| \leqq f(s)$, $f(s)$ some integrable function on $\mathfrak{S}$, and $\int x_n(s)ds = \sum_{i=1}^{k} x_{ni}m(E_{ni})$, $x_n(s) = x_{ni}$ on E_{ni}. The limit of the sequence of integrals is independent of the sequence of finite-valued functions satisfying the given conditions. The resulting integral has the usual properties of Lebesgue integration; for instance, the complete additivity of $\int_E x(s)$ as a function of measurable sets, and the convergence property: if the $x_n(s)$ are integrable, converge to $x(s)$ almost everywhere, and $\|x_n(s)\| \leqq f(s)$, for all n and s, with $f(s)$ integrable, then $\lim_n \int_E x_n(s)ds = \int_E x(s)ds$ for all measurable sets E of $\mathfrak{S}$.

5. **The Dunford (first) integral.** A clever new point of view was injected into the theory of Lebesgue integration by Dunford [15] who observed that if one normed the space of continuous functions by the condition $\|f\| = \int |f(s)| ds$, the resulting linear normed space is not complete. If one completes this metric space in the usual way, one gets exactly the Lebesgue integrable functions. By changing the norm of f to $\|f\| = |\int |f(s)|^p ds|^{1/p}$, $p \geqq 1$, the space L^p of functions for which $|f(s)|^p$ is integrable emerges. Adapting this technique, Dunford assumes a completely additive set function α defined on a σ-field of subsets of n-dimensional Euclidean space, which includes the Borel measurable sets. It is shown that if $x(s)$ is continuous on $\mathfrak{S}$, then $\int x(s)d\alpha(E)$ as well as $\int \|x(s)\| d\beta(E)$ exist, where $\beta(E)$ is the

total variation of $\alpha(E)$ on E. Then the class of Lebesgue integrable functions $x(s)$ is the completion of the class of continuous functions on $\mathfrak{S}$ normed by $\int\|x(s)\|d\beta(E)$, $x(s)$ is the limit of a sequence of continuous functions almost everywhere in terms of β, and $\int x(s)d\alpha(E)$ is the limit of $\int x_n(s)d\alpha(E)$, where $x_n(s)$ is any sequence of continuous functions such that $\lim_n \int\|x_n(s)-x(s)\|d\beta(E)=0$. In the case in which $\alpha(E)$ is Lebesgue measure, the class of integrable functions as well as their integrals agree with that of the Bochner definition. On the other hand, there are Riemann integrable functions for which $\int\|x(s)\|ds=\infty$, which are therefore not integrable by either of these definitions.

Dunford extends this definition to a metric space, probably because he started with an integral on the space of continuous functions. Later he noticed that a more elegant approach is via the space of finite-valued measurable functions, since this does not involve any topological conditions on $\mathfrak{S}$. The next step is then obvious: we take a general space $\mathfrak{S}$, a class of (measurable subsets) $\mathfrak{E}$, which forms a σ-field, a completely additive set function α on $\mathfrak{E}$ to real numbers, and define an integral for the class of finite-valued measurable functions. We norm this class of functions with $\int\|x(s)\|d\beta(E)$, where $\beta(E)$ is the total variation of α on E. The completion of this class of functions under this norm gives the Lebesgue integrable functions $L(\mathfrak{X})$ as well as an integral.

6. Other approaches to Bochner integral. As we have already noted, the building up of an integral of a function $x(s)$ on $\mathfrak{S}: a\leqq s\leqq b$ to an LNC space $\mathfrak{X}$, following the Lebesgue process runs into the difficulty that his definition of measurable functions is dependent on an order process, while no such order is postulated in $\mathfrak{X}$. Hence, other approaches are necessary. The speaker, in an unpublished paper [31] predating Bochner and Dunford, noted that the process outlined by F. Riesz [58] for the definition of a Lebesgue integral could be carried over to the more general setting without much change. Defining a step function on (a, b) in the usual way, i.e., $x(s)=x_i$ for $s_i\leqq s<s_{i+1}$; $x(b)=x_n$, for the subdivision $\sigma=(a=s_0<s_1<\cdots<s_n=b)$ of (a, b) and $\int x(s)ds=\sum_{i=1}^{n} x_i(s_{i+1}-s_i)$, an arbitrary bounded function $x(s)$ on (a, b) is integrable if it is the limit of a sequence of step functions almost everywhere, the $\int x(s)ds$ being the limit of $\int x_n(s)ds$ where $x_n(s)$ is any bounded sequence of step functions converging to $x(s)$ almost everywhere. In the case when $x(s)$ is unbounded, the sequence of step functions $x_n(s)$ converging to $x(s)$ is subject to the additional condition that $\int_E x_n(s)ds$ be uniformly ab-

solutely continuous in the sense that $\int_E\|x_n(s)\|ds$ are uniformly absolutely continuous. It follows then that $\int_a^b\|x(s)\|ds$ exists, so that this procedure is equivalent to that of Bochner.

An alternative approach is suggested by the definition of Lebesgue integral due to Hahn [28]. A function $x(s)$ is defined to be measurable if for every $\epsilon>0$, there exists a perfect set P_ϵ, such that the measure of P_ϵ is greater than $b-a-\epsilon$, and $x(s)$ is continuous on P_ϵ. This uses the Lusin property of measurable functions. A Riemann type of integral is obviously definable on P_ϵ. If $x(s)$ is bounded and measurable in this sense on (a, b), then $\int x(s)ds=\lim_{\epsilon\to 0}\int_{P_\epsilon}x(s)ds$, the P_ϵ being limited to the perfect sets on which $x(s)$ is continuous. This limit can be shown to exist. If $x(s)$ is unbounded, then it is integrable on (a, b) if $\lim_{\epsilon\to 0}\int_{P_\epsilon}x(s)ds$ exists, with the same limitation on P_ϵ. While under these hypotheses $\|x(s)\|$ is a real-valued measurable function, it is sufficient but not necessary that it be integrable in the Lebesgue sense, in order to have $x(s)$ integrable according to this definition.

7. **Birkhoff integral.** For some reason or other the Young approach used by Fréchet in setting up an integral for functions on a general space was passed over. In order to make effective use of this procedure, it was necessary to extend the notion of unconditional convergence of an infinite series of elements to an LNC space $\mathfrak{X}$. Birkhoff [4] defines this as convergence to the same element by any rearrangement of the series. There are several equivalent methods of defining unconditional convergence (see, e.g., this Bulletin vol. 46 (1940) p. 950), the most elegant method being based on a Moore-Smith limit. If π represents a finite subset: $(n_1, \cdots, n_k)$ of the integers and $\pi_1\geqq\pi_2$ is defined by set inclusion, then the π form a directed set. $\sum x_n$ is unconditionally convergent if $\lim_\pi\sum_\pi x_n$ exists. If $\mathfrak{X}$ is the space of real or complex numbers, then an unconditionally convergent series is absolutely convergent. It is obviously possible to define in a similar way the meaning of $\sum_\Omega x_q$, where the q are any general set Ω (no order needed), by setting $\pi=(q_1, \cdots, q_k)$ and defining $\sum_\Omega x_q$ as the limit as π spreads of $\sum_\pi x_q$. It is easily demonstrable that $\sum_\Omega x_q$ exists if and only if x_q vanishes except for a denumerably infinite subset of Ω, and $\sum_n x_{q_n}$ extended over this subset is unconditionally convergent. Birkhoff [4, p. 362] notes that the set of unconditionally convergent sequences of an LNC space $\mathfrak{X}$ form again an LNC space. The notion of unconditional convergence is applicable to any sequence of sets. If $X_1, \cdots, X_n, \cdots$ are subsets of $\mathfrak{X}$, then $\sum X_n$ is unconditionally convergent if $\sum x_n$ is unconditionally convergent

for any choice of the x_n in X_n. The class of elements so expressible is defined as $\sum_n X_n$.

For the definition of an integral, Birkhoff [4] assumes that the general set $\mathfrak{S}$ contains a σ-field $\mathfrak{E}$ of "measurable" subsets E, $\alpha(E)$ is a positive completely additive function on $\mathfrak{E}$, and $x(s)$ is a point function on $\mathfrak{S}$ to $\mathfrak{X}$. Then subdivisions σ of $\mathfrak{S}$ into disjoint sets $E_1, \cdots, E_n, \cdots$ of $\mathfrak{E}$ determines the set of elements $X(x, \sigma) = \sum_n x(s_n)\alpha(E_n)$, with s_n in E_n, if the series of sets is unconditionally convergent. The closed convex extension of $X(x, \sigma)$ denoted by Cl co $X(x, \sigma)$ is called an integral range of $x(s)$ corresponding to σ. This is a generalization of the interval defined by the extremes of approximating sums in the case of real numbers. If the greatest lower bound of the diameter of Cl co $X(x, \sigma)$ relative to σ is zero, $x(s)$ is said to be integrable, the integral being the common point of Cl co $X(x, \sigma)$. By the use of the Moore-Smith limit it is possible to by-pass the convex extension and closure. If one orders subdivisions σ of $\mathfrak{S}$ by assuming that $\sigma_1 \geq \sigma_2$ if σ_1 is finer than σ_2 (every subset of σ_1 is a subset of some set of σ_2) then $x(s)$ is integrable with respect to the completely additive set function $\alpha(E)$ if the $\lim_\sigma \sum_n x(s_n)\alpha(E_n)$ exists. A necessary condition for the existence of such an integral is that there exist a subdivision σ_0 such that for any $\sigma \geq \sigma_0$, $\sum_\sigma x(s_n)\alpha(E_n)$ be unconditionally convergent for any choice of the s_n in E_n. If the Birkhoff integral exists on $\mathfrak{S}$, it also exists on every subset E of $\mathfrak{E}$ and is a completely additive set function on $\mathfrak{E}$ to $\mathfrak{X}$, in the sense that if $E = \sum E_n$ (E_n disjoint) then $x(E) = \sum_n x(E_n)$, the series on the right being unconditionally convergent. The Birkhoff integral includes the Riemann type of integral, as well as the Bochner and Dunford integrals as special cases, and gives an integral value for some functions which are not integrable by these other methods.

In reading the Birkhoff paper, one notices that he could also have defined an alternative type of integral, where $f(s)$ is a real-valued point function and $x(E)$ is a completely additive set function on $\mathfrak{E}$ to $\mathfrak{X}$ as defined in the preceding paragraph. Most of the reasoning applies to yield integrals of the form $\int f(s)dx(E)$. It has bilinear properties in f and $x(E)$.

8. Bilinear integrals. Once the observation has been made that either the point function or the set function may belong to an LNC space $\mathfrak{X}$ and one recalls that the space of real numbers is an LNC space, the idea of having both of these functions in LNC spaces is imminent. But now the question of multiplication arises. We either enter the field of ring spaces, or observe that the product of a real number into an element x of $\mathfrak{X}$ is a bilinear transformation on the product space $\mathfrak{R}\mathfrak{X}$ to $\mathfrak{X}$ ($\mathfrak{R}$ being the space of real numbers). We propose to replace multiplication by real numbers by a bilinear transformation B on the product space $\mathfrak{X}\mathfrak{Y}$ to $\mathfrak{Z}$; $\mathfrak{X}$, $\mathfrak{Y}$, and $\mathfrak{Z}$ being LNC spaces. It is conceivable that one may need a boundedness condition, i.e., that there exists an $M \geqq 0$, such that $\|B(x, y)\| \leqq M\|x\|\|y\|$. Assuming, then, such a bilinear transformation, a point function $x(s)$ on $\mathfrak{S}$ to $\mathfrak{Y}$, a set function $y(E)$ on the sets E of $\mathfrak{E}$ to $\mathfrak{X}$, we can follow the usual pattern and define $\int B(x(s), dy(E))$ as that element of $\mathfrak{Z}$ which is equal to $\lim_\sigma \sum_\sigma B(x(s_n)y(E_n))$ provided the limit exists, the series involved being assumed to be unconditionally convergent. Bilinearity of B suffices to make this integral bilinear in $x(s)$ and $y(E)$. Boundedness of $B(x, y)$ in y for each x and complete additivity of the set function $y(E)$ are sufficient to guarantee that if x is a finite-valued measurable function then $\int B(x(s), dy(E))$ exists and is equal to $\sum_i B(x_i, y(E_i))$, x_i being the value of $x(s)$ on E_i. Other conditions are needed to guarantee complete additivity of the integral as a set function or the interchangeability of integral and limits.

Instances of this type were considered by Gowurin [26] and by Bochner and Taylor [8]. Gowurin assumes that $\mathfrak{S}$ is the linear interval (a, b) and $\mathfrak{Y}$ the space of linear bounded transformations T on the LNC space $\mathfrak{X}$ to the LNC space $\mathfrak{Z}$. Then $\mathfrak{Y}$ is a linear normed complete space under the modulus of T, $B(y, x) = Tx$ is a bounded bilinear transformation on $\mathfrak{Y}\mathfrak{X}$ to $\mathfrak{Z}$ with $\|B(y, x)\| \leqq \|T\| \|x\|$. Conversely if $B(y, x)$ is a bounded bilinear transformation on $\mathfrak{X}\mathfrak{Y}$ to $\mathfrak{Z}$, then for fixed y, $B(y, x)$ defines a linear bounded transformation on $\mathfrak{X}$ to $\mathfrak{Z}$, i.e. $B(y, x)$ as y ranges over $\mathfrak{Y}$ defines a subset of the linear bounded transformations on $\mathfrak{X}$ to $\mathfrak{Z}$. If $y(s)$ and $x(s)$ are point functions on $\mathfrak{S}$ to $\mathfrak{Y}$ and $\mathfrak{X}$ respectively, we can set up the Stieltjes sums $\sum_\sigma B[y(s_i'); x(s_{i+1}) - x(s_i)]$, where $\sigma = (a = s_0 \leqq s_1 \leqq s_2 \leqq \cdots \leqq s_n = b)$ is a subdivision of (a, b) and $s_i \leqq s_i' \leqq s_{i+1}$. If these sums have a limit either as the max $(s_{i+1} - s_i)$ approaches zero or as σ spreads, we get a Stieltjes integral $\int_a^b B[y(s); dx(s)]$. This Stieltjes integral possesses bilinear properties in x and y as well as an integration by parts theorem. In order to obtain the existence of this integral for all continuous functions $y(s)$ on $\mathfrak{S}$ to $\mathfrak{Y}$, Gowurin proposes a generalization of the notion of bounded variation. The function $x(s)$ on $\mathfrak{S}$ to $\mathfrak{X}$ has finite W-variation relative to B and $\mathfrak{Y}$, if there exists a positive constant M such that $\| \sum_\sigma B[y_i; x(s_{i+1}) - x(s_i)]\| \leqq M \max_i \|y_i\|$ for all subdivisions σ of (a, b) and all $y_1, \cdots, y_n$ of $\mathfrak{Y}$. This condition on $x(s)$ is necessary and sufficient to guarantee the existence of $\int B[y(s); dx(s)]$ for all continuous functions $y(s)$ if B is bilinear bounded. It may be noted that if $\mathfrak{Y}=\mathfrak{X}^*$, the space of linear continuous forms on $\mathfrak{X}$, and $\mathfrak{Z}$ is the set of real numbers, then $x(s)$ is of finite W-variation if and only if it is of strong bounded variation on $\mathfrak{S}$; if $\mathfrak{Y}=\mathfrak{R}$, the space of reals, and $\mathfrak{Z}=\mathfrak{X}$, then finite W-variation reduces to weak bounded variation. These considerations find their application in the problem of finding the form of the most general linear forms on what might be called second story spaces, whose elements are functions on a range to an LNC space.

9. **The Price integral.** The Gowurin integrals are essentially of the Riemann type. Price [54] has made an extensive study of the corresponding situation using instead the Young-Birkhoff slant. The real-valued measure function $\mu(E)$ is replaced by a function on the class of "measurable" sets to the space $\mathfrak{T}$ of linear bounded transformations τ on the LNC space $\mathfrak{X}$ to $\mathfrak{X}$ with $\|\tau\|=\text{LUB}\ [\|\tau(x)\|$ for $\|x\|=1]$. The following properties of $\tau(E)$ generalize the measure function properties: (a) if $\tau(E_0)=0$, and $E\leqq E_0$, then $\tau(E)=0$; (b) if $\tau(E)\neq 0$ then $\tau(E)$ has a reciprocal in $\mathfrak{T}$; (c) $\tau(E)$ is completely additive on $\mathfrak{E}$, i.e., if $\sum_n E_n=E$ in $\mathfrak{E}$, E_n disjoint, then $\tau(\sum_n E_n)=\sum_n \tau(E_n)$, with unconditional convergence. One could obviously follow the Birkhoff path and consider sums $\sum\tau(E_n)x(s_n)$ with s_n in E_n over subdivisions into measurable subsets of $\mathfrak{S}$. Price prefers to inject a generalization of the notion of the convex extension of a set which plays a strong role in Birkhoff's definition. The generalization rests on the observation that the convex extension of a set X_0 is the totality of elements of the form $\sum_1^n c_i x_i$, with $c_i>0$, and $\sum_1^n c_i=1$, x_i in X_0, together with the fact that cx is a linear continuous transformation on $\mathfrak{X}$ to $\mathfrak{X}$ for fixed c. Hence, Price defines a convexification transformation C^* on X_0 to $\mathfrak{X}$ by the expression $\sum_i \tau_i x_i$ in terms of a class T of finite subsets t of $\mathfrak{T}$ satisfying the following conditions: (a) if $t=(\tau_1, \cdots, \tau_n)$, then $\sum_1^n \tau_i=I$ (I the identity transformation in $\mathfrak{T}$); (b) if $t_1=(\tau_{11}, \cdots, \tau_{1n})$ and $t_2=(\tau_{21}, \cdots, \tau_{2m})$ are in T, then $t=t_1t_2=(\tau_{1i}\tau_{2j},\ i=1, \cdots, n;\ j=1, \cdots, m)$ is in T; (c) there exists a constant W such that if t is in T then $\|\sum_1^n \tau_i x_i\|\leqq W\cdot\max\|x_i\|$. This last condition is a generalization of the W-variation of Gowurin. For the integral definition the convexification transformation C^* is defined in terms of the measure transformation function $\tau(E)$. The class T consists of the elements $t=[(\sum_{i=1}^{i=n}\tau(E_i))^{-1}\tau(E_j); j=1, \cdots, n]$ and $t'=[\tau(E_j)(\sum_{i=1}^{i=n}\tau(E_i))^{-1}; j=1, \cdots, n]$, the E_i being disjoint in $\mathfrak{E}$ and $\tau(\sum_i E_i)\neq 0$, together with the extension of this set of elements to make T multiplicative in accordance with condition (b) above. Then the integration definition follows the Birkhoff line based on $\sum_k \tau(E_k)C^*x(E_k)$ with $x(E_k)$ the set

of values of $x(s)$ for s on E_k. Thus $x(s)$ is integrable if there exists a subdivision σ_0 of σ, such that $x(s)$ is bounded on E_k, $\sum_{\sigma_1} \tau(E_k) C^* x(E_k)$ is unconditionally convergent, and the greatest lower bound of the diameter of this type of expression for all σ is zero.

10. **Gelfand-Pettis integral.** The realization that the totality of linear continuous functionals or forms on a LNC space transfers the onus of convergence and related properties from the space to the space of real numbers gives rise to a convenient set of weak properties. We shall denote by x^* an element of the space $\mathfrak{X}^*$ of linear forms of $\mathfrak{X}$. As is well known $\mathfrak{X}^*$ is a linear space normed and complete if $\|x^*\| = \text{LUB}\ [|x^*(x)|$ for $\|x\| = 1]$. The expression $x^*(x)$ or (x^*, x) is a bilinear form on $(\mathfrak{X}^*, \mathfrak{X})$. We then have a sequence x_n weakly convergent to x if, for every x^*, the sequence of real numbers (x^*, x_n) converges to (x^*, x), the function $x(s)$ on (a, b) to $\mathfrak{X}$ is weakly continuous if $(x^*, x(s))$ is continuous for every x^*, the function $x(s)$ is weakly measurable if $(x^*, x(s))$ is measurable for every x^*, $x(s)$ is weakly integrable on a measurable set E if $(x^*, x(s))$ is integrable on E for every x^*, and there exists an $x(E)$ of $\mathfrak{X}$ such that $\int_E (x^*, x(s))ds = (x^*, x(E))$. Then we define $\int_E x(s)ds = x(E)$. $x(s)$ is weakly integrable if for every measurable subset E of $\mathfrak{E}$, $x(s)$ is integrable on E. This definition for integrability for $\mathfrak{S}$ the real interval (a, b) was suggested by Gelfand [22] and extended by Pettis [52] to the case when $\mathfrak{S}$ is a general space with a class of measurable sets and completely additive measure function. Pettis also discusses the properties of the integral and its relation to other integral definitions.

Usually any weak property must be supplemented by additional conditions to insure a corresponding strong property in LNC spaces. So it turns out that a necessary and sufficient condition that a function $x(s)$ be measurable in the strong (or Bochner) sense is that $x(s)$ be weakly measurable and almost separately valued (i.e., by neglecting a suitable set of measure zero in $\mathfrak{S}$ the values $x(s)$ form a separable subset of $\mathfrak{X}$).[3] The fact that the Bochner and Birkhoff integrals are special cases of the Gelfand-Pettis integral is cleverly deduced from the fact that under this type of integration, the operation of integration and linear continuous transformation on a LNC space $\mathfrak{X}$ to a LNC space $\mathfrak{Y}$ are commutative in an optimum sense, viz., that any integral reducing to Lebesgue integration if $\mathfrak{X}$ is the space of reals which has this commutative property must be Gelfand-Pettis integrable. If $x(s)$ is strongly measurable, then the Gelfand-Pettis integral reduces to and agrees with the Birkhoff integral as well as a second integral definition of Dunford [14] requiring a function $x(s)$ to be integrable on $\mathfrak{S}$ if there exists a sequence of finite-valued measurable functions $x_n(s)$ converging to $x(s)$ almost everywhere and such that $\lim_n \int_E x_n(s)ds$ exists for all measurable subsets E. This latter definition connects up with the well known theorem that if a sequence of real-valued Lebesgue integrable functions $f_n(s)$ converges to $f(s)$ almost everywhere, then $\lim_n \int_E f_n(s)ds = \int_E f(s)ds$ for all measurable sets E if and only if the $\int_E f_n(s)ds$ are uniformly absolutely continuous. Incidentally then, the properties "$\int_E x_n(s)ds$ uniformly absolutely continuous" and "$\int_E x_n(s)ds$ convergent for all measurable E" are not equivalent in this more general setting since the first of these properties would imply the Bochner integrability of an $x(s)$ approached by a finite-valued sequence $x_n(s)$, but the second would not.

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[3] An elegant proof of this fact in E. Hille, Colloquium Lectures (*Functional analysis and semi-groups*) p. 36, centers in the fact that if a LNC space $\mathfrak{X}$ is separable, the sequence $\{x_n\}$ is dense in the space, and x_n^* are selected so that $\|x_n^*\| = 1$ and $x_n^*(x_n) = \|x_n\|$, then the forms $x_n^*(x)$ define an isometric linear transformation on $\mathfrak{X}$ to the space of bounded sequences. There results the measurability of $\|x(s) - x\|$ for each x of $\mathfrak{X}$ as the least upper bound of the sequence of measurable functions $|(x_n^*, (x(s) - x))|$. This suggests another definition of measurability, viz., $x(s)$ is measurable if the function $\|x(s) - x\|$ is measurable for every x of $\mathfrak{X}$. (See G. B. Price [54, p. 25].)

If $(x^*, x(s))$ is integrable for all x^*, then this expression defines a linear continuous transformation on the space $\mathfrak{X}^*$ to the space L^1 of Lebesgue integrable functions on $\mathfrak{S}$. (See Pettis [52, p. 286].) Consequently, $\int_E (x^*, x(s)ds$ is a linear continuous form on $\mathfrak{X}^*$ for every E, that is, there exists for each measurable E an element $x^{**}(E)$ of $\mathfrak{X}^{**}$, the conjugate of $\mathfrak{X}^*$, such that $\int_E (x^*, x(s)ds = (x^*, x^{**}(E))$. It is only when $x^{**}(E)$ is an element of $\mathfrak{X}$ for every E that we get the Gelfand-Pettis integral, which seems natural since we should like our integral values in the same space $\mathfrak{X}$ with which we started. If, however, one is not averse to finding one's integral values in an extended space (after all Lebesgue integration notions involve the ideas of extension), then with Dunford [16] one has integrability of $x(s)$ if $(x^*, x(s))$ is integrable for every x^*, $\int x(s)ds$ being defined as the x^{**} indicated. Pettis [52, p. 293] shows that if $x(s)$ is also strongly measurable, then x^{**} lies in $\mathfrak{X}$ for each E and the Dunford third integral becomes a Pettis integral. There is an alternative way of looking at the Dunford third integral suggested by Phillips [53, p. 130]. Any member of $\mathfrak{X}$ is also a member of $\mathfrak{X}^{**}$. Consequently defining the integrability of $x(s)$ in terms of the integrability of $(x^*, x(s)) = (x^{**}(s), x^*)$ is a weak integrability, a stepping down from $\mathfrak{X}^{**}$ in its properties, what one might call a subweak integrability.

11. **Phillips-Rickart generalizations.** In the case of a LNC space $\mathfrak{X}$, we have in addition to the topology introduced by the norm, the

topology associated with weak convergence of linear forms (x^*, x). These give rise to a topological space, a vicinity $V(x_1^*, \cdots, x_n^*; x_0; \epsilon)$ being defined by the x for which $|x_i^*(x-x_0)| < \epsilon$; $i=1, \cdots, n$. For $\mathfrak{X}^*$ we have two types of weak convergence, that induced by $\mathfrak{X}$ and that by $\mathfrak{X}^{**}$. Both of these are special cases of linear topological spaces, spaces which are linear and have a topology in which addition and multiplication by real numbers is continuous, and which are determined by a system of vicinities of the origin satisfying the Hausdorff postulates (see v. Neumann [48]; Kolmogoroff [37]). In addition, for elegant results, it is assumed that the vicinities are convex (if x_1 and x_2 belong to a vicinity, then so does $ax_1+(1-a)x_2$ for $0 \leqq a \leqq 1$). In that case any vicinity determines, for all x, a pseudo norm $\|x\|_V$ satisfying the norm properties, excepting that $\|x\|_V=0$ does not necessarily imply $x=0$ (if $\|x\|_V=0$ implies $x=0$, then the linear topological space becomes a linear normed space with $\|x\|_V$ as norm). We shall call such a linear topological space with convex vicinities a LCT space.

The Phillips generalization of the integration problem [53] (see also Birkhoff, Ann. of Math. vol. 38 (1937) p. 51) consists in the first place in playing in LCT spaces which include the LN spaces and the weak convergence on normed spaces as used by Pettis. The properties of real-valued functions not being available in this setting, an adaptation of the method used by Birkhoff for defining integrability becomes desirable. Phillips notes that in this type of definition (as in fact in all Lebesgue integrability definitions), there is involved an iterated limit. First one defines the unconditional convergence of an infinite series $\sum_i x(s_i)mE_i$ for some subdivisions and then takes the limit of these sums as to subdivisions, i.e., $\lim_\sigma \lim_\pi \sum_\pi x(s_i)mE_i$, where σ stands for a subdivision and π a finite number of integers. One can, however, define an iterated limit without assuming the existence of the interior limit simply by replacing lim lim by lim $\overline{\underline{\lim}}$. In the definition of integral this means that the approximating sums need be in a sense only approximately convergent or summable. Thus, according to Phillips, a sequence of elements x_n of $\mathfrak{X}$ is unconditionally summable to x relative to a vicinity V if there exists a subset π_V of the integers such that if $\pi \geqq \pi_V$, then $\|\sum_\pi x_n - x\|_V < 1$. A similar definition can be given for unconditional summability of a sequence of subsets X_n of $\mathfrak{X}$ to an element x. Given now a space $\mathfrak{S}$, a class $\mathfrak{E}$ of measurable subsets E, a completely additive measure function α on $\mathfrak{E}$ to $\mathfrak{A}$, a many-valued function $x(E)$ on $\mathfrak{E}$ to $\mathfrak{X}$ (being for instance the totality of values of a point function $x(s)$ for s on E), then $x(E)$ is integrable relative to $\alpha(E)$ if for every measurable E

of $\mathfrak{E}$, there exists an element $J(x, E)$ of $\mathfrak{X}$ such that for every vicinity V, there exists a subdivision σ of E such that $\sum_\sigma x(E_i)\alpha(E_i)$ is unconditionally summable to $J(x, E)$ relative to V. In order to prove existence theorems, it is necessary to impose completeness conditions relative to a Moore-Smith mode of convergence depending on the directed set defined by the product space $\mathfrak{V}\mathfrak{P}$ of vicinities V and subsets π of the integers. Most of the usual properties of Lebesgue integration carry over, including the interchangeability of continuous linear transformation on a LCT space to a LCT space and integration.

Rickart [55], following the ideas of Phillips, and generalizing the definitions of Burkill [9] and Kolmogoroff [36], drops the measure function $\alpha(E)$ and is concerned only with a set function, which may be many-valued, on "measurable" subsets to a LCT space $\mathfrak{X}$. Further he is willing to consider a multiple-valued integral. He defines two subsets X_1 and X_2 to be equal within a vicinity V if X_1 is contained in X_2+V, and X_2 contained in X_1+V. Then a sequence of sets X_n is unconditionally summable to a set X within V, if there exists a subset of the integers π_V such that if $\pi \geqq \pi_V$, then $\sum_\pi X_n$ and X are equal within V. Then $x(E)$ is integrable on E_0 if there exists a subset $I(x, E_0)$ of $\mathfrak{X}$ such that $\lim_\sigma \sum_\sigma x(E) = I(x, E_0)$. This means that for every V, there exists a subdivision σ_V of E_0 such that if $\sigma \geqq \sigma_V$, then $\sum_\sigma x(E_i)$ is unconditionally summable to $I(x, E_0)$ within V. The closure of the set $I(x, E_0)$ is the integral set of x over E_0. The case where $I(x, E_0)$ reduces to a single element is, of course, of special interest, but many integration properties can be proved for the integral as a closed set of elements.

Rickart considers also the special case when $x(E)$ is a bilinear function on a LCT space $\mathfrak{Y}$ and the class $\mathfrak{E}$ of measurable subsets of $\mathfrak{S}$, viz., $x(E) = B(y; E)$, generalizing the integral of Gowurin [26]. Of course, in a bilinear transformation on $\mathfrak{Y}\mathfrak{Z}$ to $\mathfrak{X}$, if $y(s)$ is a point function on $\mathfrak{S}$ to $\mathfrak{Y}$, and x a single-valued set function $z(E)$ on $\mathfrak{E}$ to $\mathfrak{Z}$, explicit mention of the z function is not really required. Rickart assumes $B(y, E)$ to be single-valued linear in y for each E, and completely additive in E for each y. In addition, it is assumed that there exists a real number $a \geqq 1$, such that if Y_i, $i=1, \cdots, n$, are any subsets of $\mathfrak{Y}$, E_i are disjoint, E_i^j, $j=1, \cdots, k_i$, constitute a finite subdivision of E_i, then $\sum_i B(Y_i, E_i) \leqq V$ implies $\sum_i \sum_j B(Y_i, E_i^j) < aV$. An integral definition following the lines of the Phillips definition can be set up for $B(y(E), E)$. If $y(E)$ is a contractive function of sets, i.e., if $E' \leqq E''$ implies $y(E') \leqq y(E'')$, then this integral of $B(y, E)$ is a special case of the general definition. Rickart also shows that by a proper definition of $B(y, E)$, the integral definition of Price [**54**] is included in the above considerations as a special case.

II. GENERALIZATION OF LEBESGUE INTEGRATION BASED ON ORDER

The generalizations of Lebesgue integration which we have been considering have been only indirectly concerned with the Lebesgue postulates mentioned at the beginning of this address. They have centered more in the consequences of these postulates, the notions of measurable sets and measurable functions. More closely connected with Lebesgue postulates is the general integral of Daniell [**11**; **12**]. In the Daniell considerations, the basic idea is that the Lebesgue integral can be considered as an extension of the Riemann integral, in that the Lebesgue integral, reducing to the Riemann integral for continuous functions, applies to a larger class of functions and on this larger class has more extensive properties.

12. **The Daniell integral.** Daniell assumes as a basic class of elements the class $\mathfrak{F}$ of all real finite-valued functions on a general range $\mathfrak{S}$. From $\mathfrak{F}$ we select a subset F_0 of functions f, which is linear and with f contains also $|f|$. Under linearity, this is equivalent to the assumption that with f_1 and f_2, $f_1 \cup f_2$, the greater of f_1 and f_2, and $f_1 \cap f_2$, the lesser of f_1 and f_2, belong to F_0. In other words, F_0 is a vector or linear lattice subset of $\mathfrak{F}$. On F_0 there is defined an integral $I(f)$ to real numbers, assumed to be linear (L) and positive (P) on F_0, i.e., satisfying postulates (3) and (4) of Lebesgue. In addition, I satisfies an equivalent of the convergence property (6) of Lebesgue (Cv): if f_n is a monotonic nondecreasing sequence of functions of F_0 converging to the zero function, then $\lim_n I(f_n) = 0$. Note that from postulates (L) and (P) it follows that if $\lim_n f_n = f$ relative uniformly as to F_0 (there exists a function f_0 of F_0 such that for every $e > 0$, there exists an n_e such that if $n > n_e$, then $|f_n - f| < e\ |f_0|$), then $\lim_n I(f_n) = I(f)$. There is also assumed a "Stieltjes" integral $S(f)$, which satisfies the linearity and convergence postulates but replaces the positive property by a boundedness property: there exists a monotone transformation M on positive functions of F_0, such that if f is in F_0 then $|S(f)| \leqq M(|f|)$. This is equivalent to assuming that for $S(f)$ there exist two linear positive integrals I_1 and I_2 such that for every f of F_0: $S(f) = I_1(f) - I_2(f)$; or that $S(f)$ transforms a set bounded in F_0 into a bounded set of numbers. As a consequence, most of the theory centers on the I integral, having the (L) (P) (Cv) properties. Conceptually F_0 abstracts the class of continuous functions or the step functions on a closed interval (a, b), and I the usual Riemann integral.

The objective is to obtain a larger class $\mathfrak{L}$, a subset of $\mathfrak{F}$ and containing F_0, of "integrable" functions and an integral I on $\mathfrak{L}$ reducing to I on F_0. The class $\mathfrak{L}$ should be a linear lattice. The integral I should be linear and positive and have the following convergence properties, suggested by Lebesgue integration: (a) if $\{h_n\}$ is a monotone nondecreasing sequence of $\mathfrak{L}$, converging to h in $\mathfrak{F}$, and if $I(h_n)$ is a bounded sequence, then h is in $\mathfrak{L}$ and $\lim_n I(h_n) = I(h)$; (b) if h_n is a sequence belonging to $\mathfrak{L}$ for which there exists a function h_0 of $\mathfrak{L}$ such that $|h_n| \leq |h_0|$ for all n, and converging to a function h in $\mathfrak{F}$, then h is in $\mathfrak{L}$ and $\lim_n I(h_n) = I(h)$.

In the Daniell procedure there is introduced an auxiliary class of functions F_1, being the limits of monotone increasing sequences of functions of F_0, which may take $+\infty$ as values. If we denote functions of F_1 by g, then $I(g)$ is defined as $\lim_n I(f_n)$ where f_n is any monotone increasing sequence in F_0 converging to g, and may have $+\infty$ as value. F_1 is closed under extension by monotone increasing sequences. For elements of $\mathfrak{F}$ we now define an upper integral by the condition: if h is in $\mathfrak{F}$ and no function of F_1 dominates h, then $\bar{I}(h) = +\infty$; otherwise $\bar{I}(h) = \text{GLB}\ [I(g)$ for all g of $F_1 \geq h]$, $+\infty$ being a permissible value. It might be noted here that the subsidiary class F_1 could be omitted from this definition if we defined $\bar{I}(h) = \text{GLB}\ [\lim_n I(f_n)$ for all monotone increasing sequences $\{f_n\}$ from F_0 such that $\lim_n f_n \geq h]$. This form bears a close relationship to the definition of upper measure of a set E as the greatest lower bound of the sums of the lengths of intervals covering E.

This upper integral has the usual properties, viz., (a) $\bar{I}(h_1+h_2) \leq \bar{I}(h_1)+\bar{I}(h_2)$; (b) $\bar{I}(ch) = c\bar{I}(h)$ if $c \geq 0$; (c) if $h_n \geq 0$, and $0 \leq h \leq \sum_n h_n$, then $\bar{I}(h) \leq \sum_n \bar{I}(h_n)$. A lower integral is defined by $-\bar{I}(-h)$, and now a function h in $\mathfrak{F}$ is said to be integrable and in class $\mathfrak{L}$ if $\bar{I}(h) = -\bar{I}(-h)$ with $\bar{I}(h)$ finite, the value of the integral $\bar{I}(h)$ being the common value. The (Cv) property of I on F_0 guarantees that if f is in F_0 it is also in $\mathfrak{L}$, the value of $I(f)$ being unchanged. This class of functions $\mathfrak{L}$ has the Lebesgue properties mentioned above. Moreover, it is possible to prove that the class F_0 is dense in $\mathfrak{L}$, in the sense that h belongs to $\mathfrak{L}$ if and only if for every $\epsilon > 0$ there exists an f_ϵ in F_0 such that $\bar{I}(|h-f_\epsilon|) < \epsilon$. Moreover, if the class $\mathfrak{L}$ is suitably extended to include functions assuming $\pm\infty$ as values, then this extended class can be shown to be complete under the metric $I(|h_1-h_2|)$.

Banach [1] has given another approach to the Daniell type of integral generalization. F_0 is again a linear lattice subset of the space $\mathfrak{F}$ of finite-valued functions on $\mathfrak{S}$, I is linear and positive on F_0, but the convergence condition (Cv) is replaced by one suggested by the

Osgood theorem or the convergence property (b) of Lebesgue integrals, viz., if the sequence f_n is bounded in F_0 and converges to zero, then $\lim_n I(f_n)=0$. This is equivalent to: if f_n is bounded below in F_0 and $\underline{\lim}_n f_n \geqq 0$, then $\underline{\lim}_n I(f_n) \geqq 0$. It is possible to show that these conditions on I are equivalent to those of Daniell, though to deduce the Banach condition (Cv) from the Daniell ones the extension theory may be needed. The definition of upper and lower integrals is applied to a subset H of functions of $\mathfrak{F}$, H being the class of functions $h(s)$ for which there exist sequences of functions f_n' and f_n'' of F_0 such that

$$\overline{\lim}_n f_n' \leqq h \leqq \underline{\lim}_n f_n''.$$

It is easy to show that the class H agrees with the functions h of $\mathfrak{F}$ for which in the Daniell formulation there exists a function g_1 of F_1 and a function g_2 with $-g_2$ in F_1 such that $g_2 \leqq h \leqq g_1$. For functions h of H, $\bar{I}(h)$ is the greatest lower bound of $\underline{\lim}_n I(f_n)$ for all sequences f_n bounded below in F_0 such that $\underline{\lim}_n f_n \geqq h$; the definition of a lower integral follows the complementary procedure. The ultimate class $\mathfrak{L}$ of integrable functions for which the upper and lower integrals are equal to the same finite number agrees with the class $\mathfrak{L}$ as defined by Daniell. A procedure similar to that of Banach is presented by Goldstine [23]. It hardly deserves mention that the Daniell procedure centering on monotone sequences is conceptually simpler than the Banach procedure involving $\underline{\lim}$ and $\overline{\lim}$.

[Since this address was given another approach to the Daniell integral has been given by M. H. Stone [61, I]. The only change in the postulates on F_0 and the integral I on F_0 is in the convergence postulate (Cv) which in the Stone version reads: If f_n and f belong to F_0 and $|f| \leqq \sum_n |f_n|$, then $|I(f)| \leqq \sum_n I(|f_n|)$. These postulates are actually equivalent to the Daniell set if the latter's (Cv) postulate is stated in the form: If f_n is any monotonic increasing sequence from F_0 converging to f in F_0, then $\lim_n I(f_n)=I(f)$. Stone does not introduce the class F_1 but defines an upper integral for positive functions h in $\mathfrak{F}$ by the condition $\bar{I}(h)$ is the greatest lower bound of $\sum_n I(f_n)$ for all sequences $f_n \geqq 0$ from F_0 such that $\sum_n f_n \geqq h$. Since under these conditions $\sum_n f_n$ belongs to F_1, it is obvious that the upper integrals for positive-valued functions for Stone and Daniell agree. The interesting new thing is that if $\mathfrak{F}$ is extended to include functions $\bar{h}$ having $+\infty$ as values and $\bar{I}(\bar{h})$ is defined as above, for positive functions, then $\bar{I}(|\bar{h}|)<\infty$ only if $\bar{h}$ differs from an h in $\mathfrak{F}$ by a null function, one for which $\bar{I}(|\bar{h}-h|)=0$. Then $\bar{h}-h$ vanishes excepting on a null set E, one for which $\bar{I}(\chi_E)=0$ where χ_E is the characteristic function of E. Further, it develops that the space for which $\bar{I}(\bar{h}) < \infty$ is a linear normed complete space with $\bar{I}(|\bar{h}|)$ as norm. The space of L-integrable functions is then the completion of the space F_0 under the norm $I(|f|)$. This is reminiscent of the definition of Lebesgue integration due to Dunford [**15**].

A more general approach to an integral of the Daniell type has been suggested by the Bourbaki group via Stone [61, IV] and independently by McShane [**43**]. The principal change is in the class F_1 of limits of monotone increasing sequences f_n. The sequence is replaced by a generalized monotonic or directed set, such a set being assumed to be monotone according to the order relation "$\leqq$" denoted by Δ if for any two functions f_1 and f_2 of the set there exists a function f_3 in the set dominating f_1 and f_2: $f_3 \geqq f_1$, $f_3 \geqq f_2$. Obviously, the least upper bound of such a directed set agrees with its Moore-Smith limit with Δ (or $\leqq$) as the order relation. The convergence postulate (Cv) is now strengthened to apply to Δ-monotone sets, i.e., if $\{f_\Delta\}$ is a Δ-monotone set in F_0 converging to an f in F_0, then $\lim_\Delta I(f_\Delta) = I(f)$. The class F_1 of Daniell is replaced by the class F_1' of limits of Δ-monotone sets of functions of F_0, and the upper integral defined in terms of functions of F_1'. The difference between the class of functions integrable by the Daniell-Stone sequential procedure and by Δ-monotone sets is a matter of null functions in the sense that if h belongs to $\mathfrak{L}_\Delta$, then $h = h_s + h_\nu$, where h_s is a sequential $\mathfrak{L}$ function and $\bar{I}_\Delta(|h_\nu|) = 0$.]

13. **Measurability of functions and sets in Daniell integrals.** In the case of Lebesgue integrals, measurability of functions is tied up with measurable sets. Since for Daniell integrals sets play no role, measurability must be characterized in terms of integrability. We note that for Lebesgue integrals, integrability and measurability are equivalent for bounded functions on a closed interval; for unbounded functions, measurability is equivalent to the integrability of every truncated part of a function, i.e., f is measurable if for every $a \leqq b$ the function $f_{ab} = a \cup f \cap b$ (equal to a for $f \leqq a$, equal to f for $a \leqq f \leqq b$, equal to b for $f \geqq b$) is integrable. Since in the case of Daniell integrability it is not assured that $f = 1$ is integrable (Daniell purposely avoided this) we get measurability relative to any positive integrable finite-valued function h, f being measurable relative to h if for all $a \leqq b$ the function $f_{ab} = ah \cup f \cap bh$ is integrable. Daniell [12, p. 210] shows that if a function $f \geqq 0$ is measurable relative to h, if $\underline{I}(f) < \infty$, and there exists a finite-valued function θ such that $f = \theta h$, then f is integrable. [Stone [61, II] defines a function f to be measurable in case for any two integrable functions h_1 and h_2, the median function of f, h_1 and h_2 is

integrable, the median of three numbers a, b, c being obviously the number between the two others, in symbols $\bigcup[a\cap b, a\cap c, b\cap c] = \bigcap[a\cup b, a\cup c, b\cup c]$. Then any integrable function is measurable. Obviously, a function, in order to be measurable in the Stone sense must be measurable relative to every positive integrable function in the Daniell sense. The converse is also true.]

For *measurable sets* there are various modes of procedure, each centering in the characteristic function χ_E of the set. On the one hand (a) we can say that E is measurable if the corresponding characteristic function is integrable; or (b) we can say that E is measurable if χ_E is measurable; again, we can observe (c) that if for every E we consider the upper integral $\bar{I}(\chi_E)$, we have defined on E a Carathéodory upper measure, which gives rise to a class of measurable sets via the usual condition: M is measurable if and only if for all E, $\mu^*E = \mu^*EM + \mu^*(E-EM)$, where μ^* is the upper measure, and EM is the product of the sets E and M. Obviously, the class of measurable sets may prove quite trivial. In case the unit function $h \equiv 1$ is integrable, Goldstine [23] has shown that a function h is integrable if and only if its integral is expressible in the form $\int h(s)d\mu$, where μ is the Carathéodory measure function defined by $\bar{I}(\chi_E)$. [Stone [61, II] has shown that measurability, as defined by him, of the unit function $h \equiv 1$ is a necessary and sufficient condition for the equivalence of measurability of a set E according to (b) and (c) above. In that case it is also possible to show that a finite-valued function is measurable if and only if the sets $E[a \leqq f \leqq b]$ are measurable for all $a < b$, and the integral of an integrable function can be defined after the Lebesgue manner.]

14. **Extensions of Daniell integrals.** An examination of the Daniell procedure reveals the fact that the real-valued functions on a general set $\mathfrak{S}$, for which an integral is sought, could be replaced by functions to more general spaces, having the linearity and order properties of the real numbers. The most convenient such spaces are linear partially ordered and lattice spaces $\mathfrak{X}$. A linear partially ordered space is one in which an order $\geqq$ is defined between some elements of the space, subject to the conditions (a) if $x_1 \geqq x_2$, then for every x: $x_1 + x \geqq x_2 + x$, (b) $a \geqq 0$ and $x \geqq 0$ imply $ax \geqq 0$. The space $\mathfrak{X}$ is a lattice if for each x_1, x_2 the greatest lower bound $x_1 \cap x_2$ and the least upper bound $x_1 \cup x_2$ exist in the space; $\mathfrak{X}$ is a σ-complete linear lattice if any sequence x_n bounded above (bounded below) has a least upper bound (greatest lower bound), a complete lattice if sequence is replaced by set. (See Kantorovic [34] and Birkhoff [5, p. 238].) If we assume $\mathfrak{F}$ to be the set of all functions on a general set $\mathfrak{S}$ to a linear complete

lattice $\mathfrak{X}$, F_0 to be a linear sublattice of $\mathfrak{F}$, I on F_0 to real numbers, linear, positive and having the (Cv) property of Daniell, then the Daniell extension theory is possible on elements of $\mathfrak{F}$, a Lebesgue type of integral exists having the Lebesgue convergence properties on a subset $\mathfrak{L}$ of $\mathfrak{F}$ of "integrable" functions, convergence of the functions being of the order type. (Cf. Birkhoff [5, pp. 59 ff.].) By a proper treatment of functions taking on "extended" values it would seem to be possible to assert the completeness of the space of integrable functions under the norm $I(|h|)$, where $|h| = h^+ + h^-$, with $h^+ = h \cup 0$ and $h^- = -h \cup 0$. The theory indicated has been carried through by M. Orihara and G. Sunouchi [51].

From these considerations it is obviously a simple matter to drop the idea that $\mathfrak{F}$ is a class of functions and replace it by a set of elements having an adequate number of the properties of the class of all finite real-valued functions on a general range. Such a set of elements is a linear σ-complete lattice. Then F_0 is again a linear sublattice of $\mathfrak{F}$, I is a linear positive form on F_0 to real numbers, having the convergence property (Cv): if f_n of F_0 is a monotone decreasing sequence such that $\bigcap_n f_n = 0$, then $\lim_n I(f_n) = 0$. A "Stieltjes" integral S would be linear, have the (Cv) property, but replace the positive property by a boundedness condition, e.g., if F_0' is a bounded subset of F_0, then the set of numbers $S(F_0')$ is bounded. The class of functions F_1 used in the definition of upper integral can be by-passed by defining the upper integral $\bar{I}(h)$ as the greatest lower bound of $\lim_n I(f_n)$ for all monotone sequences f_n chosen from F_0 for which $\bigcup_n (h \cap f_n) = h$. If no such sequence exists, then $I(h) = +\infty$. It is possible to introduce a parallel to the class F_1 of Daniell which may include functions having $+\infty$ as a value. We simply assume F_1 to consist of all monotone nondecreasing sequences in F_0. Two such sequences f_{1n}, f_{2m} define the same element if $\bigcup_n (f_{1n} \cap f_{2m}) = f_{2m}$ for every m and $\bigcup_m (f_{1n} \cap f_{2m}) = f_{1n}$ for every n. If a sequence has a least upper bound in $\mathfrak{F}$, the sequence is equivalent to this element. The definition of order in F_1 and relative to $\mathfrak{F}$ is obvious, as is the extension of the integral I to F_1. It can be shown, that the upper integral $\bar{I}$ has the usual properties. This also holds for the resulting I integral for which $\bar{I}(h) = -\bar{I}(-h)$ finite, at least as applied to elements in $\mathfrak{F}$. There is, however, some difficulty about the completeness of the class of integrable functions since it is not possible to speak of a "null" set, only of null elements for which $\bar{I}(|h|) = 0$. Izumi and Nakamura [32, III] have suggested the above extension process based on the postulates and procedure of Banach [1]. Matsuyama [44] has followed the Goldstine [23] presentation and given more detail. Reference should also be made to Nakano [47, II].

The next obvious generalization is to replace the value space of the integral I on F_0 by a more general space, having the real numbers as a special case, for instance, by a linear lattice $\mathfrak{Y}$ assumed to be complete. The I-integral then becomes a linear order preserving transformation satisfying the (Cv) postulate. The S-integral replaces the order preserving condition either by a boundedness condition or equivalently assumes it to be the difference of two positive I-transformations. An upper integral is definable on elements of $\mathfrak{F}$ and selects from $\mathfrak{F}$ a linear subclass by the condition that $\bar{I}(h)$ be in $\mathfrak{Y}$. It has the properties $\bar{I}(ah)=a\bar{I}(h)$ for $a\geqq 0$, and $\bar{I}(h_1+h_2)\leqq\bar{I}(h_1)+\bar{I}(h_2)$, but there is doubt whether the convergence property: if $h_n\geqq 0$, and $0\leqq h\leqq\sum_n h_n$, then $\bar{I}(h)\leqq\sum_n\bar{I}(h_n)$ holds. For the class of integrable functions for which $\bar{I}(h)=-\bar{I}(-h)$, it is possible to assert linearity, but the Lebesgue convergence properties may not be valid. We might, however, note that the Hahn-Banach extension theorem [3, p. 28] on linear order preserving transformations on a σ-complete linear lattice $\mathfrak{X}$ to a complete linear lattice $\mathfrak{Y}$ is valid. [See Izumi, Matuyama, and Orihara [32, V].] Hence, it would be possible to extend the linear order preserving integral I on F_0 to a linear order preserving integral on the linear subset $\bar{F}$ of $\mathfrak{F}$ for which $\bar{I}(\bar{h})$ is in $\mathfrak{Y}$, and satisfying the condition $I(\bar{h})\leqq\bar{I}(\bar{h})$ for all $\bar{h}$ in $\bar{F}$. This would be uniquely determined on the set of $\bar{h}$'s for which $\bar{I}(\bar{h})=-\bar{I}(-\bar{h})$.

15. **Special cases.** The Daniell integral theory, considering the integral as an order preserving transformation on a linear lattice to a linear lattice, assumes that the "integral" on the basic set F_0 is given. Bochner and Fan [7] develop such an integral in connection with the problem of determining the most general linear order preserving transformation on the space of continuous functions on a finite interval to a linear partially ordered monotonely σ-complete space $\mathfrak{X}$ (if x_n is a monotone increasing sequence bounded in $\mathfrak{X}$, then $\bigcup_n x_n=\lim_n x_n$ exists in $\mathfrak{X}$). They find that such a transformation takes the form of $\int f(s)\alpha(dI)$, where $\alpha(I)$ is a non-negative additive interval function on (a, b) to $\mathfrak{X}$, the integration being the order limit in the sense of successive subdivisions of (a, b). The generalization to order preserving linear transformations on the set of all bounded real-valued functions on a general space $\mathfrak{S}$ is easily made, the result being $T(f)=\int f(s)\alpha(dE)$ where $\alpha(E)$ is a non-negative additive set function on all subsets E of $\mathfrak{S}$ to $\mathfrak{X}$, and $\mathfrak{X}$ is a linear partially ordered space with a Moore-Smith monotone complete property (if x_p is a directed set monotone increasing in p, and bounded in $\mathfrak{X}$, then the least upper bound of x_p exists in $\mathfrak{X}$). The integral can be defined as the common value of $\bigcap_\sigma\sum_\sigma M_i\alpha(E_i)$ and $\bigcup_\sigma\sum_\sigma m_i\alpha(E_i)$, where M_i

and m_i are respectively the least upper bound and greatest lower bound of f on E_i and $\sigma = (E_1, \cdots, E_n)$ stands for any subdivision of $\mathfrak{S}$ into a finite number of disjoint sets.

These special instances suggest that it should be possible to work out theories of integration paralleling to some extent that on normed linear spaces. We could consider the case where the integrand is of the form $f(s)\alpha(E)$, f real-valued on a general set $\mathfrak{S}$, $\alpha(E)$ defined on "measurable" subsets of $\mathfrak{S}$ to a linear partially ordered, or linear lattice space with proper completeness properties, $\alpha(E)$ either finitely additive or if completely additive then in the sense of unconditional convergence. We might note that if for a real-valued function $f(s)$ we define the function of intervals $\alpha(t_1, t_2) = \chi_E(s)$, where $E = [s \mid t_1 \leqq f < t_2]$, then $\int_{-\infty}^{\infty} t\,d\alpha(I)$ is such an integral and reproduces the function f. The Lebesgue integral, when it exists, permits one to interchange integrations: $\int f = \int_{-\infty}^{\infty} t\,d\int \alpha(I)$. The resolution of the identity for bounded Hermitian transformations in Hilbert space is another such instance.

The complementary situation where f is on $\mathfrak{S}$ to $\mathfrak{X}$, a linear partially ordered or lattice space and $\alpha(E)$ a real-valued function on certain subsets of $\mathfrak{S}$ with proper additive properties, is obviously possible also. Izumi [32, VII] following Bochner's integration theory on LNC spaces has carried through this type of definition assuming $\mathfrak{S}$ to be the linear interval (a, b). He also points out that the functions on $\mathfrak{S}$ could be replaced by a linear σ-complete lattice $\mathfrak{B}$ with a sublattice V_0 (step functions) on which I is defined with values in a σ-complete lattice $\mathfrak{Y}$. The extension of I to a larger subset of $\mathfrak{B}$ follows the extension method suggested by MacNeille [41], an element f of $\mathfrak{B}$ being integrable if there exists a sequence of positive elements of V_0: u_n such that $f = \sum_n u_n$ and $\sum_n I(u_n)$ converges in the order sense.

16. **Carathéodory theory of integrals over spaces without points.** Most of the integrals we have been considering operate on functions, but are functions on a general class of elements. In particular measure functions are on certain classes of subsets of a given class. In this connection there is an obvious suggestion to replace the subsets of a given space by a class of elements, which has some of the properties of the class of all subsets of a given class. This suggestion was made by Carathéodory [10] and expanded by others, e.g., Wecken [64], Ridder [57], Olmsted [50], and A. Pereira Gomes [24].

We assume a basic class $\mathfrak{E}$ of elements E, which form a Boolean algebra, having a null element 0, an all element 1, addition or join, multiplication or meet, and a complementing process. If the sum or join of any sequence of elements exists, we shall call $\mathfrak{E}$ σ-complete. If the join of any set of elements exists, we shall call $\mathfrak{E}$ complete. A set of "measurable" sets then would correspond to a σ-complete subalgebra containing the null and all element. Obviously various combinations are possible as in set theory.

A Carathéodory upper measure function μ^* on $\mathfrak{E}$ satisfies the conditions (a) $0 \leqq \mu^* E \leqq \infty$; (b) $\mu^*(E_1 \cup E_2) \leqq \mu^* E_1 + \mu^* E_2$; (c) if $E \leqq \bigcup_n E_n$ (which if $\bigcup_n E_n$ does not exist might be replaced by the condition $E = \bigcup_n (E_n \cap E)$), then $\mu^* E \leqq \sum_n \mu^* E_n$. A class of measurable elements M would be defined by the condition $\mu^* E = \mu^* E \cap M + (E \cap CM)$, CM complementary of M, for all E of $\mathfrak{E}$. Such an outer measure function can be deduced from any non-negative function $p(E)$ on $\mathfrak{E}$ with $\mu^* E$ the greatest lower bound of $\sum_n p(E_n)$ for all $\bigcup_n E_n \geqq E$, which is similar to the Daniell definition of upper integral. The resulting measure function and class of measurable sets may be trivial. Usually one postulates a set $\mathfrak{M}$ of measurable elements M and a completely additive measure function on $\mathfrak{M}$, with the additional condition that there exists a sequence of elements M_n in $\mathfrak{M}$ such $\mu(M_n) < \infty$ for all n and $\bigcup_n M_n = 1$.

One is faced with the problem of defining a "point" function in spite of the fact that no "point" elements are postulated in $\mathfrak{E}$. For bounded functions, Carathéodory notes that in a sense a "point" function is determined by its least upper bound and greatest lower bound on the sets to which the "point" belongs. He therefore assumes that a point function is determined by two functions α, β on $\mathfrak{E}$ to real numbers, α decreasing and β increasing. There is also assumed a dense sequence of values $\{y_n\}$ on the finite interval to which α and β are defined and a corresponding sequence of elements E_n such that $\beta(E_n) \leqq y_n$ and for every element E such that $E \cap E_n = 0$ we have $\alpha(E) \geqq y_n$. This definition is conceptually rather complicated, due perhaps to the desire of defining a "point" function in terms of real-valued set functions. Another somewhat simpler approach to the Carathéodory type of point function (function on $\mathfrak{E}$ to real numbers) has been given by Kappos [35]. Kappos defines first simple functions (essentially denumerably-valued step functions) and obtains a general point function as a Dedekind cut in this class of functions which constitutes a linear lattice. The process of defining a point function is much simplified if we reverse our desires and define a point function as a function on the reals $-\infty < y < \infty$ to elements of $\mathfrak{E}$. We note, for instance, that for a real-valued point function the sets (a) $E_y = E[f(s) = y]$; (b) $E_y = E[f(s) \geqq y]$; (c) $E_y = E[f(s) > y]$; (d) $E_y = [f(s) \leqq y]$; and (e) $E_y = [f(s) < y]$, defining functions on y to $\mathfrak{E}$, all determine the point function f. Corresponding to (a) we would define a function E_y on $-\infty < y < \infty$ a single-valued point function if it satisfies the additional conditions (1) $\bigcup_y E_y = 1$, (2) $y_1 \neq y_2$ implies $E_{y_1} \cap E_{y_2} = 0$. Condition (b) would require E_y to satisfy the following additional conditions (1) $\bigcup_y E_y = 1$, $\bigcap_y E_y = 0$, (2) $y_1 \leqq y_2$ implies $E_{y_1} \geqq E_{y_2}$, (3) $\bigcap_{y<y_1} E_y = E_{y_1}$ or $E_{y-0} = E_y$ for all y. There are corresponding obvious conditions for the forms (c), (d), (e). The product involved in (3) exists if $\mathfrak{E}$ is σ-complete, because of the monotonic assumption (2). If $\mathfrak{E}$ is complete, it is obvious that one can define a point function of type (b) from one of type (a). Point functions of type (b) with the basic class forming a σ-complete Boolean algebra are usually taken as a basis for study. See, e.g., Olmsted [50]. Obviously, such a function is completely determined by its values at a denumerably dense set on $-\infty < y < \infty$. By following through the methods used in proving that the ordinary class of measurable functions is a linear σ-complete lattice, it is possible to show that the set of all point functions form such a linear σ-complete lattice, but even more that they form a ring of functions (one easily defines f^2; then $4fg = (f+g)^2 - (f-g)^2$). The road to integrability is obvious. If $\mathfrak{M}$ is the class of "measurable" elements M with the completely additive measure function μM, and E_y is a point function on $(-\infty, \infty)$ to $\mathfrak{E}$, then E_y is integrable if and only if $\int_{-\infty}^{\infty} y d\mu(E_y)$ exists.

It is possible to define a "characteristic" point function of an element E of $\mathfrak{E}$ by the condition $\chi(E, y) = 1$ if $y \leqq 0$, $\chi(E, y) = E$ if $0 < y \leqq 1$; $\chi(E, y) = 0$ if $y > 1$. These characteristic functions form a Boolean algebra of the same type as $\mathfrak{E}$. Moreover, it is possible if $\mathfrak{E}$ is σ-complete to express any point function in the form $\int_{-\infty}^{\infty} y d\chi(E_y)$, where the integral is defined after the manner of Lebesgue, first for bounded functions giving $\int_a^b y d\chi(E_y)$ when $E_y = 0$ for $y < a$, and $E_y = 1$ for $y > b$, and then as $\lim_n \int_{-n}^{n} y d\chi(E_y)$ if E_y ranges over the entire infinite interval, i.e., is unbounded. See Birkhoff [5, p. 251]. The linear lattice of all point functions is consequently determined by the Boolean algebra of characteristic functions via an integration process.

An important result of the latter type is due to Freudenthal [21]. Let $\mathfrak{X}$ be a σ-complete linear lattice. Assume that $\mathfrak{X}$ contains a unit u, an element of $\mathfrak{X}$ satisfying the conditions (a) $u \geqq 0$, (b) $u \cap |x| = 0$ implies $x = 0$. Then the subset of $\mathfrak{X}$ which consists of the elements e such that $e \cap (u - e) = 0$ form a σ-complete Boolean algebra $\mathfrak{E}$. Then to every element x of $\mathfrak{X}$ there corresponds a "point" function e_y on $-\infty < y < \infty$ to $\mathfrak{E}$ such that $x = \int_{-\infty}^{\infty} y d(e_y)$.

17. **An additional approach to abstract integrals.** Izumi [32, IV] has suggested another aspect of ordinary integration from which the

notion of abstract integration might be considered. In defining the Riemann integral of a function $f(s)$ on $a \leqq s \leqq b$, we consider the approximating sums $\sum_i f(s_i')(s_i - s_{i-1})$ for a subdivision σ of (a, b). Considered from the point of view of linear forms we have defined for every σ a set of linear forms $I_\sigma(f)$ valid for all finite-valued functions on (a, b). The subdivisions σ form a directed set by inclusion. The class of Riemann integrable functions is that for which $\lim_\sigma I_\sigma(f)$ exists. This leads to the following abstraction. Given a linear space $\mathfrak{X}$ of elements x, and a set of linear forms $L_\alpha(x)$ on $\mathfrak{X}$ to real numbers. Assume that the α form a directed set to which a Moore-Smith limit is applicable. Then the space of "integrable" elements is that subset X_0 of $\mathfrak{X}$ for which $\lim_\alpha L_\alpha(x)$ exists, and the integral $L(x)$ is the limit. Because of the linearity property of the limit notion, it follows at once that X_0 is linear and L is linear on X_0. Additional conditions on $\mathfrak{X}$ and the forms $L_\alpha(x)$ must be added to obtain properties of $L(x)$, similar to those of Riemann and Lebesgue integration. Obviously, if $\overline{\lim}_\alpha L_\alpha x$ exists as a finite number for all x of $\mathfrak{X}$, then this defines a semi-additive function $\overline{L}(x)$, i.e. $\overline{L}(x_1+x_2) \leqq \overline{L}(x_1)+\overline{L}(x_2)$, and $\overline{L}(ax) = a\overline{L}(x)$ for $a \geqq 0$, making the Hahn-Banach theorem applicable, and assuring us of the existence of a linear form $L(x)$ on all $\mathfrak{X}$ which can be regarded as an extension of $L(x)$ on X_0. Extensions of these considerations when the real number system as the values of the linear forms are replaced by linear normed or topological spaces, or linear lattices, are possible.

I must bring this meager survey of the development of abstract integration to a close. I feel apologetic about not including many important phases such as the generalization of the Radon-Nikodym theorem [49] starting with the Riesz paper on *Sur quelques notions fondamentales dans la théorie des opérations linéaires* [59] and effectively continued by Dieudonné [13]. Other points would be derivative properties, Denjoy types of integrals, and integrals of nonlinear type. If there is a unifying thread running through this survey, it would be that integration is essentially a linear process operating on a linear space. As suggested by the Lebesgue postulates, one abstracts and considers integration as a transformation from one space to another preserving certain properties, such as linearity, order, boundedness, and convergence. Of course, one could make the rash statement that an integral is a linear "continuous" form, a statement which is quite in keeping with recent trends (A. Weil, L. Schwartz). This is a logical sequel to the expression of the most general linear form on continuous functions on a finite interval as a Stieltjes integral or that on the space of bounded functions as a generalized integral. A second notion which plays a prominent role is that of extension. The value of an "integral" is known for a certain group of elements, subset of a larger group. One tries to extend to a larger group preserving certain properties, obtaining other properties for the larger group. For instance, the completion procedures of the rational numbers to the real number systems may be usable. There is, of course, the question of what limitations one should put on the notion of extension. Thus, for instance, the space of all linear forms on a linear space forms again a linear space, the first conjugate of the original space. The second conjugate includes the original space as a subset, which might be considered as an extension of the original linear space. In case the original space is not reflexive, the process can continue. Where stop? Perhaps this is a good place to conclude.

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UNIVERSITY OF MICHIGAN

THE ANNUAL MEETING IN ST. LOUIS

The fifty-ninth Annual Meeting of the American Mathematical Society was held at Washington University, St. Louis, Missouri, Saturday through Monday, December 27–29, 1952, in conjunction with meetings of the American Association for the Advancement of Science, the Association for Symbolic Logic, and the Mathematical Association of America.

Almost 500 people registered for the meetings, among whom were the following 339 members of the Society:

V. W. Adkisson, L. W. Akers, A. A. Albert, E. S. Allen, C. B. Allendoerfer, R. D. Anderson, B. M. Armstrong, Nachman Aronszajn, M. C. Ayer, W. G. Bade, B. J. Ball, I. A. Barnett, Iacopo Barsotti, R. G. Bartle, M. A. Basoco, P. T. Bateman, J. D. Baum, E. G. Begle, Stefan Bergman, R. H. Bing, A. H. Black, R. L. Blair, H. D. Block, H. W. Bode, Raoul Bott, Joseph Bram, R. W. Brink, A. R. Brown, R. H. Bruck, H. D. Brunk, P. B. Burcham, Sister M. Leonarda Burke, R. S. Burington, Jewell H. Bushey, R. K. Butz, S. S. Cairns, Eugenio Calabi, J. M. Calloway, H. H. Campaigne. K. H. Carlson, Maria Castellani, Lamberto Cesari, M. N. Chase, J. O. Chellevold, Y. W. Chen, R. V. Churchill, D. E. Coffey, Harvey Cohn, L. A. Colquitt, Damian Connelly, P. F. Conrad, Sister Rose Margaret Cook, Byron Cosby, Jr., W. A. Couch, N. A. Court, C. H. Cunkle, J. C. Currie, C. W. Curtis, J. M. Danskin, Robert Davies, W. E. Deskins, A. H. Diamond, J. L. Doob, H. P. Doole, Jim Douglas, T. L. Downs, W. C. Doyle, Nelson Dunford, J. P. Dunnett, W. L. Duren, Jr., William H. Durfee, S. E. Dyer, J. M. Earl, J. C. Eaves, Samuel Eilenberg, R. E. Ekstrom, J. D. Elder, H. M. Elliott, J. G. Elliott, D. O. Ellis, Bernard Epstein, D. H. Erkiletian, M. E. Estill, H. J. Ettlinger, G. M. Ewing, W. H. Fagerstrom, F. D. Faulkner, William Feller, W. T. Fishback, D. A. Flanders, Harley Flanders, W. C. Foreman, G. E. Forsythe, J. S. Frame, R. E. Fullerton, H. M. Gehman, J. J. Gergen, F. J. Gerst, Murray Gerstenhaber, Leonard Gillman, A. M. Gleason, Casper Goffman, H. E. Goheen, V. D. Gokhale, Michael Goldberg, S. H. Gould, R. F. Graesser, L. M. Graves, F. L. Griffin, J. S. Griffin, Jr., H. C. Griffith, Simon Gruenzweig, D. L. Guy, Felix Haas, D. T. Haimo, Franklin Haimo, Edwin Halfar, P. C. Hammer, J. R. Hanna, Frank Harary, H. G. Harp, Melvin Hausner, N. A. Haynes, I. L. Hebel, G. A. Hedlund, L. J. Heider, A. E. Heins, Henry Helson, Melvin Henriksen, I. N. Herstein, Fritz Herzog, Edwin Hewitt, T. H. Hildebrandt, T. W. Hildebrandt, Sister M. Laetitia Hill, J. J. L. Hinrichsen, F. E. Hohn, D. L. Holl, L. A. Hostinsky, J. A. Hratz, S. T. Hu, G. B. Huff, Ralph Hull, M. G. Humphreys, M. H. Ingraham, W. H. Ito, L. K. Jackson, H. G. Jacob, Nathan Jacobson, Fritz John, L. W. Johnson, B. W. Jones, F. B. Jones, P. S. Jones, W. C. Kalinowski, Irving Kaplansky, M. E. Kellar, J. L. Kelley, M. R. Kenner, J. R. F. Kent, Mother Marie Kernaghan, D. E. Kibbey, V. L. Klee, S. C. Kleene, George Klein, Erwin Kleinfeld, L. A. Knowler, J. C. Koken, L. A. Kokoris, G. L. Kreezer, R. J. Lambert, C. E. Langenhop, R. E. Langer, Paolo Lanzano, Leo Lapidus, H. D. Larsen, W. G. Leavitt, Joseph Lehner, R. A. Leibler, Walter Leighton, Benjamin Lepson, W. J. LeVeque, Sister Mary Teresine Lewis, H. D. Lipsich, Lee Lorch, G. G. Lorentz, L. L. Lowenstein, C. I. Lubin, H. T. McAdams, W. C. McDaniel, E. J. McShane, G. R. MacLane, Saunders MacLane, H. M. MacNeille, Szolem Mandelbrojt, C. G. Maple, Morris Marden, R. H. Marquis, J. M. Marr, E. C. Marth, W. S. Massey, Gaylord Merriman, E. A. Michael, R. J. Michel, R. R. Middlemiss, W. L. Miser,

E. E. Moise, M. A. Moore, R. A. Moore, T. W. Moore, W. L. Moore, Marston Morse, T. A. Mossman, H. T. Muhly, G. G. Murray, Zeev Nehari, David Nelson, W. J. Nemerever, John von Neumann, Albert Newhouse, C. V. Newsom, T. A. Newton, C. O. Oakley, H. W. Oliver, W. R. Orton, Morris Ostrofsky, T. P. Palmer, O. O. Pardee, S. T. Parker, G. W. Patterson, L. E. Payne, W. H. Pell, H. P. Pettit, C. G. Pitner, J. C. Polley, G. B. Price, O. J. Ramler, J. F. Randolph, L. T. Ratner, L. M. Reagan, M. S. Rees, R. F. Reeves, Francis Regan, Haim Reingold, R. W. Rempfer, J. G. Renno, P. R. Rider, J. K. Riess, J. D. Riley, R. F. Rinehart, L. A. Ringenberg, R. A. Roberts, L. V. Robinson, L. D. Rodabaugh, Hartley Rogers, Jr., I. H. Rose, Alex Rosenberg, P. C. Rosenbloom, J. B. Rosser, W. C. Royster, L. A. Rubel, H. S. Ruse, Charles Saltzer, Hans Samelson, R. G. Sanger, H. M. Schaerf, R. D. Schafer, M. M. Schiffer, E. R. Schneckenburger, Lowell Schoenfeld, J. T. Schwartz, L. L. Scott, W. R. Scott, R. R. Seeber, Jr., George Seifert, J. B. Serrin, M. A. Seybold, Seymour Sherman, Harold Shniad, Marlow Sholander, Edward Silverman, Annette Sinclair, F. B. Sloss, D. M. Smiley, M. F. Smiley, James C. Smith, W. S. Snyder, E. J. Specht, H. D. Sprinkle, S. K. B. Stein, R. L. Sternberg, P. C. Suppes, Gabor Szegö, C. T. Taam, T. T. Tanimoto, Alfred Tarski, M. E. Taylor, H. P. Thielman, A. J. Tinkley, E. W. Titt, A. W. Tucker, A. R. Turquette, J. L. Ullman, W. R. Utz, Jr., B. O. Van Hook, G. B. Van Schaack, D. E. Van Tijn, H. E. Vaughan, D. C. Velez, D. F. Votaw, T. L. Wade, Jr., G. L. Walker, R. J. Walker, H. S. Wall, C. Y. Wang, H. C. Wang, J. A. Ward, L. E. Ward, Jr., A. M. Wedel, J. V. Wehausen, L. M. Weiner, B. A. Welch, F. J. Weyl, George Whaples, P. M. Whitman, G. T. Whyburn, L. S. Whyburn, R. L. Wilder, A. B. Willcox, W. L. Williams, R. A. Willoughby, R. H. Wilson, Jr., G. N. Wollan, Oswald Wyler, C. T. Yang, Ti Yen, L. C. Young, J. W. T. Youngs.

The Josiah Willard Gibbs Lecture, on *Topology and geometrical analysis*, was presented by Professor Marston Morse of the Institute for Advanced Study on Saturday evening. President Detlev Bronk of Johns Hopkins University, president of the National Academy of Sciences, presided.

Professor John von Neumann of the Institute for Advanced Study gave his Retiring Presidential Address, *A logical theory of automata*, on Sunday afternoon. Professor S. C. Kleene presided.

At a joint session of Section A and the Society at 3:30 P.M., Monday, Professor R. L. Wilder of the University of Michigan gave an address, *Origin and growth of mathematical concepts*, as a retiring Vice President of the American Association for the Advancement of Science. Professor L. M. Graves presided.

Professor A. M. Gleason of Harvard University gave an address, *Natural co-ordinate systems*, on Saturday afternoon, on invitation of the Committee to Select Hour Speakers for Annual and Summer Meetings. Professor Nelson Dunford presided. Professor Gleason's paper, which dealt with Hilbert's fifth problem, was awarded the Newcomb Cleveland Thousand Dollar Prize Award for the outstanding paper at the meeting of the American Association for the Advancement of Science.

The Registration Headquarters for this meeting were in the

Women's Building and adjacent to various lounges which were available at all times. This feature together with tea served in the afternoons contributed to the success of the meeting.

More formal entertainment was provided by an Opera Workshop Performance on Sunday evening and a Sightseeing Tour of Saint Louis on Monday.

A dinner for members of the mathematical organizations and their guests was held on Monday evening, December 29, in the Women's Building. Professor T. L. Downs was the toastmaster. The mathematical organizations were welcomed by Leslie J. Buchan, Vice-Chancellor of Washington University. Professor G. T. Whyburn, President Elect of the American Mathematical Society, Professor L. M. Graves. Vice President and Chairman of Section A of the American Association for the Advancement of Science, Professor C. B. Allendoerfer, Editor of the Mathematical Association of America, and Professor J. B. Rosser, President of the Association for Symbolic Logic, spoke for their respective organizations. A resolution of appreciation and thanks was offered by Professor J. S. Frame to Washington University, the Committee on Arrangements, and all who had contributed to the success of the meeting. The resolution was adopted by a rising vote.

The annual Business Meeting of the Society was held on Monday afternoon, December 29, 1952. The Secretary reported on the affairs of the Society.

At the annual election, in which 820 votes were cast, the following officers were elected.

Vice President, Dr. D. H. Lehmer.

Secretary, Professor E. G. Begle.

Treasurer, Dean A. E. Meder, Jr.

Associate Secretary, Professor L. W. Cohen.

Member of the Editorial Committee of the Bulletin, Professor G. B. Price.

Members of the Editorial Committee of the Transactions and Memoirs, Professors J. L. Doob and L. V. Ahlfors.

Member of the Editorial Committee of the Proceedings, Professor A. C. Schaeffer.

Member of the Editorial Committee of the Colloquium Publications, Professor Deane Montgomery.

Member of the Editorial Committee of Mathematical Reviews, Professor Einar Hille.

Member of the Editorial Committee of Mathematical Surveys, Professor Max Shiffman.

Members of the Committee on Printing and Publishing, Professors

M. R. Hestenes, J. R. Kline, and C. J. Rees.

Representative on the Board of Editors of the American Journal of Mathematics, Professor Samuel Eilenberg.

Members-at-large of the Council, Professors F. B. Jones, E. E. Moise, B. J. Pettis, R. M. Thrall, and G. W. Whitehead.

Members of the Board of Trustees, Professors B. P. Gill, J. R. Kline, and W. T. Martin.

The Council met on Sunday evening, December 28, 1952.

The Secretary announced the election of the following forty-two persons to ordinary membership in the Society.

Mr. Robert MacColl Adams, Research Mathematician, University of Texas;
Mr. Philip Nye Armstrong, Engineer, International Business Machines, Inc., Endicott, New York;
Mr. Clarence F. Barr, Head, Department of Mathematics, University of Wyoming;
Mr. Jose de Barros Neto, Assistant, University of São Paulo, São Paulo, Brazil;
Mrs. Barbara Beechler Blair (Mrs. R. L.), Assistant Instructor, State University of Iowa;
Mr. Robert L. Blair, Instructor, State University of Iowa;
Dr. Claude W. Burrill, University of Manchester;
Mr. Harold Davis, Engineer, University of California, Los Angeles, California,
Mr. Roy Bryan Deal, Jr., Instructor, University of Oklahoma;
Mr. Robert Diaz-Fernandez, University of California, Berkeley, California;
Mr. Victor John Doberly, Accessocraft Products Corporation, New Rochelle, New York;
Mr. Rudolf F. Drenick, Manager, Analytical Group, RCA Victor Division, Camden, New Jersey;
Mr. Cecil Eugene Duncan, Stanford University;
Mr. Frederick William Gehring, Instructor, Harvard University;
Mr. William Lewis Gordon, Duke University;
Mr. James Ray Hanna, Associate Professor, University of Wichita;
Mr. Walter Julius Huebner, Jr., St. Louis University;
Mrs. Mary E. Knight, Mathematician, White Sands Proving Ground, Las Cruces, New Mexico;
Mr. Alfred Baker Lehman, University of Florida;
Rev. Charles Joseph Lewis, Brown University;
Mr. John Howard Maecher, Teaching Fellow, University of Miami;
Mr. Arne Magnus, Instructor, University of Kansas;
Dr. Lucien Massé, Vice President, Weiss Geophysical Corporation of Canada, Calgary, Alberta, Canada;
Mr. Edward Joseph Murphy, Member of Technical Staff, Bell Telephone Laboratories, Summit, New Jersey;
Mr. James Pachares, Research Assistant, University of North Carolina;
Mr. Prom Panitchpakdi, University of Kansas;
Mr. Robert Lee John Penn, General Manager, Mathematics Research and Publishing Company, Austin, Texas;
Mr. Donald Jeanne Ritchie, Mathematician, Bendix Aviation Corporation Research Laboratories, Detroit 3, Michigan;
Miss Anne Elizabeth Scheerer, University of Pennsylvania;
Mr. Ben Moore Seelbinder, Part-time Instructor, University of North Carolina;

Mr. Edwin Seymour Shapiro, Mathematician, U. S. Naval Radiological Defense Laboratory, San Francisco 22, California;
Dr. Frank L. Spitzer, New York, New York;
Mr. Raimond Aldrich Struble, Aerodynamicist, Douglas Aircraft Company, Santa Monica, California;
Professor Patrick Colonel Suppes, Stanford University;
Dr. Robert Flemming Tate, Lecturer, University of California, Berkeley, California;
Mr. Morris Tenenbaum, Cornell University;
Dr. Jose Pasquel Tola, Director, Instituto de Matematicas, Universidad Mayor de San Marcos, Lima, Peru;
Mr. Tung Tsang, Assistant Chemist, Minneapolis-Honeywell Regulator Company, Minneapolis, Minnesota, and Department of Chemical Engineering, University of Minnesota;
Mr. Bernard Weitzer, Mathematician, Republic Aviation Corporation, Farmingdale, Long Island, New York;
Dr. Robert Lewis Westhafer, Associate Professor, New Mexico College of Agriculture and Mechanic Arts, State College, New Mexico;
Mr. J. Hunter White, Jr., District Representative, International Business Machines Corporation, River Edge, New Jersey;
Miss Irma Marian Wyman, Research Associate, Willow Run Research Center, Ypsilanti, Michigan.

It was reported that the following thirteen persons had been elected to membership on nomination of institutional members as indicated:

Harvard University: Mr. Shreeram Shankar Abhyanuar, Mr. Edward Holton Bensley, Mr. Gerald Gustave Bilodeau, Mr. Alexander Blair, Miss Marian Boykan, Mr. Richard Robinson Goldberg, and Mr. Seth L. Warner.

Lehigh University: Mr. Chester B. Sensenig.

State University of Iowa: Mr. Wendall Don Lindstrom.

University of New Hampshire: Mr. Armand Raymond Lamontagne.

Tulane University of Louisiana: Mr. Charles Edward Capel, Mr. William Munroe Faucett, and Mr. John Sanders Griffin.

The Secretary announced that the following had been admitted to the Society in accordance with reciprocity agreements with various mathematical organizations: London Mathematical Society: Mr. David George Kendall, Princeton University, Professor Douglas Geoffrey Northcott, the University of Sheffield, Sheffield, England, and Professor Edward Maitland Wright, King's College, Aberdeen, Scotland; Schweizerische Mathematische Gesellschaft: Dr. Alfred Otto Huber, Institute for Fluid Dynamics and Applied Mathematics, University of Maryland; Unione Matematica Italiana: Professor Lorenzo Calabi, Boston College, Chesnut Hill, Massachusetts.

Mount Holyoke College, South Hadley, Massachusetts, and the University of Utah, Salt Lake City, Utah, were elected to institutional membership.

The Secretary is pleased to report at this time that the ordinary membership of the Society is now 4483, including 346 nominees of institutional members and 40 life members. The membership of the Society continues to show a slow, but steady, growth. There are also 124 institutional members. The total attendance at all meetings in 1952 was 1998; the number of papers read was 733; there were 13 hour addresses, 4 Colloquium Lectures, 1 Gibbs Lecture, 1 Retiring Presidential Address, and 15 papers at the Applied Mathematics Symposium. The number of members attending at least one meeting was 1348.

The following appointments by the President were reported: as a committee to recommend policies for a separate Memoirs Editorial Committee: J. L. Doob, Chairman, R. P. Boas, and S. C. Kleene; as a committee to recommend to the Council the award of the Bôcher Memorial Prize: L. V. Ahlfors, Chairman, Philip Franklin, and D. C. Spencer; as a committee to nominate members for the Policy Committee: P. A. Smith, Chairman, G. A. Hedlund, and A. W. Tucker; as a committee on translations from Russian and other foreign languages: R. P. Boas and Irving Kaplansky for a term of three years beginning July 1, 1952, R. E. Bellman and Hans Samelson for a term of two years beginning July 1, 1952, J. L. Doob for a period of one year beginning July 1, 1952, and R. P. Boas as chairman for a period of one year beginning July 1, 1952; as an organizing committee for summer institutes: A. A. Albert, Chairman, and Salomon Bochner for a term of three years beginning July 1, 1952, H. P. Robertson and Hassler Whitney for a term of two years beginning July 1, 1952, and Nathan Jacobson and Oscar Zariski for a period of one year beginning July 1, 1952; as an arrangements committee for the Sixth Symposium in Applied Mathematics to be held at the Corona Laboratories of the National Bureau of Standards on August 26–28, 1953: D. H. Lehmer, Chairman, H. F. Bohnenblust, J. H. Curtiss, J. W. Green, George Pólya, and John Todd; as Editor of Volume VI of the Proceedings of the Symposium in Applied Mathematics: J. H. Curtiss; as tellers for the 1952 annual election: Zeev Nehari and G. B. Van Schaack; as a member of the Committee on Applied Mathematics (for a period of three years beginning January 1, 1953): Shizuo Kakutani (committee now consists of M. H. Martin, Chairman, R. V. Churchill, F. J. Murray, Eric Reissner, and Shizuo Kakutani); as a member of the Committee on Places of Meetings (for a period of three years beginning January 1, 1953): E. R. Lorch (committee now consists of Orrin Frink, Chairman, M. F. Smiley, and E. R. Lorch); as members of the Committee on Visiting Lectureships: Garrett Birkhoff for a period of two years beginning January 1, 1953, and Einar Hille for a period of three years beginning January 1, 1953 (committee now consists of R. P. Boas, Jr., Chairman, Garrett Birkhoff, and Einar

Hille); as liaison officer to the Quarterly of Applied Mathematics for a term of three years beginning January 1, 1953: A. H. Taub; as a member of the Committee to Select Hour Speakers for Summer and Annual Meetings for the period 1953–1954: Oscar Zariski (committee now consists of E. G. Begle, Chairman, Tibor Radó, and Oscar Zariski); as a member of the Committee to Select Hour Speakers for Eastern Sectional Meetings for the period 1953–1954: A. M. Gleason (committee now consists of L. W. Cohen, Chairman, R. D. Schafer, and A. M. Gleason); as a member of the Committee to Select Hour Speakers for Southeastern Sectional Meetings for the period 1953–1954: J. J. Gergen (committee now consists of W. M. Whyburn, Chairman, Wallace Givens, and J. J. Gergen); as a member of the Committee to Select Hour Speakers for Western Sectional Meetings for the period 1953–1954: E. E. Moise (committee now consists of J. W. T. Youngs, Chairman, Marshall Hall, Jr., and E. E. Moise); as a member of the Committee to Select Hour Speakers for Far Western Sectional Meetings for the period 1953–1954: R. D. James (committee now consists of J. W. Green, Chairman, R. P. Dilworth, and R. D. James).

The Secretary reported that Professor J. A. Dieudonné had accepted an invitation to deliver an hour address at the April 1953 meeting in Chicago.

The Council approved the following places of sectional meetings in 1953: The Midwest November meeting at Northwestern University and the Far West November meeting at the California Institute of Technology.

The Secretary reported that the Policy Committee for Mathematics had voted to join the incorporators of a Scientific Manpower Commission. This Commission will gather and publicize information concerning present and future needs for scientific manpower.

The Secretary reported that a reciprocity agreement has been concluded with the Swedish Mathematical Society.

The Secretary reported that the invitation to hold the 1953 Summer Meeting at Laval University in Quebec had been withdrawn because of difficulties in obtaining housing. The Council voted to accept an invitation from Queen's University and the Royal Military College to hold the Summer Meeting in Kingston, Ontario.

The Council approved the substitution of Professor R. P. Boas for Professor William Feller as a member of the Mathematical Reviews Editorial Board from February to August 1953.

The Bulletin Editorial Committee reported that the Journal used 718 pages in 1952, but that the increased size of reports of meetings and other material not under the control of the Editors of the Bulletin had resulted in fewer pages of invited addresses. The Council voted to recommend to the Board of Trustees that a total of 625 pages be authorized for the 1953 Bulletin.

The Transactions and Memoirs Editorial Committee reported that the interval between receipt of a manuscript and publication is approximately nine months. The Council voted to recommend to the Trustees that 1100 pages be published in the Transactions for 1953.

The Mathematical Reviews Editorial Committee reported that 1014 pages had been published in 1952. The subscription list as of November 1952 was 2222.

The Proceedings Editorial Committee reported that the interval between receipt of a manuscript and publication had decreased to nine or ten months. The Council voted to recommend to the Board of Trustees that 1000 pages be authorized for the 1953 Proceedings. Professor R. C. Buck, E. A. Coddington, E. R. Kolchin, and E. E. Moise were reported as new assistant editors for the Proceedings.

The Colloquium Editorial Committee recommended that there be no Colloquium lectures in 1954, since the International Congress in Amsterdam will be held September 2–9, partially overlapping our summer meeting. The Council voted to approve this recommendation.

The Council elected Professor A. A. Albert as a representative of the Society on the Policy Committee for Mathematics for a four-year term beginning January 1, 1953.

The Council voted to invite Professor Wassily Leontieff of Harvard University to deliver the Gibbs Lecture for 1953.

Professor A. A. Albert reported that the National Science Foundation had granted $20,000 to the Society for a Summer Institute on Lie Groups and Lie Algebras. Professor Nathan Jacobson has been named Chairman of this Institute.

The Council voted to approve amendments to our by-laws which would: (1) empower the Council, under certain conditions, to speak in the name of the Society with respect to matters affecting the status, dignity, and effective position of mathematics or mathematicians; (2) require a referendum on such Council actions when requested by 200 or more members of the Society; (3) require a business meeting at the Summer as well as the Annual Meeting; (4) allow final action to be taken at a business meeting only when business is accepted by unanimous consent or notified to the full membership of the Society in the call for the meeting. These by-laws will be presented to the Society for approval at a business meeting at the time of the Summer Meeting.

Abstracts of the papers read follow. Presiding officers at the sessions for contributed papers were Dr. Seymour Sherman, Professor Saunders MacLane, Dr. A. H. Diamond, Professors G. B. Price, J. L. Kelley, Nathan Jacobson, D. L. Holl, A. A. Albert, E. E. Moise, Szolem Mandelbrojt, W. L. Duren, Jr., Ralph Hull, and Morris Marden.

Papers whose abstract numbers are followed by the letter "*t*" were read by title. Paper number 159 was presented by Professor Herstein, 179 by Professor Stefan Bergman, 186 by Professor Schiffer, 193 by Professor Lorentz, 221 by Professor P. R. Rider, 238 by Mr. Minagawa, 241 by Professor Suppes, 252 by Professor Samelson, 255 by Mr. Sprinkle, and 258 by Dr. Harary.

Dr. Huber was introduced by Professor Alexander Weinstein, Mr. Griffin by Professor S. T. Hu, Professor Toralballa by Professor J. W. T. Youngs, Mr. Fettis by Professor P. R. Rider, Dr. Spitzer by Professor D. A. Darling, Dr. Jaeger by Professor S. S. Cairns, Dr. Bremermann by Professor Stefan Bergman.

ALGEBRA AND THEORY OF NUMBERS

146*t*. H. W. Becker: *The Rignaux and Lebesgue transforms of Pythagorean tetrahedrons.*

Let the six edges of a Pythagorean tetrahedron be u, v, x, y, z, t, and the corresponding edges of tet. under transform H be tx, uv, ux, xz, ty, tu; and under the Petrus transform of the first kind P, be $t^2-y^2=x^2+z^2$, $2uv$, $u^2-v^2=x^2-z^2$, $2xz$, $2ty$, $t^2+y^2=u^2+v^2$, Dickson's *History* II, p. 446. Rignaux, L'Intermediaire des Math. vol. 25 (1918) p. 129, gave a transform R on Petrus tets. under which (putting $TU+TX+UX=\Delta$, and including the three edges $\not\supset\Delta$ unnecessary to his purposes) the edges $U,\cdots,T\to T^2-Y^2+\Delta$, $2V((T+U)(U+X))^{1/2}$, $U^2-V^2+\Delta$, $2Z((T+X)(U+X))^{1/2}$, $2Y((T+U)(T+X))^{1/2}$, $T^2+Y^2+\Delta$. The transform P also iterates on Petrus tets., where obviously $R\supset P$. Lebesgue, Dickson ibid., p. 637, showed that a certain quartic $=\square$ is a relative invariant under his transform L. Applying this to the tet. $u,\cdots,t=((e^2g^2+f^2h^2)(e^2h^2+f^2g^2))^{1/2}$, in the form e, f, g, $h\to e$, f, $e^2f^2(g^4-h^4)$, $2tefgh$, then $u,\cdots,t\to t^2y^2+u^2x^2$, $2tuvx$, $x^4-v^2y^2$, $2tuxy$, $2tuxz$, $t^4-v^2z^2$, and $L=RP=PH$. Notwithstanding that $RPI\gg P^2I$, due to the rich possibilities of cancellation inherent in H, $R^2=H^2=I$, the identity, but $L^2=P^2\gg I$. A Pythagorean tet. also depends on $x=((\epsilon^2\eta^2-\zeta^2\theta^2)(\epsilon^2\theta^2-\zeta^2\eta^2))^{1/2}$. Under transform Λ: ϵ, ζ, η, $\theta\to\epsilon$, ζ, $\epsilon^2\zeta^2(\eta^4-\theta^4)$, $2x\epsilon\zeta\eta\theta$; $u,\cdots,t\to 2tuvx$, $t^2y^2+v^2z^2$, $2tvxz$, $2tvyz$, $z^4-u^2y^2$, $t^4-u^2x^2$. There are analogous transforms of the second kind: G', P', R', L', Λ', in a general way under permutations $(uv)(xz)$, with analogous inter-relations. But quite unexpectedly, $\Lambda=L'$, $\Lambda'=L$. (Received September 22, 1952.)

147*t*. R. H. Bruck: *Difference sets in a finite field.*

Let F be a finite field of P elements and let $H(m)$ be the multiplicative group of nonzero mth powers in F, where m divides $P-1$. The group $H(m)$ is a difference set in F if the number of ordered pairs of elements h, k of $H(m)$ such that $h-k=c$ is the same for each nonzero c in F. A necessary condition is that $P\equiv 1+m \bmod m^2$; this assumed, the following conditions are necessary and sufficient that $H(m)$ be a difference set for the stated value of m: (i) $m=2$: none. (ii) $m=4$: $P=1+4x^2$ for an integer x. (iii) $m=8$: $P=1+8x^2=9+64y^2$ for integers x, y. There are no difference sets $H(m)$ for $m=3, 5, 6$. Case (i) is trivial. In case (ii), P must be prime; for P prime the result was essentially proved by Gauss and reproved by Chowla (Proc. Nat. Acad. Sci. India, Sect. A, vol. 14 (1944) pp. 45–46). Case (iii) allows $P=9, 73$; other P's (if any) must exceed 10^{10}. For indications of proof see the abstract on Gauss's method. (Received November 12, 1952.)

148. R. H. Bruck: *Gauss's method for the number of solutions of certain equations in a finite field.*

Let g be a multiplicative generator of the finite field $F=GF(P)$ and let $H(m)$ be the multiplicative group of mth powers, where m divides $P-1$. The number of solutions in F of $ax^m+\cdots+cz^m=d$, for any finite number of nonzero unknowns, $x, \cdots, z$, is expressible in terms of $(u, v)_m$, the number of ordered pairs of elements h, k of $H(m)$ such that $g^uh+g^vk+1=0$. Gauss (for these and other references see A. Weil, Bull. Amer. Math. Soc. vol. 55 (1949) pp. 497–508) gave an elementary determination of $(u, v)_m$ for P prime, $m=3, 4$. Later writers preferred to work with Gaussian or Jacobian sums. The present paper characterizes Gauss's method by giving a complete set of postulates for the number-theoretical functions $(u, v)_m$; the postulates concern P but contain no reference to $GF(P)$. The group of mappings $(u, v)_m\to(ru, rv)_m$, r prime to m, plays a central role, the function $(u, v)_m$ being expressible in terms of invariants of the group. The detailed determination of $(u, v)_m$ is given for $m=2, 3, 4, 5, 6, 8$. (Received November 12, 1952.)

149. J. M. Calloway: *On the discriminant of arbitrary algebraic number fields.*

An application of Minkowski's lemma on linear forms proves the result that the discriminant d of every algebraic number field of degree $n>1$ is divisible by a prime number. An identity of C. L. Siegel (Nachr. Ges. Wiss. Göttingen (1922) pp. 17–24) gives this result for totally real fields. A generalization of Siegel's identity is obtained which shows that $|d|$ of an arbitrary algebraic number field of degree $n>1$ is greater than $(\pi/3)^{2r_2}$ $(n=r_1+r_2)$. This is a special case of a second identity which also yields the Fourier expansion for the $(k+1)$st Bernoulli polynomial. The identity is obtained in two ways. The first, a direct analogue of Siegel's method, consists of expanding a function in an n-fold Fourier series, multiplying both sides by a certain expression, and integrating. The second method consists of expanding a different function in an n-fold Fourier series. The second method while more elegant involves a more complicated argument as to the convergence of the Fourier series. (Received November 17, 1952.)

150*t*. Leonard Carlitz: *Congruences for the coefficients of the Weierstrass $\wp$-function and related functions.*

Let $\wp(x)=\wp(x; g_2, g_3)$ denote the Weierstrass $\wp$-function, where g_2, g_3 are indeterminates. Put $f(x)=1/\wp(x)=h^2(x)$; also let $h(x)=\sum_1^\infty A_mx^m/m!$, $f(x)=\sum_2^\infty B_mx^m/m!$, $\wp(x)=x^{-2}+\sum_2^\infty C_mx^m/m!$, so that the A_m, B_m, C_m are polynomials in g_2, g_3 with integral coefficients. Let p be a fixed odd prime. The principal object of the paper is to prove congruences of the type $\sum_{s=0}^{r} (-1)^{r-s}C_{r,s}A_p^{r-s}B_{m+s(p-1)}\equiv 0$ (mod p^r) for $m\geq r\geq 1$. For the A_m we are unable to prove quite so much. (Received October 20, 1952.)

151t. Leonard Carlitz: *Kummer's congruences and the Schur derivative.*

Let p be a fixed prime and $\{a_m\}$ a sequence of rational numbers that are integral (mod p). Define $\Delta a_{p^m} = (a_{p^{m+1}} - a_p^{p^m} a_{p^m})/p^{m+1}$, $\Delta^r a_{p^m} = (\Delta^{r-1} a_{p^{m+1}} - a_p^{p^{m+r-1}} a_{p^m})p^{m+1}$. The main result of the paper is that if $\sum_{s=0}^{r} (-1)^{r-s} C_{r,s} a_p^{r-s} a_{m+s(p-1)} \equiv 0 \pmod{p^r}$ for $m \geqq r \geqq 1$, then $\Delta^r a_{p^m}$, $1 \leqq r \leqq p-1$, are integral for all m. (Received October 18, 1952.)

152t. Leonard Carlitz: *Some theorems on Kummer's congruences.*

Let $\{a_m\}$ be a sequence of rational numbers that are integral (mod p), where p is a fixed prime. The relation $\sum_{s=0}^{r} (-1)^{r-s} C_{r,s} a_{m+s(p-1)} a_p^{r-s} \equiv 0 \pmod{p^r}$ for all $m \geqq r \geqq 1$ may be called Kummer's congruence for $\{a_m\}$. In the present paper it is proved for example that if $\{a_m\}$ and $\{b_m\}$ satisfy Kummer's congruence, then the same is true of $\{a_m b_m\}$; also if $a_m^{(k)} = m^k a_m$, then $a_m^{(k)}$ satisfies a congruence of a similar kind. If some additional hypotheses are introduced then one can assert that $\{c_m\}$ satisfies Kummer's congruence, where $c_m = \sum_0^m C_{m,s} a_s b_{m-s}$. (Received November 10, 1952.)

153t. Eckford Cohen: *Rings of arithmetic functions.* II. *Quadratic congruences.*

Let r be an arbitrary odd number greater than 1, and let $a_1, \cdots, a_s$ be integers prime to r. The arithmetic part of this paper is concerned with the number of solutions $N = N_s(n, r)$ of the congruence $n \equiv a_1 x_1^2 + \cdots + a_s x_s^2 \pmod{r}$. The author defines for this problem a finite "singular series" $\mathfrak{S} = \mathfrak{S}_s(n, r)$, analogous to the Hardy-Ramanujan series for sums of squares in the nonmodular case. The function $\mathfrak{S}$ is a divisor sum involving the familiar Ramanujan sum in case s is even and a Gauss character sum in case s is odd. Using orthogonality properties of these exponential sums, it is shown on the basis of induction that $N = \mathfrak{S}$. A direct proof of this fact is also given. The function $\mathfrak{S}$ is evaluated completely to give explicit arithmetic formulas for N, and several corollaries arising from special cases are obtained. Algebraic properties of arithmetic functions required in the above problem are discussed along the lines of Part I of this paper (Duke Math. J. vol. 19 (1952)). (Received December 29, 1952.)

154t. Eckford Cohen: *On the number of solutions of cubic congruences.*

Certain aspects of the method employed in the above paper with regard to quadratic congruences are used to determine the number of solutions $N = N(n, p^\lambda)$ of the cubic congruence (I), $n \equiv ax^3 + by^3 \pmod{p^\lambda}$, where p is a prime $\equiv 1 \pmod{3}$, $(\lambda \geqq 1)$, a, b are integers prime to p, and n is arbitrary. Denote by t the largest integer $\leqq \lambda$ such that $n \equiv 0 \pmod{p^t}$. Exact formulas for N are obtained, with particularly simple results in case $t \equiv 1$ or $2 \pmod{3}$. From these formulas the following solvability criterion is deduced: *If* $p \neq 7$, *then* (I) *is insolvable if and only if* $t \not\equiv 0 \pmod{3}$, $t < \lambda$, *and* a *and* b *have a different cubic character* (mod p). The case $p = 7$ is exceptional. A similar approach is applied to obtain the exact number of solutions $N' = N'(n, p^\lambda)$ of the congruence (II), $n \equiv ax^3 + by^3 + cz^3 \pmod{p^\lambda}$, where p and λ are defined as above and $(abc, p) = 1$. From this result is is verified that (II) *is always solvable.* Several corollaries, including approximations and recursion relations, are deduced as special cases of the results for N and N'. The author also obtains the number of primitive solutions of (I) and (II) ($xyz \not\equiv 0 \pmod{p}$) in the case of a prime modulus p. (Received December 29, 1952.)

155t. Harvey Cohn: *Sign characterisation of continued fractions.*

Consider triples of vectors of a two-dimensional plane lattice, V_1, V_2, V_3, with the property that any two are a basis and $V_1+V_2+V_3=0$. Call the basis reduced if not all three vectors are in a pair of (opposing) quadrants. This is the sign condition. Then under the neighbor process of the type $V_1 \rightarrow V_1 - V_3$, $V_2 \rightarrow V_2 + V_3$, $V_3 \rightarrow V_3$, all such reduced bases are connected. This is proved by a modification of Smith's continued fraction algorithm. (Research sponsored by the U. S. Army O.O.R.) (Received November 13, 1952.)

156. P. F. Conrad: *On ordered division rings.*

Some well known properties of ordered groups are used to prove theorems on ordered division rings. A division ring D can be ordered if and only if there exists a valuation f of D for which the residue class ring R can be ordered so that all elements of the form $P+a_1^2 \cdots a_n^2$ are positive, where P is the prime ideal of f and $a_1^2 \cdots a_n^2$ is a unit. D can be ordered so that f is the natural valuation (i.e., the valuation ring is the set of all elements that are comparable to 1 or infinitely smaller than 1) if and only if the ordering of R is Archimedean. For each division ring with valuation there exists a maximal immediate extension. These results are generalizations of those obtained by W. Krull for ordered fields (J. für Math. vol. 167 (1932) pp. 160–196). For each ordered division ring there exists an a-closed a-extension (i.e., extension that preserves the value group). Every ordered field can be embedded in the field of formal power series over the value group with real coefficients. An a-closed ordered field is completely determined by its value group. (Received November 10, 1952.)

157. C. W. Curtis: *Noncommutative extensions of Hilbert rings.*

Let S be a ring containing a ring R in its center, having a common identity with R, such that S is a finite R-module, and S satisfies the ascending chain condition for R-submodules. It is proved that each prime ideal in S is an intersection of primitive ideals if and only if R is a Hilbert ring in the sense of Goldman [Math. Zeit. vol. 54 (1951) pp. 136–140]. If R contains a field K, and if each field homomorphic to R has finite degree over the image of K, then each primitive algebra homomorphic to S is finite-dimensional over K. The results are applied to prove, for example, that every irreducible representation of the universal associative of a Lie algebra over a field K of prime characteristic is finite-dimensional over K. (Received November 10, 1952.)

158t. C. W. Curtis: *On commutative linear transformations in infinite-dimensional spaces.*

Let A be a linear transformation (l.t.) acting in a vector space M of countable dimension over an algebraically closed field K. It is proved that if A is (i) locally algebraic and if (ii) for each $\lambda \in K$, whenever x is in the range of $(A-\lambda I)^n$ for every positive integer n, there exists a vector y, also in the range of $(A-\lambda I)^n$ for every n, such that $y(A-\lambda I)=x$, then each l.t. commutative with every l.t. commutative with A is in the closure in the finite topology of the polynomial algebra $K[A]$. This result includes as a special case the theorem of Wedderburn which states that if M is finite-dimensional over K, then a l.t. commutative with every l.t. commutative with a given l.t. A is a polynomial in A. The proof is in part an application of Prüfer's well known decomposition theorem for countable primary abelian groups. Examples are given which show that the theorem is false if condition (i) is omitted, and that the method of proof fails if condition (ii) is not satisfied. (Received November 10, 1952.)

159. Gerard Debreu and I. N. Herstein: *Nonnegative square matrices.*

In a series of three papers in the Sitzungsberichte der Berliner Akademie, 1908, 1909, and 1912, Frobenius obtained some very powerful theorems about the form of the largest (in absolute value) characteristic roots of matrices all of whose elements were nonnegative. These asserted, among other things, that for a certain fundamenta type of nonnegative matrix the largest characteristic root in absolute value was actually positive, simple, and had associated with it a positive eigenvector. In the present paper there is given a simple derivation of the key Frobenius' results. At the same time these are somewhat sharpened. Using these theorems the nonnegativity properties of $(sI-A)^{-1}$ are characterized for large s where A is a nonnegative matrix. A necessary and sufficient condition is given that the powers A^n of the matrix A, having complex elements, converge as $n\to\infty$. Of late these types of results have found many applications in mathematical economics; the results of this paper seem to encompass most of the theorems that have been used in that discipline so far. General techniques are given to obtain further results along these lines. (Received November 14, 1952.)

160. D. O. Ellis: *Remarks on Boolean functions.*

One considers functions $f(x)=(a\wedge x)\vee(b\wedge x')$ in a Boolean algebra B. The special case is that when $b\leqq a$. The principal remarks are: 1. *The set of Boolean functions on B forms a ring of transformations on B in which the group of multiplicative units is precisely the group of autometrized motions of B. The constant functions form an ideal in this ring and the resulting factor ring is isomorphic to the associated Boolean ring of B.* 2. *$f(x)$ maps B onto $[a\wedge b, a\vee b]$.* 3. *Let $f(x)$ be in the special case and let B_f denote the space of the orbital topology induced by $f(x)$. Then the principal ideal of $a\oplus b$ is a group of homeomorphisms of B_f onto itself and the join of orbital topologies for the special case is the strongest T_1 topology in B.* One also considers the minimazation of the "triangle function" of B and the fitting of graphs of Boolean functions to finite point sets in B^2. (Received November 7, 1952.)

161. Melvin Henriksen: *Rings of entire functions of finite order.*

Let R_λ, R_∞, R denote respectively the ring of entire functions of order $\leqq\lambda$, the ring of entire functions of finite order, and the ring of entire functions. In 1940, Helmer (Duke Math. J. vol. 6, pp. 345–356) showed that every finitely generated ideal of R is principal, or equivalently, if f, g in R have no zeros in common, there exist s, t in R such that $sf+tg=1$. He asked if this held in R_∞. A negative answer to this question existed already in 1936 (Proc. London Math. Soc. vol. 40, p. 256). In this paper, sufficient conditions that the residue class field of a maximal ideal of R_λ or R_∞ be isomorphic to the complex field are given and the existence of nonmaximal prime ideals is shown. The tools developed in a paper by the author (Pacific Journal of Mathematics vol. 2, pp. 179–184) on the ideal structure of R are used. (Received November 12, 1952.)

162. S. T. Hu: *Singular homology theory of associative algebras with unity elements.*

Let W be any given associative algebra over the real field with a unity element e. By means of the algebras A_n of continuous real functions defined on the unit n-simplex Δ_n in the $(n+1)$-dimensional euclidean space, $n=1, 2, \cdots$, a semisimplicial complex $S(W)$ in the sense of Eilenberg and Zilber is defined in the present paper called the singular complex of the algebra W. The homology and cohomology groups of the complex $S(W)$ are defined to be the singular homology and cohomology groups of the algebra W. For each two-sided ideal J of the algebra W, an associated subcomplex $\sigma(J)$ of $S(W)$ is defined. The relative homology and cohomology groups of $S(W)$ modulo $\sigma(J)$ are defined to be the relative singular homology and cohomology groups of the algebra W modulo the ideal J. If $W=C(X)$ is the algebra of all continuous real functions defined on a compact Hausdorff space X, then the singular complex $S(X)$ of the space X is simplicially isomorphic with the singular complex $S(W)$ of the algebra W, that is to say, there is a simplicial map h: $S(X)\rightarrow S(W)$ which maps $S(X)$ onto $S(W)$ in a one-to-one fashion. If A is any closed subspace of X and J denotes the ideal of W consisting of the elements w with $w(A)=0$, then h maps $S(A)$ onto $\sigma(J)$. This implies that the relative singular homology and cohomology groups of X modulo A are isomorphic with the corresponding groups of W modulo J. (Received November 10, 1952.)

163. Arno Jaeger: *On rings of differential operators over fields of prime number characteristic.*

Let D be an *iterative differentiation* in the sense of F. K. Schmidt (cf. J. Reine Angew. Math. vol. 190 (1952)) of a field F of characteristic $p\neq 0$ with the property $Dx=1$, $D^n x=0$ $(1<n<p^i)$ for a certain element $x\in F$. Let F_i be the field of all constants of orders $\geqq i$ in F. Then, according to a basis theorem, the additive group F^+ of F can be regarded as a vector space of dimension p^i over F_i, the basis elements being the x^n $(0\leqq n<p^i)$. Let R_i be the ring of endomorphisms of F^+ of the type $\alpha=\sum_{n=0}^{p^i-1} a_n D^n$ $(a_n\in F)$ where αy $(\alpha\in R_i,\ y\in F)$ and $\alpha+\beta$ $(\alpha, \beta\in R_i)$ are defined in the usual way and $\alpha\beta$ by $\alpha\beta=\sum_n a_n D^n \sum_m b_m D^m=\sum_j \sum_n \sum_{k\leqq n,j} C_{j,k} a_n (D^{n-k} b_{j-k}) D^j$. If an element of R_i is an automorphism of F^+, then the inverse automorphism is an element of R_i, too. Every element of R_1 can be decomposed in a product of linear factors left multiplied by a unit. These and more complicated properties of R_i (especially for $i>1$) are applied to solve linear differential equations in F explicitly and to discuss the conditions for solvability in the nonhomogeneous case. (Received November 17, 1952.)

164t. R. V. Kadison: *Infinite general linear groups.*

The uniformly closed, normal subgroups of the general linear group (group of all invertible operators) of the various types of factors are studied. In the case of a factor of type II_1, the determinant function comes into play, and it is shown that each closed, noncentral, normal subgroup of the general linear group of such a factor is the complete inverse image under the determinant map of some closed subgroup of the positive reals. The general linear group of a factor of type III contains no proper, closed, noncentral normal subgroups as is the case with the determinant 1 subgroup in a factor of type II_1. The uniform closure in the general linear group of the set of invertible operators in a factor of type I_∞ or II_∞ which act as the identity on the orthogonal complement of some subspace of finite relative dimension is a proper, closed, noncentral, normal subgroup. In a factor of type I_∞ each closed, noncentral normal subgroup of the general linear group is the direct sum of the group just defined and the nonzero scalars. There are results which indicate that the same situation prevails in a factor of type II_∞ but the complete result is not obtained in this case. (Received November 13, 1952.)

165. Irving Kaplansky: *Dual modules over a valuation ring.* II.

This paper is chiefly devoted to locally linearly compact modules over a complete discrete valuation ring. After the requisite duality is set up, sundry structure theorems are proved; in particular, modules annihilated by p^2 admit complete analysis. Stated for a locally compact abelian group G satisfying $p^2G=0$, the result asserts that G is a direct sum of a discrete group, a compact group, and a third group which is characterized by a certain pair of dual vector spaces over the integers mod p. If G satisfies the second axiom of countability, the vector spaces have countable dimension and the inner product between them is unique by a theorem of Mackey; this recaptures a theorem of Vilenkin. (Received November 13, 1952.)

166. Erwin Kleinfeld: *Simple alternative rings.* II.

Let R be a simple alternative ring, not a nil ring. Then using methods developed by R. H. Bruck and the author (see Proc. Amer. Math. Soc. vol. 2 (1951) pp. 878–890) it is shown that the fourth power of every commutator of R is in the nucleus of R. If R is not associative then its nucleus and center coincide. A. A. Albert's result on simple alternative rings with nontrivial idempotents (see Canadian Journal of Mathematics vol. 4 (1952) pp. 129–135) reduces the situation to the case where the nilpotent elements can be shown to form an ideal. The final result is that R is either associative or a Cayley-Dickson algebra of order eight over its center. (Received November 13, 1952.)

167. W. J. LeVeque: *On the distribution modulo* 1 *of sequences involving periodic functions.*

Suppose that $\phi\in C^2$ is periodic, of period ω, and that both ϕ and its derivative ϕ' are functions which have only finitely many zeros in $(0, \omega)$, and which have no zeros in common with their respective first derivatives. Then it is shown, using the method developed in earlier papers, that the sequence $\{u_n\phi(v_nx-c)\}$ is uniformly distributed (u.d.) modulo 1 for almost all $x\in(0, \omega)$ if either (A) $\{u_n\}$ is a positive lacunary sequence and $\{v_n\}$ is a positive nondecreasing sequence, or (B) $\{v_n\}$ is a positive lacunary sequence and $\{u_n\}$ is a positive nondecreasing sequence such that $n^\epsilon=O(u_n)$ for some $\epsilon>0$. Also, bounds are given for the associated exponential sum in the two cases. A corollary of case (A) is that the complex numbers ξ, ξ^2, $\cdots$, when reduced modulo 1 and modulo i, are uniformly distributed over the unit square $0\leqq \operatorname{Re} z<1$, $0\leqq \operatorname{Im} z<1$ for almost all complex ξ with $|\xi|>1$. This extends a result due to Koksma. If neither $\{u_n\}$ nor $\{v_n\}$ is lacunary, the method fails for general periodic ϕ, but it is shown that the following sequences are u.d. (mod 1) for almost all $x\in(0, 2\pi)$: (C) $\{an^j \cos(bn^kx-c)\}$, where b and k are positive integers with $j\geqq k$, and (D) $\{u_n \cos(u_nx-c)\}$, where u_n is an increasing sequence of positive integers. (Received October 31, 1952.)

168*t*. W. J. LeVeque: *On uniform distribution modulo* Δ.

Let Δ be a subdivision of the interval $(0, \infty)$: $\Delta=(z_0, z_1, \cdots)$, where $0=z_0<z_1<\cdots$ and $\lim_{n\to\infty} z_n=\infty$. For $z_{n-1}\leqq x<z_n$, put $[x]_\Delta=z_{n-1}$, $\delta(x)=z_n-z_{n-1}$, and $(x)_\Delta=(x-[x]_\Delta)/\delta(x)$, so that $0\leqq(x)_\Delta<1$. Then the positive increasing sequence $\{x_k\}$ may be said to be uniformly distributed modulo Δ if the sequence $\{(x)_\Delta\}$ is u.d. (mod 1). Various criteria are given for uniform distribution (mod Δ), including extensions of Fejér's theorem for sequences $\{x_k\}$ which increase slowly relative to $\{x_n\}$ (i.e., such that $[x_k]_\Delta$ is constant for long blocks of indices k), and the assertion that if $\delta(x)$ is a nonincreasing function such that $\delta(x)=O(x^{-1})$, then $\{k\theta\}$ is u.d. (mod Δ) for almost all $\theta>0$. Finally, a transfer theorem is proved, making it possible to deduce uniform distribution (mod Δ_2) from uniform distribution (mod Δ_1) in certain cases. (Received October 31, 1952.)

169*t*. T. S. Motzkin and Olga Taussky: *Pairs of matrices with property* L. II.

A large class of pairs A, B of $n\times n$ matrices with property L is characterized in a manner different from the one introduced originally (see T. S. Motzkin and O. Taussky, Bull. Amer. Math. Soc. vol. 58 (1952) p. 41 and Trans. Amer. Math. Soc. vol. 73 (1952) pp. 108–114). Consider the pencil P of matrices $\lambda A+B$, and in particular the matrices with multiple characteristic roots in P. Assume that P is not singular, i.e. not every matrix in P has a multiple characteristic root. Then there are $n(n-1)$ values of λ for which the corresponding matrix has a multiple characteristic root, namely the roots of the discriminant of $|\lambda A+B+xI|=0$. We say that the pair of matrices A, B has property D if all these roots are double. It is proved that a pair of matrices A, B with property D which generate a pencil, no matrix in which has a characteristic root of multiplicity $\geqq 3$ nor two different double characteristic roots, has property L. (Received November 13, 1952.)

170. Alex Rosenberg: *The number of irreducible representations of simple rings with no minimal ideals.*

An associative ring is said to be simple if it is semi-simple and has no proper two-sided ideals. A simple ring is called a Zorn ring if every non-nil left ideal contains a nonzero idempotent. Let A be a simple Zorn ring with no minimal ideals, unit element, and exactly $\aleph_0$ elements. Then it is shown that A admits precisely $2^{\aleph_0}$ distinct (i.e. non-isomorphic) faithful irreducible modules. Further let L be the ring of all linear transformations on a vector space of dimension $\aleph$ over a division ring of at most $\aleph$ elements, and let F be the maximal two-sided ideal in L. Then it is shown that $L-F$ admits precisely $2^{2^{\aleph}}$ distinct, irreducible modules. (Received November 12, 1952.)

171. R. D. Schafer: *A generalization of a theorem of Albert.*

A. A. Albert's recent theorem, that any finite power-associative division ring of characteristic $p>5$ is a field, is generalized as follows. Let R be a finite power-associative ring without elements of additive order 2, 3, or 5. If every element a of R satisfies an equation of the form $a^{n(a)}=a$, $n(a)$ an integer >1, then either (i) R is a direct sum of finite fields, or (ii) the attached commutative ring R^+ contains an ideal which is the unique 3-dimensional classical central simple Jordan algebra without nilpotent elements $\neq 0$ over some finite field; in this latter case, R^+ is a direct sum of fields and such 3-dimensional algebras (over possibly different finite fields). (Received November 3, 1952.)

172. L. M. Weiner: *A factorization theorem concerning sums of homogeneous functions.*

In this paper is begun the study of the ring R of functions which are representable as sums of homogeneous functions. Any element of R is $F=F_n+F_{n-1}+\cdots+F_0$, where F_i is a real-valued, continuous, homogeneous function of degree i of m real variables. This ring R is not an integral domain but contains the polynomial integral domain as a proper subset. Associated with each point p of the region of definition of the elements of R and each F of R is the polynomial $P_F(p) = F_n(p)x^n + F_{n-1}(p)x^{n-1} + \cdots + F_0(p)$. In general, an element F of R cannot be factored uniquely into irreducible factors. The following theorem is established. If F is an element of R defined in a region $\mathfrak{R}$ such that the discriminant of $P_F(p)$ is distinct from zero and $F_n(p)$ and $F_{n-1}(p)$ are not both zero for any point p of $\mathfrak{R}$, then F is factored uniquely into irreducible factors in $\mathfrak{R}$ aside from the order of the factors and the units. (Received November 12, 1952.)

173. George Whaples: *Additive polynomials*. II.

Let k be a field with characteristic p, no inseparable extensions, and infinitely many elements. All decompositions (1) $u(x) = v(f(x)) + g(x)$ and (2) $u(x^{p^m}) = h(w(x))$, where u, v, w denote arbitrary polynomials over k with constant term zero and f, g, h additive polynomials ($=$a.p.$=p$-polynomials as defined by O. Ore, Trans. Amer. Math. Soc. vol. 35 (1933) pp. 559–584), are consequences of the following maximal ones. In (1), $f(x)$ may be chosen as the least degree a.p. whose zeros are the set of all Γ in the algebraic closure of k with $u(x+\Gamma) = u(x) + u(\Gamma)$. In (2), $h(x)$ may be chosen as the least degree a.p. for which an identity (3) $h(x^{p^n}) = \sum_\nu u(l_\nu(x))$ holds. If furthermore k has one extension of degree p and for each n at most one extension of degree n, then (3) may be replaced by the condition: $h(x) =$ the least degree a.p. with $h(K^+)$ $=$additive group generated by $u(K^+)$ for all finite extensions K of k. (Received November 24, 1952.)

ANALYSIS

174. W. G. Bade: *Unbounded spectral operators*. Preliminary report.

N. Dunford has shown (see Bull. Amer. Math. Soc. Abstract 59-2-185) that if $S = \int \lambda E(d\lambda)$, where $E(\beta)$ is a compact projection valued measure on the Borel subsets β of the complex plane, then a bounded operator T commuting with all the $E(\beta)$ is a spectral operator (i.e. satisfies $\sigma(T, E(\beta)X) \subset \bar{\beta}$ for each β) if and only if $T = S + N$ where N is a bounded quasinilpotent commuting with S. The corresponding problem for unbounded operators is investigated where $E(\beta)$ is a measure on the complex sphere K and $Sx = \lim \int_{C_n} \lambda E(d\lambda)x$, $C_n = \{\lambda; |\lambda| \leq n\}$ where this limit exists. A closed unbounded operator T is a spectral operator (i.e. $\sigma(T, E(\beta)X \subset \bar{\beta}$, the closure of β in K) in case N, defined by $Nx = Tx - Sx$, is bounded on the dense set $\{x; x = E(\beta)y, y \in X, \beta \text{ bounded}\}$. However, examples are constructed to show T may be an unbounded spectral operator with (a) N unbounded with no resolvent set, (b) N bounded but not quasinilpotent in X. An operational calculus for T and S is investigated. (Received December 15, 1952.)

175. R. G. Bartle: *Solutions of equations in Banach spaces*. Preliminary report.

A study is made of the equation (*) $v = H(u)$, where H maps a Banach space $\mathfrak{U}$ into $\mathfrak{B}$, and has on $\mathfrak{U}$ a Fréchet differential mapping onto $\mathfrak{B}$. The following results are cited. Given v_0 such that (*) has k distinct solutions, conditions are given to assure that (1) points in a neighborhood of v_0 have (1a) at least k distinct solutions, or (1b) exactly k solutions, and that (2) every point in $H(\mathfrak{U})$ has exactly k distinct solutions. If the inverse image under H of a line segment in $\mathfrak{B}$ is compact, then H maps onto $\mathfrak{B}$; if, in addition, H is locally 1-1, then H is a homeomorphism of $\mathfrak{U}$ onto $\mathfrak{B}$. If $\mathfrak{X}$ is a compact metric space and $\Gamma_0 = \{x_0(t); t \in \mathfrak{X}\}$ is a curve in $\mathfrak{U}$ which H sends into v_0, then for v_1 sufficiently near v_0 there will be a curve $\Gamma_1 = \{x_1(t); t \in \mathfrak{X}\}$

mapping into ν_1 such that $x_1(t)$ is uniformly close to $x_0(t)$, for $t\in\mathfrak{T}$. (Received November 13, 1952.)

176t. S. D. Bernardi: *On an inequality of Prawitz.* Preliminary report.

Recent results of the author (see *Two theorems on schlicht functions*, Duke Math. J. vol. 19 (1952); *A survey of the development of the theory of schlicht functions*, Duke Math. J. vol. 19 (1952)) are improved through the application of the inequality $\sum_{n=1}^{\infty}(2n-\alpha)|b_n|^2/\alpha\leq 1$, where $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$ is regular and schlicht for $|z|<1$ and $(f(z)/z)^{-\alpha/2}=1+\sum_{n=1}^{\infty}b_nz^n$, α real. The inequality is applied to obtain information in the form of inequality relations concerning functions $f(z)$ with coefficients a_n belonging to special algebraic fields. The methods are similar to those used in the two aforementioned papers. (Received November 4, 1952.)

177t. H. D. Block: *Construction of solutions and propagation of errors in nonlinear problems.*

It is intended that the results of this paper provide usable theoretical tools for numerical computation of the solutions of a class of nonlinear problems. If X is an additive group with a metric which is invariant under translation, then we call X a b-space. Let X and Y be b-spaces and let F map X into Y. With certain assumptions concerning F, the equation $Fx=y$ has, for each y in Y, a unique solution x, the iterative construction of which is presented. Bounds on the error due to using only a finite number of iterations are presented, as well as some other useful estimates. Bounds on the errors involved due to using the perturbed equation: $(F+f)(x+\xi)=(y+\eta)$ are also given. If T is an arbitrary set and G maps $T\times X$ into Y then, under suitable assumptions on G, the equation $G(t,x)=0$ defines an implicit function $x=x(t)$. The iterative determination of $x(t)$ is also treated. Finally an iterative method for the eigenvalue problem $Fx=\lambda x$, where X and Y are now the same normed vector space, is also presented. (Received November 13, 1952.)

178t. F. F. Bonsall and Morris Marden: *The zeros of rational functions with self-inversive polynomial factors.*

This paper generalizes the authors' previously announced results [Bull. Amer. Math. Soc. Abstract 58-2-157]. Let $M[\Phi,T]$ denote the total multiplicity of the zeros of a function Φ in the region T and $N[\Phi,T]$ the number of distinct zeros of Φ in T. The principal result of this paper may be stated as follows: Let $\phi=(fg)/(FG)$, where f and F are self-inversive polynomials (i.e., symmetric in the unit circle C), where g is a polynomial with all its zeros inside C and G one with all its zeros outside C; and where degree $(fg)>$degree (FG); then, for the derivative ϕ' of ϕ, $M[\phi',|z|>1]=M[\phi,|z|>1]+N[1/\phi,|z|\geq 1]$. Thus, if $h(z)$ is any polynomial having all its zeros in C and if $\psi(z)=h(z)\phi(z)$, then $M[\psi',|z|>1]=M[\phi',|z|>1]$. These results are also extended to certain cases where degree $(fg)=$degree (FG). (Received November 26, 1952.)

179. H. J. Bremermann: *The kernel function and invariant metric in domains of regularity.*

A domain of existence of an analytic function of complex variables is denoted as a "domain of regularity." Any domain G possesses an "envelope of regularity" $R(G)$. The Bergman kernel function $K_G(z,\bar{\zeta})$ $(z=(z_1,z_2),\ \zeta=(\zeta_1,\zeta_2))$ for any domain G is analytic in $R_z(G)\times R_{\bar{\zeta}}(G)$. If $K_G(z, \bar{z})$ becomes infinite at all boundary points of G, then G is a domain of regularity. (Corollary: The domains $K_G(z, \bar{z}) < M$ are domains of regularity.) Examples show that for an arbitrary domain of regularity R the domain $R_z\times R_{\bar{\zeta}}$ is not necessarily the domain of existence of the kernel function $K_R(z, \bar{\zeta})$. But if R can be approximated by a sequence of domains of regularity containing $\bar{R}$, then $R_z\times R_{\bar{\zeta}}$ is the domain of existence of $K_R(z, \bar{\zeta})$. If in addition the boundary of R is "smooth" or if R is an analytic polyeder, then $K_R(z, \bar{z})$ becomes infinite everywhere at the boundary of R. In this case all boundary points of R are infinite points with respect to the Bergman metric. On the other hand, if one can introduce in a domain G a Kähler metric with analytic coefficients in such a way that all boundary points are "infinite," then G is a domain of regularity. (Received November 17, 1952.)

180. H. D. Brunk: *On functions having isolated singularities on the positive real axis.* Preliminary report.

In a strip region, Δ, let $f(s)$ be holomorphic except for isolated singularities at points $s=\lambda_n$ (real), $n=1, 2, \cdots, 0<\lambda_n\uparrow\infty$. If $f(s)$ is suitably restricted on the boundary of Δ, an elementary observation involving the residue theorem yields a function $F(s)$ represented by the partial sums $\sum_{k=1}^{n} a_k \exp(-\lambda_k s)$ asymptotically with respect to a sequence of positive constants related to the growth of $f(s)$ in Δ; here a_k is the residue of $f(s)$ at $s=\lambda_k$. Theorems on asymptotic Dirichlet series ((1) Mandelbrojt, Ann. École Norm. (3) vol. 43 (1946) pp. 351–378; (2) Mandelbrojt and Brunk, Duke Math. J. vol. 18 (1951) pp. 297–306; (3) Brunk, Bull. Amer. Math. Soc. vol. 55 (1949) pp. 204–212) then give theorems relating the magnitudes of the residues of $f(s)$ to its growth in Δ. For example, one can say, roughly, that if $f(s)$ fails to grow sufficiently rapidly in Δ, the fundamental inequality of (1) shows that $\limsup_{n\to\infty} (\log |a_n|)/\lambda_n < \infty$ (if $\liminf_{n\to\infty} \lambda_{n+1}-\lambda_n>0$). Similarly, the theorem of (2) has application to a situation in which at each point $s=\lambda_k$ one or the other of two functions holomorphic in a certain right half-plane except for isolated singularities at points $s=\lambda_k$ has residue 0. (Received November 12, 1952.)

181*t*. P. L. Butzer: *On the extensions of Bernstein polynomials to the infinite interval.*

Let the function $f(x)$ be defined over the infinite interval $0\leqq x<\infty$. Szász [Jour. Research Nat. Bur. Stand. vol. 45 (1950) pp. 239–245] introduced the transform $P^f(u; x)=e^{-ux}\sum_{\nu=0}^{\infty} [(ux)^\nu/\nu!]f(\nu/u)$, for which $\lim_{u\to\infty} P^f(u; x)=f(x)$ uniformly in $(0, \infty)$ if $f(x)$ is continuous in $(0, \infty)$. $P^f(u; x)$ is the analogue for the interval $[0, 1]$ of the familiar Bernstein polynomials. The author considers a generalization of $P^f(u; x)$, namely the transform $W^f(u; x)=ue^{-ux}\sum_{\nu=0}^{\infty}[(ux)^\nu/\nu!]\int_{\nu/u}^{(\nu+1)/u} f(s)ds$ where $f(x)$ is Lebesgue integrable over the interval $0\leqq x<\infty$, i.e. $f(x)\in L(0, \infty)$. Let $M^f(x)=\sup_{0<s<\infty} (1/(s-x))\int_x^s|f(y)|dy$. If $f(x)$ and $|f(x)|^p$, $p>1$, both belong to $L(0, \infty)$, then it is shown that (i) $\lim_{u\to\infty} W^f(u; x)=f(x)$ almost everywhere in $(0, \infty)$, (ii) there exists a function $M^f(x)$ in the space L^p such that $\sup_{u>0} |W^f(u; x)| \leqq 3M^f(x)$ almost everywhere in $(0, \infty)$. As a corollary $\int_0^\infty |W^f(u; x)-f(x)|^p dx\to 0$ as $u\to\infty$. (Received November 13, 1952.)

182. Lamberto Cesari: *On the analytical representation problem of Frechet surfaces.*

After a series of research (C. B. Morrey, J. W. T. Youngs, E. J. Mickle) L. Cesari proved the following theorem: Given any continuous mapping of finite Lebesgue area

$(T, A): p = T(w)$, $w = (u, v) \in A$, from a closed 2-cell A of the w-plane into the p-space E_3, $p = (x, y, z)$, there is Frechet equivalent mapping $(T', B): p = T'(w)$, $w \in B$, from the unit circle B into E_3, $T'(w) = [x(u, v), y(u, v), z(u, v)]$, where the functions x, y, z have a.e. in B^0 ordinary first partial derivatives L^1-integrable in B^0; the Lebesgue area is given by the classical integral, and $E = G$, $F = 0$ a.e. in B^0. The author has given a new proof of the theorem above, under more general conditions, where no use is made of the Dirichlet integral and of the topological structure properties of continuous mappings. Consistent use is made of tangential properties of mappings of finite Lebesgue area (L. Cesari, Comment. Math. Helv. vol. 22 (1949) pp. 1–17) and of the existence and properties of an everywhere dense system of rectifiable contours (L. Cesari, Bull. Amer. Math. Soc. vol. 57 (1951) p. 168; R. E. Fullerton, Ibid. vol. 58 (1952) p. 57, and p. 180). The set A is supposed to be any of the following sets: any open set A'; any finite sum A'' of disjoint finitely connected Jordan regions; any subset A''' of a set A'' open in A''. The set B is supposed to be any set topologically equivalent to A. (Received November 12, 1952.)

183. J. M. Danskin: *An extension of the Brown-Robinson iterative process for finding the value of a game.*

Let $M(x, y)$ be a continuous function defined on the unit square. We define sequences $\langle x_n \rangle$ and $\langle y_n \rangle$ as follows: $\langle x_n \rangle$: x_0 is arbitrary; x_n maximizes the sum $\sum_{k=0}^{n-1} M(x, y_k)$ $(n \geq 1)$; $\langle y_n \rangle$: y_0 is arbitrary; y_n minimizes the sum $\sum_{k=0}^{n-1} M(x_k, y)$ $(n \geq 1)$. It is proved that $\lim_{n \to \infty} (1/n) \sum_{k=0}^{n-1} M(x_n, y_k) = \lim_{n \to \infty} (1/n) \sum_{k=0}^{n-1} M(x_k, y_n) = v$, where v is the "value of the game" in the sense of the theory of games. This generalizes a result of Julia Robinson, proved for matrices (Ann. of Math. vol. 54 (1951) 296–301). The result remains valid when the unit square is replaced by the direct product of two compact spaces. (Received November 13, 1952.)

184. Jim Douglas, Jr.: *On the expansion of measures in series.*

Let (X, S) be a measurable space, and let $\{z_\alpha\}$ be the class of all measures defined on (X, S) each of which assumes the values zero and one only. Let measures m_1 and m_2 be absolutely continuous with respect to n with density functions f_1 and f_2. Then the Kakutani quasi-metric is $\rho(m_1, m_2) = \int (f_1 f_2)^{1/2} dn$. For each totally finite signed measure m, define $m^* = \sum \rho^2(m, z_\alpha) z_\alpha$. Various necessary and sufficient conditions for $m = m^*$ are given. As for ordinary Fourier series, there exist spaces (X, S) and measures m for which $m^* \neq m$. Conditions on S are given to insure $m = m^*$ for all m defined on S. (Received November 10, 1952.)

185. Nelson Dunford: *Spectral operators.*

An operator T in a complex Banach space X is called a spectral operator in case it has a resolution of the identity E defined on the Borel sets in the plane. For such operators the vector valued function $(\xi I - T)^{-1} x$ has a maximal single-valued analytic extension $x(\xi)$ defined on an open set $\rho(x)$. The spectrum $\sigma(x)$ of x is defined to be the complement of $\rho(x)$. The resolution of the identity is unique and satisfies the relation $E(\sigma)X = [x \mid \sigma(x) \subset \sigma]$, providing σ is closed. A corollary is that $E(e)$ commutes with every bounded linear operator which commutes with T. The operator S is said to be of scalar type in case it is spectral and satisfies $S = \int \lambda E(d\lambda)$. An operator T is spectral if and only if it is the sum $T = S + N$ of a scalar type operator S and a generalized nilpotent N commuting with S. This canonical decomposition is unique and for functions f analytic on the spectrum of T, $f(T) = \sum (N^n/n!) \int f^{(n)}(\lambda) E(d\lambda)$. Commutative algebras of spectral operators are of the form $A = C(M) \oplus R$ where M is the space of maximal ideals in A and R is the radical in A. (Received November 13, 1952.)

186. P. R. Garabedian and M. M. Schiffer: *Convexity of domain functionals.*

The second variations of the capacity γ, virtual mass α, and principal frequency λ of a region D are derived rigorously for Hadamard shifts of the smooth surface S of D and for interior variations of D in space of two and three dimensions. When S is shifted along level surfaces of a harmonic function u, it is shown that the capacity γ of S is a convex function of u. The inverse square $1/\lambda^2$ of the principal frequency of a domain D obtained as the image of a fixed plane region by a conformal mapping $z+\epsilon f(z)$ is shown to be a convex function of ϵ. Convexity theorems such as these are used to prove that domains for which certain combinations q of capacity, virtual mass, and principal frequency are stationary actually minimize q and are unique. For example, the result of Pólya and Szegö that λs^7 is a maximum for a circular disc is rederived, and it is shown that the virtual mass of an axially symmetric parallel flow past a surface S enclosing two circular rings is a minimum when S reduces to a vortex sheet spanning the two rings. (Received December 5, 1952.)

187. Felix Haas: *A theorem about differential equations on two-dimensional manifolds.*

Let M be a closed, orientable, two-dimensional manifold. V, a vector field on M, defines a differential equation in a way which insures uniqueness. Let the singular points of V form a compact set. Finally, let C^+ be a semi-characteristic of the differential equation defined by V, and let $\overline{C}$ be the set of ω-limit points of C^+ which we assume free of singular points. Then the following theorem holds: If $\overline{C}$ is dense anywhere on M, M is a torus and V is free of singular points. This is the best theorem possible in this direction since the work of Denjoy shows that if M is a torus and V is free of singular points, the set $\overline{C}$ may certainly be dense in some set G of M. The proof proceeds along the following lines: If $\overline{C}$ does contain a closed curve or a curve of infinite arc length which is a characteristic of the orthogonal vector field, it is shown that M is a torus and V is free of singular points. If $\overline{C}$ does not contain such a curve, it is shown that $\overline{C}$ cannot be dense anywhere on M. (Received October 28, 1952.)

188. Henry Helson: *Lacunary sets of integers.*

Classical theorems of Sidon and of Zygmund on lacunary trigonometric series were proved under the hypothesis that the frequencies $n_1, n_2, \cdots$ occurring in the series satisfy a relation $n_{\nu+1}/n_\nu \geq k > 1$. Two conditions weaker than this one are given, one necessary for the truth of Sidon's theorem, the other sufficient for Zygmund's theorem. The proofs depend on theorems of Hardy and Littlewood. (Received November 10, 1952.)

189*t*. Alfred Huber: *On the uniqueness of generalized axially symmetric potentials.*

Let G be a region lying in the half-plane $y>0$, the boundary of which contains a segment S of $y=0$. Let $u(x, y)$ be a solution of $y\Delta u+ku_y=0$ (k an arbitrary real constant), defined in G, assuming the boundary value 0 on S. The results are: (1) For $k\geq 1$: $u=0$ throughout G. (2) For $k<1$: $u=y^{1-k}v(x, y)$, where v is analytic in $G+S$ and satisfies $y\Delta v+(2-k)v_y=0$. Inversely each function u of this form satisfies the above conditions. The proof involves the use of the identification principle for solutions even in y and analytic on S (A. Weinstein, *Discontinuous integrals and generalized* *potential theory*, Trans. Amer. Math. Soc. vol. 63 (1948) p. 344) and a theorem on removable singularities on $y=0$. This work was done under contract N7onr 397(05). (Received November 21, 1952.)

190. J. L. Kelley: *Extension theorem for finitely additive measures.*

Let p be a non-negative, monotonic function defined on a boolean algebra $\mathcal{A}$ and satisfying: $p(0)=0$ and (1) $p(x)+p(y)\geqq p(x\vee y)+p(x\wedge y)$ (for example, Lebesgue outer measure on the algebra of all subsets of the real numbers). Let m be a non-negative, additive function, defined on a subalgebra of $\mathcal{A}$, such that $m(x)\leqq p(x)$ for each x. Then there is a non-negative additive extension of m to all of $\mathcal{A}$ which is also dominated by p. If condition (1) is replaced by (2) $p(x)+p(y)\geqq p(x\vee y)$, the corresponding theorem is false. (Received November 17, 1952.)

191*t*. M. S. Klamkin: *Sums and asymptotic sums of series related to the harmonic series.* II.

$\sum_{p=1}^{p}[S_{m,p}/p^n+S_{n,p}/p^m]=S_{m,p}S_{n,p}+S_{m+n,p}+\sum_{p=1}^{p}[\sum_{m,n,r\ \mathrm{cyclic}}S_{m,p}S_{n,p}/p^r-S_{m,p}/p^{n+r}]$ $=S_{m,p}S_{n,p}S_{r,p}-S_{m+n+r,p}$, two identities of 2nd and 3rd order respectively, are extended to the Nth order where $S_{m,p}=\sum_1^p 1/p^m$. These identities are also valid if $S_{m,p}=\sum_1^p 1/a_p^m$. If $T_{r,p}=\sum_1^p 1/b_p^r$ we also have $\sum_1^p [S_{n,p}/b_p^m+T_{m,p}/a_p^n]=S_{n,p}T_{m,p}$ $+\sum_1^p 1/a_p^n b_p^m$. This is also extended to higher order. The above identities are then specialized to obtain sums and asymptotic sums of series whose pth terms are given by $(-1)^{p+1}S_{1,p}/p$; $(-1)^{p+1}R_{1,p}/p$; $p^n S_{1,p}$; and $S^m_{r,p}R^n_{s,p}/p$ where $R_{s,p}=\sum_1^p (-1)^{p+1}/p^s$. (Received November 13, 1952.)

192. V. L. Klee, Jr.: *Invariant extension of linear functionals.*

There is established a theorem on extension of linear functionals, subject to being majorized by a given positively homogeneous subadditive functional and to being invariant under a given set of linear transformations. As corollaries are obtained an improvement of the results of Agnew and Morse (Ann. of Math. vol. 39 (1938) pp. 20–30) and Woodbury (Bull. Amer. Math. Soc. Abstract 56-2-168) on invariant extension of functionals, and also the principal results of Dunford and Yood (Proc. Amer. Math. Soc. vol. 2 (1951) pp. 225–233) on existence of a common nonzero fixed point for the adjoints of a semi-group of bounded linear operators. Besides the previously known cases of commutative semi-groups and solvable or finite groups, the desired extension and existence theorems are established for compact groups of bounded linear operators. (Received November 14, 1952.)

193. G. G. Lorentz and D. G. Wertheim: *Linear functionals on Köthe spaces.*

Let $X_C=X$ be the ("Köthe") space of all real functions $f(t)$ such that $\|f\|=\sup\int_0^1|f|c\,dt<+\infty$, where the supremum is taken for all $c(t)\in C$. Properties of C are discussed which ensure that X_C is a Banach lattice with some simple abstract properties. Let $X(B)$ be the space of all weakly measurable functions $\bar{f}(t)=\bar{f}_t$ from (0, 1) to a Banach space B such that $\|\bar{f}\|=\|\|\bar{f}(t)\|_B\|_*<+\infty$. Under simple assumptions on X and for separable B the general form of a continuous linear functional $L(f)$ on $X(B)$ is given by $L(f)=\int_0^1\bar{F}_t(\bar{f}_t)dt$ where $\bar{F}_t$ is an element of the space $X^*(B^*)$. The proof uses operators of class $(b, 0)$ of Kantorovitch and is different from that given by Dieudonné [Canadian Journal of Mathematics vol. 3 (1951) pp. 129–139] for the special case $X=L^p$. (Received November 14, 1952.)

194. G. R. MacLane: *Distortions of the Riemann surface of the inverse sine.*

The Riemann surface of $z=\arcsin w$ may be split into "sheets" S_n, where each S_n is a w-plane cut along all the real axis except for the interval $(-1, 1)$. Let S_n be distorted as follows: the two cuts, $(1, +\infty)$ and $(-\infty, -1)$, are pushed, remaining parallel to the real axis, into any two cuts parallel to the real axis. This distortion is limited by the following requirement: let d be a constant, $0<d<\pi/2$, and let there exist points w_n such that in each distorted S_n one cut lies in $|\arg (w-w_n)|<\pi/2-d$ and the other cut lies in $|\arg (w_n-w)|<\pi/2-d$. Then the distorted surface is parabolic and is the image of $|z|<\infty$ by an entire function $w=f(z)$. Assuming that $0\in S_0$ is not a branch point of the surface, there exists a particular normalization of this map, $f(0)=0\in S_0$, $f'(0)=p_0+q_0i$, $p_0^2+q_0^2=1$, such that $f'(z)$ has infinitely many distinct representations of the form $f'(z)=(\exp (-az^2))\{c_1e^{rz}\pi_1(z)+c_2e^{sz}\pi_2(z)\}$, where a, r, and s are real constants, $a\geqq 0$, and $\pi_i(z)=\Pi'\{(1-z/b_{i,k}) \exp (z/b_{i,k})\}$, with $\cdots b_{i,-k}<b_{i,-k+1}<\cdots<b_{i,-1}<0<b_{i,1}<\cdots<b_{i,k}<\cdots$ and the zeros of π_1 and π_2 are approximately interlaced: $b_{2,k-1}<b_{1,k}<b_{2,k+1}<b_{1,k+2}$. (Received November 12, 1952.)

195*t*. Hazleton Mirkil: *Translation-invariant function algebras over compact groups.* II.

Let A be a function algebra (pointwise multiplication) which is "left-homogeneous" over a compact group G in the sense of our previous abstract (Bull. Amer. Math. Soc. Abstract 58-6-642). Theorem: The following three lattices are isomorphic: (1) the left-quotients of G by closed subgroups, (2) the left-invariant self-adjoint subalgebras of $Q(G)$, (3) the left-invariant self-adjoint closed subalgebras of A. These isomorphisms are the natural ones, e.g. the subalgebra corresponding to a given left-quotient G/G_0 consists of all functions which are constant on each left coset, or, equivalently, which admit the elements of G_0 as right periods. In particular, the theorem states that left-invariant self-adjoint subalgebras of Q are automatically closed in Q for all reasonable norms, and that left-invariant self-adjoint closed subalgebras A_0 of A are generated by $A_0\cap Q$. (Received October 16, 1952.)

196*t*. Hazleton Mirkil: *Translation-invariant function algebras over compact groups.* III.

Let A be left-homogeneous over compact G, and let J be the ideal consisting of elements which vanish on neighborhoods of $e\in G$. A is "of type C" if (1) for all $s\neq e$ there exists $x\in J$ with $x(s)\neq 0$, and (2) there exists $k>0$ with $\|x+y\|>k\|x\|$ for all $y\in J$ and all x. $A/\bar{J}$ is then a "local" algebra, i.e., it has a unique maximal ideal, and A is completely determined by the restriction to $Q(G)$ of the mapping $A\rightarrow A/\bar{J}$. Conversely, given an abstract local banach algebra P with maximal ideal M and a homomorphism h: $Q\rightarrow P$ satisfying $h(u_{ij})\equiv\delta_{ij}$ (mod M) for every irreducible matrix element u_{ij}, a condition which is clearly necessary, one can construct in a natural way a left-homogeneous algebra of type C. (Received October 16, 1952.)

197*t*. Hazleton Mirkil: *Translation-invariant function algebras over compact groups.* IV.

Let G be a compact group and let A be the finite linear combinations of continuous positive-definite functions. It is known that the pointwise product of two elements of A belongs to A, and that A has a norm (topologically unique) which makes it a complete normed algebra, in fact a "left-homogeneous" algebra over G. Theorem: The closure of the ideal J in A (see the preceding abstract) contains all elements of A which vanish at e. Equivalently, every closed ideal lies in more than one maximal ideal unless it is itself maximal. When G is the circle, A is the absolutely convergent fourier series with $\|\sum c_n e^{inx}\| = \sum |c_n|$; and for this case the theorem was first proved by Wiener. Kaplansky generalized Wiener's result to arbitrary commutative (even noncompact) groups G. (Received October 16, 1952.)

198t. Hazleton Mirkil: *Translation-invariant function algebras over compact groups.* V.

Let G be the multiplicative group of complex numbers of absolute value one, and let A be a homogeneous algebra over G. Then A must have one of two mutually exclusive separation properties: it must be either "interval-quasi-analytic" in the sense of Mandelbrojt (definition: every $x \in A$ constant on an open subset of G is constant on G), or else "regular" in the sense of Silov (definition: every two disjoint closed subsets S_0 and S_1 of G can be separated by some $x \in A$, $x = 0$ on S_0, $x = 1$ on S_1). This dichotomy does not hold for homogeneous algebras A over an arbitrary compact group G. (Received October 16, 1952.)

199t. T. S. Motzkin and J. L. Walsh: *Least pth power polynomials on a real finite point set.*

Let E be the set of m real distinct points $\{x_1, x_2, \cdots, x_m\}$, and for variable $\xi_1 \leqq \xi_2 \leqq \cdots \leqq \xi_n$ let the polynomial $T_n(x) = \prod_1^n (x - \xi_i)$ minimize $\sum_1^m \mu_i |T_n(x_i)|^p$, $\mu_i > 0$, $0 < n < m$. For $p > 1$, $T_n(x)$ is unique and (1) $x_0' < \xi_1 < x_1' < \cdots < \xi_n < x_n'$ where the x_i' belong to E. (2) For $0 < p < 1$ every $T_n(x)$ has distinct roots ξ_i in E. For $p = 1$ there exist fundamental polynomials $T_n^{(\alpha)}(x)$ with distinct roots in E such that the set $T_n(x)$ is precisely the set $\{\sum_\alpha \lambda_\alpha T_n^{(\alpha)}(x), \sum \lambda_\alpha = 1, \lambda_\alpha \geqq 0\}$; every $T_n(x)$ fulfills (3), namely (1) with the signs $\leqq$. For given $p > 0$, any polynomial $\prod_1^n (x - \xi_i)$ satisfying the corresponding condition (1), (2), or (3) is a minimizing polynomial $T_n(x)$ for duly chosen weights. (Received December 9, 1952.)

200. G. G. Murray: *Measures in Boolean algebras.*

A theorem of Nakamura and Sunouchi (Proc. Imp. Acad. Tokyo vol. 18 (1942) pp. 333–335) gives a decomposition of a bounded, finitely additive measure defined on a Boolean algebra with respect to a σ-ideal contained in the algebra, i.e. the measure is split into the sum of two measures, one of which is concentrated in a maximal element of the ideal. By means of Stone's compactification any Boolean algebra B can be represented isomorphically as the collection of open-and-closed sets of a compact Hausdorff space Ω, and a measure m defined on B can be made to correspond to a unique countably additive Baire measure m^* in Ω. Taking the Baire sets of the first category in Ω to be the ideal of the Nakamura-Sunouchi theorem, it is possible to decompose m into the sum of a largest countably additive measure and a purely finitely additive measure. This decomposition was obtained in a different manner by Yosida and Hewitt (Trans. Amer. Math. Soc. vol. 72 (1952) pp. 46–66), but the above method characterizes more precisely the Baire measure corresponding to a purely finitely additive measure since it can be seen to be concentrated in a Baire set of the first category. When the Boolean algebra is taken to be the collection of all subsets of a given set, problems such as the existence of nontrivial, countably additive, two-valued measures and Banach's general measures can be transformed into topological problems in compact Hausdorff spaces. (Received November 17, 1952.)

201*t*. Pasquale Porcelli: *Concerning uniform completeness of sets of reciprocals of linear functions.* II.

This paper is a continuation of the study, begun in an earlier paper of the same title (Bull. Amer. Math. Soc. Abstract 58-6-649), of sets K: $\{(1+h_p x)^{-1}\}_{p=0}^{\infty}$. To the three characterizations in (I) of sets K which are uniformly complete in $F[0, 1]$ is added a fourth, namely: a necessary and sufficient condition for K to be uniformly complete in $F[0, 1]$ is that K should be closed in $L^2[0, 1]$. Furthermore, if K is not closed in $L^2[0, 1]$, then the closed linear manifold generated by K in $L^2[0, 1]$ is nowhere dense in $L^2[0, 1]$. The theorems established in (I) and in this paper concerning K hold also for the sets K_n: $\{(1+h_p x)^{-n}\}_{p=0}^{\infty}$ where n is a positive integer. (Received November 13, 1952.)

202*t*. M. S. Robertson: *Multivalently star-like functions.*

Let $f(z)=a_1z+a_2z^2+\cdots+a_nz^n+\cdots$ be regular and multivalently star-like of order 2 with respect to the origin for $|z|<1$. A. W. Goodman (Trans. Amer. Math. Soc. vol. 68 (1950) pp. 204–223) has shown that $|a_3|\leqq 5|a_1|+4|a_2|$ when the coefficients are all real. The author, using a different method of approach, shows that, even when the coefficients are complex, the following sharp inequalities hold for all positive integers n: $6|a_n|\leqq 2n|n^2-4||a_1|+n(n^2-1)|a_2|$. Analogous sharp inequalities are obtained for multivalently star-like functions of order p of the form $f(z)=\sum_{n=1}^{\infty} a_nz^n$ which further substantiate the Goodman conjecture (Trans. Amer. Math. Soc. vol. 63 (1948) pp. 175–192; also vol. 70 (1951) pp. 127–136) for the coefficients of multivalent functions. (Received November 10, 1952.)

203*t*. L. B. Robinson: *Solution of a system of equations which helps us to compute a complete system of semitensors.*

Given the system of equations $-y_j\partial f/\partial y_i-y_j'\partial f/\partial y_i'-y_j''\partial f/\partial y_i''-I_j\partial f/\partial I_i=0$, $i\neq j$, $i, j=1, 2, 3$, $-y_1\partial f/\partial y_1-y_1'\partial f/\partial y_1'-y_1''\partial f/\partial y_1''+I_2\partial f/\partial I_2+I_3\partial f/\partial I_3=0$, $-y_2\partial f/\partial y_2-y_2'\partial f/\partial y_2'-y_2''\partial f/\partial y_2''+I_1\partial f/\partial I_1+I_3\partial f/\partial I_3=0$, $-y_3\partial f/\partial y_3-y_3'\partial f/\partial y_3'-y_3''\partial f/\partial y_3''+I_1\partial f/\partial I_1+I_2\partial f/\partial I_2=0$. Write the matrix with first row y_1, y_1', y_1'', I_1, second row y_2, y_2', y_2'', I_2, and third row y_3, y_3', y_3'', I_3. The fundamental solutions of the above system are given by the determinants of the above matrix which involve the I. They are three in number. These solutions enable one to compute a complete system of semitensors. (Received October 27, 1952.)

204. L. V. Robinson: *Generalized calculus as related to partial differential equations.*

Basically, the operational and more formal sides of ordinary calculus and ordinary differential equations depend on such operators as $D=d/dx$, $D_x=\partial/\partial x$, $D_y=\partial/\partial y, \cdots$, and on their respective inverses. Hence, to be able to replace these and their inverses with an operator, $\lambda=p_1\partial/\partial x_1+p_2\partial/\partial x_2+\cdots+p_n\partial/\partial x_n$, and its inverse, where the p_k $(k=1, 2, \cdots, n)$ are understood to be complex-valued functions of of the x_k with continuous derivatives in a space S^* in which $|p_1|\neq 0$, is then to extend the scope of ordinary calculus. Such an extension may be referred to as a generalized calculus. It is necessary, however, to define the inverse λ^{-1} and to establish basic rules of operations with λ and λ^{-1}. Having done this, it is then found that, between these rules and those of ordinary calculus, there is a very close parallelism. To this generalized calculus, it further develops that partial differential equations bear the same relation as ordinary differential do to ordinary calculus. In fact, a one-to-one correspondence is found to exist between λ and D, whence for every ordinary differential equation solvable by any means, an infinite number of partial differential equations which can likewise be solved exists. A means is therefore afforded for greatly increasing the number of types of partial differential equations for which formal solutions can be found; and ordinary linear differential equations correspond only to the class of parabolic linear partial differential equations. (Received November 14, 1952.)

205. P. C. Rosenbloom: *Studies on Maxwell's equations.* I.

Solutions of Maxwell's equations of the form exp $(i\omega t)$ $F(x, y, z)$ are studied. It is well known that these equations may be reduced to (1) $LQ=\nabla\times Q+\beta Q=A$, where A is a given vector function, β is a constant, and $Q=H+i\alpha E$, and α is a suitable constant. It is obvious that if R is a solution of $\Delta R+\beta^2 R=A$, then $Q=MR=\beta R-\nabla\times R+\beta^{-1}\nabla(\nabla\cdot R)$ is a solution of (1). It is shown conversely that every solution of (1) in a domain D can be expressed in this form, where R is given explicitly in terms of the tangential components of Q on the boundary of D. In this way one can give an analogue for (1) of the mean value theorem for harmonic functions and the analogue of spherical harmonics in terms of which all solutions of (1) can be expanded locally. The methods can be applied to quite general systems of equations of elliptic type with constant coefficients. (Received November 17, 1952.)

206t. H. M. Schaerf: *Some topological consequences of the existence of measures.*

Let m be a σ-finite countably additive measure on a family S of subsets of a neighborhood space R and N the family of its null sets. *Theorem* 1: If R is of uniform first countability and S the family of all open sets, then a representation $R=R_0+R_1$ with R_0 in N and R_1 separable exists if and only if N is closed under formation of unions. *Theorem* 2: If R is a topological group, m is finite, and S and N are left invariant, then either R is locally compact or of the first category or N contains all measurable compact sets. *Theorem* 3: If R is a metric group of the second category and m is a Borel measure which is positive on all nonvoid open sets, then $m(A)m(B)>0$ implies that $B^{-1}A$ has a nonvoid interior and that every neighborhood of the unit contains an inner point of $A^{-1}A$. (Received December 11, 1952.)

207. J. T. Schwartz: *Abstract boundary values.*

By making use of the fact that every symmetric extension of an operator T is a restriction of its adjoint and by considering the hermitian bilinear form $i(T^*x, y)-i(x, T^*y)$, an alternate approach to the extension theory of symmetric operators can be made. This has the advantage of dealing more directly with the domains of the operators involved than the usual Cayley transform treatment. The method is convenient for the theory of symmetric differential operators, since the abstract and the concrete theory of these operators are brought into adjustment. An account of details will be given in the series of technical reports to be prepared in connection with the ONR spectral theory project at Yale. (Received December 15, 1952.)

208t. J. T. Schwartz: *Essential spectra of differential operators.*

The essential spectrum of a self-adjoint operator is the complement in its spectrum of the set of all isolated points of finite multiplicity. A characterization of the essential spectrum is given which yields many properties of this spectrum for self-adjoint operators derived from ordinary differential operators. The essential spectrum is independent of boundary conditions, an additive function of the interval on which the differential operator is defined, and positive for formally positive operators. A general criterion can be given which yields many particular cases in which the coefficients can be "slightly" modified without changing the essential spectrum. In this way, nth order generalizations of well known results relating the spectrum of a second order linear differential operator to its coefficient functions are obtained. (Received December 22, 1952.)

209*t*. W. S. Snyder: *On derivatives of set functions*. Preliminary report.

Let Γ be a class of closed subsets of a bounded open set G in the plane, each set T in Γ having parameter of regularity greater than a fixed constant $\rho>0$. Let f be a real-valued, non-negative function defined on Γ, with upper derivative $\overline{D}(f, x)$ defined on a set A of positive measure. Let S be a subset of A with positive measure. We say a constant M is in the upper class if, given $\epsilon>0$, there is a $\delta>0$ such that if $\{T_i\}_1^n$ are disjoint sets in Γ with diam $(T_i)<\delta$ and $\left|\sum_1^n T_i-S\right|+\left|S-\sum_1^n T_i\right|<\delta$, then for some subfamily $\{T_{i_j}\}_{j=1}^l$ with $\left|\sum_1^n T_i-\sum_{j=1}^l T_{i_j}\right|<\epsilon$ we have $\sum_{j=1}^l f(T_{i_j})<M$. It is proved that the infimum of numbers M in the upper class is equal to the integral of $\overline{D}(f, x)$ on S. (Received November 14, 1952.)

210. C. T. Taam: *Non-oscillation and comparison theorems of linear differential equations with complex-valued coefficients*.

Consider (A): $W''+Q(x)W=0$, where $Q(x)$ is a complex-valued function defined for $x\geqq 0$ and belonging to $L(0, R)$ for every large R. Consider only those nontrivial solutions. Let I be an interval which need not be closed or bounded. (A) is said to be disconjugate on I if no solution of (A) has more than one zero on I. This criterion is proved: If there exists a real-valued function $m(x)$ AC on every closed interval contained in I and if $m'+m^2+q_1+kq_2\leqq 0$ AE on I for some k, where q_1 and q_2 are the real and imaginary parts of Q, then (A) is disconjugate on I. From this result other explicit criteria are obtained. Among others, the following comparison theorem is given: Let $f(x)$ be real and belonging to $L(0, R)$ for every large R, and let I_0 be an interval which is either closed or open. If $y''+f(x)y=0$ is disconjugate on I_0 and if $f\geqq q_1+kq_2$ AE on I_0 for some k, then (A) is disconjugate on I_0. Application to the complex domain is discussed. (Received November 13, 1952.)

211*t*. C. T. Taam: *On the complex zeros of functions of Sturm-Liouville type*.

Let $Q(z)$ be an analytic function of the complex variable z in a region D. Consider only those solutions of (A): $W''+Q(z)W=0$ which are distinct from the trivial solution ($\equiv 0$). The following theorem is proved: Let S be a closed region contained in D, the boundary C of S being a closed contour. Let M be the maximum value of $|Q(z)|$ on C. If S can be divided into n subregions such that each subregion has a diameter not greater than $\pi M^{-1/2}$ and that for any two points z_1 and z_2 of a subregion the linear segment z_1z_2 lies in S, then, when $Q(z)$ is not a constant, each solution of (A) has at most one zero in each subregion and at most n zeros in S. The proof of this theorem depends on a previous result of the author (Theorems 5.1, 5.2, Amer. J. Math. vol. 74 (1952) pp. 317–324). Another estimate of the number of zeros in a circle is also given. (Received November 13, 1952.)

212. J. L. Ullman: *Eigenvalues of a Toeplitz form*. II.

Let $f(s)$ be a continuous, positive function on $0 \leq s \leq 2\pi$, with Fourier series $\sum_{-\infty}^{\infty} c_n z^n$, $z = \exp(is)$. The form $T_n(f, x) = \sum_0^n \sum_0^n c_{j-i} x_i \bar{x}_j$ is the Toeplitz form associated with $f(s)$. It has determinant $T_n(f)$, and its eigenvalues, which are real, are $t_1^n, \cdots, t_{n+1}^n$, arranged in ascending order. Let $g(u)$ be a continuous, non-negative function on $0 \leq u \leq 1$. The form $T_n(f, g, x) = \sum_0^n \sum_0^n (c_{j-i} + g(i/n)\delta_{i,j}) x_i \bar{x}_j$, with determinant $T_n(f, g)$ and eigenvalues $v_1^n, \cdots, v_{n+1}^n$, clearly generalizes $T_n(f, x)$. In Bull. Amer. Math. Soc. Abstract 58-6-667, a new proof of the distribution of the t_i^n was presented. In this paper, that method is extended to find the distribution of the v_i^n. This result yields a proof of a conjecture communicated by Professor Kac that $\lim_{n\to\infty} (T_n(f, g))^{1/n} = \exp\ ((1/2\pi)\int_0^1 \int_1^{2\pi} \log f(s) + g(u))ds du)$, a generalization of the formula $\lim_{n\to\infty} (T_n(f))^{1/n} = \exp\ ((1/2\pi)\int_0^{2\pi} \log f(t) dt)$, due to Szegö. (Received November 13, 1952.)

213. W. R. Utz: *The connectedness of limit sets.*

The principal theorems of the paper are that in a regular continuous flow on a metric space each orbit-closure is a minimal set and that each non-null α or ω-limit set is a minimal set. As a corollary one has that each α and ω-limit set is connected. It is generally known, and easily shown, that in any continuous flow on a compact metric space the α and ω-limit sets are connected. An example is given to show that "compact" cannot be replaced by "complete." Other corollaries to the principal theorems are given and it is pointed out that the theorems hold for a regular discrete flow on a metric space. (Received November 12, 1952.)

214. H. S. Wall: *Hausdorff means with convex mass functions.*

In order for the regular Hausdorff mean $[H, \phi]$ to include the Cesaro mean $(C, 1)$ it is necessary and sufficient that $\phi = \phi_1 - \phi_2 + i\phi_3 - i\phi_4$, where ϕ_j is non-negative, nondecreasing and convex in $[0, 1]$. If $L(f)$ denotes $f(x) + x\int_x^1 (1/t) df(t)$, then ϕ is non-negative, nondecreasing and convex in $[0, 1]$ if and only if $\phi = L(f)$, where f is non-negative and nondecreasing in $[0, 1]$. If n is a positive integer, then $[H, \phi] \supset (C, n)$ if and only if $\phi = \phi_1 - \phi_2 + i\phi_3 - i\phi_4$, where $\phi_j = L^n(f_j)$ and f_j is non-negative and nondecreasing in $[0, 1]$. (Received November 12, 1952.)

APPLIED MATHEMATICS

215. Nachman Aronszajn: *Equilibrium deflection of a clamped rectangular plate.*

The deflection for any given load is calculated by using a formula for the projection on $\mathfrak{H}_1 \oplus \mathfrak{H}_2$ in terms of projections on $\mathfrak{H}_1$ and $\mathfrak{H}_2$ (see N. Aronszajn, Trans. Amer. Math. Soc. vol. 68 (1950) pp. 375–380). This method allows convenient a priori and a posteriori evaluations of error. (Received December 16, 1952.)

216*t*. H. D. Block: *A note on contact transformations.*

Many contact transformations: $Q_i = f_i(q_1, \cdots, q_n, p_1, \cdots, p_n)$, $P_i = g_i(q_1, \cdots, q_n, p_1, \cdots, p_n)$ can be generated by a suitable function $S(q_1, \cdots, q_n, P_1, \cdots, P_n)$ and the equations $Q_i = \partial S/\partial P_i$, $p_i = \partial S/\partial q_i$. It is well known that a given contact transformation can be so generated if and only if the variables $(q_1, \cdots, q_n, P_1, \cdots, P_n)$ form an independent set. Similar results are true for the generating functions $S(q_1, \cdots, q_n, Q_1, \cdots, Q_n)$, $S(p_1, \cdots, p_n, Q_1, \cdots; Q_n)$, and $S(p_1, \cdots, p_n, P_1, \cdots, P_n)$. It is a common misconception that every contact transformation can be generated by at least one of these four methods (cf. MacMillan, *Theoretical mechan-*

ics: Dynamics of rigid bodies, p. 364 or Goldstein, *Classical mechanics*, p. 240 and the footnote on p. 248). The purpose of this note is to present examples of contact transformations which can not be generated by any of these four methods. The details will appear in the Journal of Mathematics and Physics. (Received November 13, 1952.)

217. Y. W. Chen: *A mathematical analysis of linearizations of supersonic flow equations.*

This paper is concerned with supersonic flows around the axis of rotational symmetry, viscosity and heat conduction are neglected. Relations between the exact and linearized solutions are investigated for the following initial value problem. Let θ be the flow angle, $\tan^2\beta = M^2-1$, M being the Mach number. In the neighborhood of a point $P(x=x_0,\ y=0)$ on the axis of rotational symmetry, the velocity distribution is assigned by $\theta=0$, $\beta=\beta_0+O(x_0-x)^\gamma$ for $x_0>x$, and $\theta=0$, $\beta=\beta_0+O(x-x_0)^\delta$ for $x>x_0$. Two types of distributions are studied: type (a) with $\gamma>1/2$, $\delta>1/2$, and type (b) with $\gamma>1/2$, $\delta=1/2$. For type (a) there is a continuous flow in the neighborhood of P and the customary linearization gives the correct first approximation. For type (b), however, the customary linearization is completely wrong after the leading Mach line C_- issued from P. The streamlines undergo a jump of curvatures in crossing C_-. In the case of a compression, i.e. $\beta=\beta_0+B(x-x_0)^{1/2}+O(x-x_0)^\delta$ with $\delta>1/2$, $B<0$, the continuous flow breaks down in the neighborhood of the reflected Mach line C_+ issued from P, and continuous solution exists only in terms of the characteristic variables. In the case of an expansion, i e. $B>0$, the flow remains continuous in crossing C_+, and the space derivatives of θ and β satisfy two Riccati differential equations along the same Mach line C_+. A correct linearization of type (b) is obtained by making use of the differential equations set up by the author in an early paper published in Communications on Pure, Appl. Math. vol. 5, pp. 52–86. (Received November 12, 1952.)

218. R. V. Churchill: *The operational calculus of Legendre transforms.*

The finite linear integral transformation $T\{F(x)\}=\int_{-1}^{1}F(x)P_n(x)dx$, where $P_n(x)$ are the Legendre polynomials, is the basis of an operational calculus that transforms the differential operator $D(1-x^2)D$ into an algebraic operator. New operational properties of that transformation are presented and illustrated in the solution of boundary value problems. The transforms of several functions are also noted. (Received November 14, 1952.)

219*t*. Nathaniel Coburn: *Some intrinsic relations involving the velocity and vorticity vectors in fluid dynamics.*

Three results involving intrinsic properties of the velocity and vorticity vectors are derived. First, in the case of plane flows, it is well known that the magnitude of the vorticity is equal to the negative of the rate of change of the magnitude of the velocity with respect to displacement in a direction normal to the stream lines plus the magnitude of the velocity times the curvature of the stream lines. It is shown that, for three-dimensional flows, this last term is the component of the vorticity vector along the binormal direction of the stream lines. The components of the vorticity vector along the stream lines and in the direction of the principal normal to these lines are determined. Secondly, it is shown that by use of this result, an intrinsic necessary and sufficient condition for parallel Bernoulli surfaces can be given. Finally, it is shown that if the stream lines are orthogonal to ∞^1 surfaces, then the rate of change of the magnitude of the velocity with respect to a displacement along the stream lines depends upon the mean curvature of these surfaces. (Received October 8, 1952.)

220. Frank Faulkner: *The degenerate problem of Bolza with separated end conditions.*

The problem treated is that of finding a curve with the following properties: a differential equation $Pdx+Qdy+Rdz=0$ is to be satisfied along the curve; certain relations of the form $G_i(x_0, y_0, z_0)=0$ and $H_i(x_1, y_1, z_1)=0$ are to be satisfied at the end points of the curve; an expression $F(x_1, y_1, z_1)-F(x_1, y_1, z_1)$ is to be maximized; the differential dx is required to be non-negative along the curve. It is shown that there is a unique solution with the desired properties, under certain conditions. In general, there are no solutions if the last restriction is removed. (Received November 13, 1952.)

221. H. E. Fettis: *On an integral involved in the theory of high speed scanning antenna.*

In a recent paper by Harry Gruenberg (Journal of Applied Physics, July 1952) a formula is derived for the power radiated by a narrow slot of approximately half a wave length, radiating into a parallel plate region. It is the purpose of this paper to indicate how the function $Q(\xi)$ which is involved in the above formula can be expressed in terms of the functions J_e and J_s defined and tabulated by L. Schwartz (*Investigation of some functions related to the Bessel function of zero order*, Luftfahrtforschung, February 8, 1944). (Received October 20, 1952.)

222. G. E. Forsythe: *Convergence of the conjugate gradient method in Hilbert space.* Preliminary report.

Let A be a self-adjoint, bounded, positive-definite operator in a real Hilbert space $\mathfrak{H}$; let the lower, upper bounds of A be m, M $(0<m<M<\infty)$. The conjugate gradient method [M. Hestenes and E. Stiefel, J. Research Nat. Bur. Standards vol. 49 (1952) to appear] is an algorithm for solving an equation $Ax=b$, where $b\in\mathfrak{H}$. Let $f(x)=(x, Ax)-2(x, b)+(b, A^{-1}b)$. Starting from an arbitrary $x_0\in\mathfrak{H}$, the points $\{x_n\}$ $(n=1, 2, \cdots)$ are chosen so that $x_n=x_0-\alpha_0 r_0-\cdots-\alpha_{n-1}A^{n-1}r_0$, where $r_0=b-Ax_0$, and so that $f(x_n)$ is minimized as a function of the real parameters α_i. To study the convergence, let $\tau_n^n=\min_{P_n}\max_{\lambda\in S}|P_n(\lambda)|$ over all nth degree polyhomials with $P_n(0)=1$, where $S\subset[m, M]$ is the spectrum of A; and let $\tau=\tau(S)=\lim_n \tau_n$. Two new results are easily proved: I. $\limsup_n [f(x_n)/f(x_0)]^{1/n}\leqq\tau$. II. Let $\int_S\lambda d\sigma_\lambda$ be the spectral decomposition of A. Suppose $S=\bigcup_{i=1}^r I_i$, where each I_i is a closed interval, and for each i suppose that $\lambda, \mu\in I_i$ implies $|(\sigma_\lambda r_0, r_0)-(\sigma_\mu r_0, r_0)|\geqq\epsilon|\lambda-\mu|$, $\epsilon>0$. Then $\lim_n [f(x_n)/f(x_0)]^{1/n}=\tau$. Previously known convergence estimates of W. Karush and of R. M. Hayes follow from I and from known connections between τ and the transfinite diameter of S. (Received November 12, 1952.)

223. H. E. Goheen: *On a nonlinear boundary value problem.*

The author establishes the existence and uniqueness of the boundary value problem, $\phi(0)=a_0<1$, $\phi(\infty)=0$, for the ordinary differential equation $\phi^2\phi''+2\phi^2\phi'^2+\nu\phi'=0$. (Received November 6, 1952.)

224. Michael Goldberg: *Rotors within rotors.*

The curve of Ribaucour can rotate continuously within an equilateral triangle while remaining tangent to its three sides. A convex involute of the deltoid can rotate continuously within a square while remaining tangent to its four sides. It is shown that the curve of Ribaucour can be inscribed in the deltoid involute and turned continuously in a constrained manner through all orientations. These curves have technical application in the theory of testing for roundness. (Received November 6, 1952.)

225. L. E. Payne: *An equality involving virtual mass and polarization.*

Let the symbols W_{ij} and P_{ij} denote the components of the virtual mass and polarization tensors respectively (see Schiffer and Szegö, Trans. Amer. Math. Soc. vol. 67, pp. 130–205). In the case of an axially symmetric body let the subscript 1 refer to the direction parallel to the axis of symmetry and the subscript 2 to a direction perpendicular to the axis of symmetry. For an axially symmetric body with a simply-connected meridian section the following relationship can be shown to exist: $2W_{11} = P_{22} - V$. From this equality follow a number of important inequalities: (a) $P_{11} + 2P_{22} \geqq 6V$; (b) $W_{11} + W_{22} \geqq 3V/2$, etc. The existence of the relationship between W_{11} and P_{22} was observed in certain special cases by Schiffer and Szegö but no proof was given. (Received November 10, 1952.)

226. Charles Saltzer: *An abridged block method for the solution of the Laplace difference equation.*

A procedure for solving the boundary value problem is developed which is analogous to Schwartz's alternating procedure for the solution of the Laplace equation. The use of this method is based on the known formula for the solution of the difference equation for a rectangular net, and the convergence of the iteration is proved. (Received November 12, 1952.)

227. J. B. Serrin, Jr.: *Some comparison theorems for subsonic flow of gases.*

Let F and $\bar{F}$ be irrotational flows (both plane or both axi-symmetric) of an incompressible ideal fluid, uniform at infinity, and bounded, respectively, by streamlines S and $\bar{S}$. If S and $\bar{S}$ have a common arc MN, while to the right of MN, $\bar{S}$ is above S, and to the left of MN, $\bar{S}$ is below S, then the following inequality persists: $q(M)/q(N) \leqq \bar{q}(M)/\bar{q}(N)$, where $q(P)$, $\bar{q}(P)$ denote respective flow speeds at a point P (Journal of Rational Mechanics vol. 1 (1952) p. 563). A similar inequality is shown to hold for the following cases of subsonic flow of a compressible medium; (1) when there is a point P on MN where $q(P) = \bar{q}(P)$; (2) when either flow is uniform; (3) when both flows are given by the linearized equations. These results imply explicit inequalities for the velocity gradient along straight arcs and at inflection points of a streamline. Similar conclusions are obtained for flows taking place in channels. (Received November 10, 1952.)

228t. D. V. Widder: *Positive temperatures on a semi-infinite rod.*

In 1944 the author proved that the only non-negative solution $u(x, t)$ of the heat equation $u_{xx} = u_t$ which vanishes as t approaches $0+$, $u(x, 0+) = 0$, $-\infty < x < \infty$, is the identically vanishing one. An analogous result is proved here for the semi-infinite rod. That is, if a solution $u(x, t)$ is non-negative for $0 < x < \infty$, $0 < t < \infty$, and if $\lim u(x, t)$ is 0 as (x, t) approaches (two-dimensionally) each boundary point of that quadrant, then $u(x, t)$ is identically zero in the quadrant. Also an integral representation is obtained for every solution non-negative in the quadrant. (Received October 10, 1952.)

GEOMETRY

229. Iacopo Barsotti: *Equivalence theories on algebraic varieties and the definition of the Picard variety*. Preliminary report.

Let V be an n-dimensional algebraic variety over a field k (usually algebraically closed). The customary definitions and properties of *continuous* (or algebraic) *equivalence* between the integral cycles of V are established by algebraic means. It is proved that, under slightly restrictive conditions, the elements of an irreducible m-dimensional algebraic system $\mathfrak{C}$ of $n-1$-dimensional cycles cut on a generic irreducible cycle of $\mathfrak{C}$ a linear system of dimension $m-1$ (the *characteristic system*). It is also proved that the cycles of $\mathfrak{C}$ are all linearly equivalent to each other if either (1) $\mathfrak{C}$ is unirational, or (2) $\mathfrak{C}$ has index 1 and dimension >1. It is proved that if $\mathfrak{C}$ is maximal, and G is the representative variety of $\mathfrak{C}$, then G is birationally equivalent to the product of a variety F and a projective space of dimension equal to the dimension of the largest linear system containing the general element of $\mathfrak{C}$. By induction on the dimension of V it is proved that, for a given V, the dimension of F has a maximum q (the *irregularity* of V), and that all the F's having dimension q are birationally equivalent to each other. It follows that if F is any of these, it is an abelian variety (the *Picard variety* of V), and that F is, as a group, isomorphic to the factor group of the group of $n-1$-dimensional cycles of V linearly equivalent to 0, in the group of those continuously equivalent to 0. The proofs are of an algebraic nature, and rest heavily on the recent work by the author on algebraic correspondences. (Received November 3, 1952.)

230. Eugenio Calabi: *On tangent bundles, holonomy groups, and torsion*. Preliminary report.

Let M^n be a compact, differentiable, n-dimensional, affinely connected manifold. The associated holonomy group with respect to a point $p\in M^n$, i.e. the group of linear transformations of the tangent space at p obtained from parallel displacements along rectifiable loops originating at p, is a subgroup of $GL(n, R)$, whose closure we call H; then the tangent bundle of M^n admits H as structural group. Conversely, if the tangent bundle admits a subgroup H of $GL(n, R)$ as structural group, then there exists an affine connection, but not generally a symmetric one, in M^n, whose holonomy group is contained in H. The existence of a symmetric affine connection in M^n with holonomy group contained in H is considered as a general problem, whose applications include the following important cases. Let $n=2k$; M^n is an almost-complex manifold if the structural group of the tangent bundle is isomorphic to a subgroup of any one of the following: (a) $GL(k, C)$, (b) $Sp(2k, R)$, or (c) $U(k)$. *Theorem*. If M^{2k} is an almost-complex manifold, then there exists a symmetric affine connection with holonomy group contained in the above named groups respectively, if and only if M^{2k} is: (a) a complex manifold, (b) a symplectic manifold, or (c) a Kaehler manifold. (Received November 14, 1952.)

231. N. A. Court: *Pascal's theorem in space*.

The following two propositions may be considered as extensions of Pascal's theorem to three-dimensional Euclidean space. I. The four triads of coterminous edges of a tetrahedron (T) are cut, respectively, by four planes in four triads of points lying on the same quadric surface if, and only if, the four secant planes meet the faces of (T)

opposite the respective vertices in four lines which are (i) coplanar or (ii) hyperbolic. II. If the three pairs of opposite edges of a tetrahedron (T) taken two-by-two are cut, respectively, by three planes, the six pairs of points of intersection lie on a quadric surface if, and only if, the three lines determined in the secant planes by the traces of the respective third pairs of opposite edges of (T) are coplanar. (Received November 4, 1952.)

232*t*. John DeCicco: *Extension of Kasner's inverse of Meusnier's theorem to Riemannian space.*

In a certain region of a Riemannian space V_n, consider a unit vector $\mu_i(x;\dot{x})$ whose components possess suitable differentiability conditions relative to the lineal elements $(x;\dot{x})$ in the region, and which are homogeneous of degree zero in the $\dot{x}$. A system S of curves in this region of V_n is said to have the Meusnier property relative to this vector $\mu_i(x;\dot{x})$ if every curve C of S through the point x in the direction $\dot{x}$ is orthogonal to the vector $\mu_i(x;\dot{x})$, and if its radius of geodesic curvature ρ at this lineal element $(x;\dot{x})$ varies directly as the cosine of the angle θ between its principal normal and the fixed direction $\mu_i(x:x)$. Thus $\rho=R\cos\theta$, where $R=R(x;\dot{x})$, is homogeneous of degree zero in the $\dot{x}$. Such a system S is defined by the differential equation $\mu_i\ddot{x}^i+\nu=2$, where $\nu=\nu(x;\dot{x})$ is homogeneous of degree two in the $\dot{x}$. The auxiliary condition $\mu_i\dot{x}^i=0$ must be satisfied. Systems S of this type are invariant under the group of arbitrary point transformations. For the classical three-dimensional case, see Kasner, *The inverse of Meusnier's theorem*, Bull. Amer. Math. Soc. vol. 14 (1908) pp. 461–465. (Received November 12, 1952.)

233. Harley Flanders: *Development of an extended exterior differential calculus.*

The purpose of this paper is to set up an algebraic machinery for the theory of manifolds with affine connections. Tensor spaces of p-vectors with q-form coefficients are introduced and their algebraic relations studied. An affine connection on a manifold M is defined as a certain type of linear operator on the space of vector fields to the space of vectors with one form coefficients. It is proved that such an operator induces further operators on each space of p vector fields with q form coefficients to the space of p vectors with $(q+1)$ form coefficients. Application is made to the systematic derivation of identities between the displacement vector, connection forms, torsion tensor, etc. Also further application is made to the Chern curvature integra in the case of Riemannian manifolds. (Received November 5, 1952.)

234*t*. P. C. Hammer: *Class-equivalence of functions and curves.* I. Preliminary report.

Let $f(x)$ and $g(x)$ be two real-valued functions defined on an open interval (a, b). If for every null sequence of nonvanishing terms, h_n, such that $[f(x+h_n)-f(x)]/h_n$ has a unique limit in the compactified real number system, the sequence $[f(x+h_n)-f(x)-g(x+h_n)+g(x)]/h_n$ has the limit zero, then f and g are said to be *class-equivalent at* x. A necessary and sufficient condition that f and g be class-equivalent for all x in (a, b) is that they differ by a constant. Extensions of this result are made to vector-valued functions defined on a linear space and to curves and surfaces. The theory provides a new tool for the establishment of uniqueness of solution curves and surfaces when Lipschitz type conditions do not prevail. (Received December 4, 1952.)

235. P. C. Hammer: *Inversion of a planar symmetrization process.*

Let C be a planar convex body (a closed bounded convex set with positive area). The convex body B, obtained by translating diameters of C so that their midpoints fall on a fixed point, is called the *symmetroid* of C and the process is called *point symmetrization* or *central symmetrization*. The symmetroid of C has the same breadth as C in each direction. Given a point symmetric planar convex body B, a construction of all convex bodies which have B as a symmetroid is given, thus completely inverting point symmetrization process. For a given point symmetric convex body B and an outwardly simple line family F there exists a one-parameter family of convex bodies with a common subfamily of diameters which extend to give F and such that the symmetroid of each such convex body is similar to B. (Received September 29, 1952.)

236*t*. Edward Kasner and John DeCicco: *The special physical systems in Riemannian space of dynamical trajectories, catenaries, brachistochrones, and velocity curves.*

A physical system S_k of ∞^{2n-1} curves in a Riemannian space V_n is defined as follows. For an arbitrary field of force in V_n, a curve C is in S_k if a constrained motion along C is possible such that the osculating geodesic surface at each point x of C possesses the force vector at x as a tangent, and the pressure P along the principal normal to C at x is proportional to the principal normal component N of the force vector. Thus $P=kN$ where k is the constant factor of proportionality. Starting from fundamental physical principles, it is proved that dynamical trajectories, general catenaries, brachistochrones, velocity systems are all special systems S_k for $k=0, 1, -2, \infty$, respectively. Also every natural family is a special velocity system. In a conservative field of force, any system S_k is a natural family and the associated calculus of variations problem is discussed. (Received November 13, 1952.)

237. Paolo Lanzano: *Tensorial connections.*

Tensorial connections were introduced by Bompiani (Rendiconti Lincei (VIII) vol. I): the purpose of this paper is to generalize them for any tensor field. Given a tensor field $A(x)$ $(p; q)$, (p indices of contravariance, q of covariance) defined on V_n, it is asked that the combinations $\nabla_i A = \partial_i A + \sum LA$ be tensors $(p; q+1)$ (tensor derivatives); the L's, functions of V_n with $2(p+q)+1$ indices, are coefficients of a tensorial connection: its transformation law is established. Coincidence of covariant and tensor derivatives is taken as basis for equivalence of tensorial and vectorial (affine) connections: it is proved that the vanishing of a tensor $(p+q; p+q+1)$ is the necessary and sufficient condition for the equivalence of a given tensorial and vectorial connection. The integrability conditions of the transformation law of the L's give rise to a curvature tensor $(p+q; p+q+2)$. Second order tensor derivatives can be obtained by use of the L and another connection M with $2(p+q+1)+1$ indices: the necessary and sufficient condition for them to be independent of the order of derivation is the vanishing of the curvature tensor of L and of a new tensor, the torsion of L and M. (Received November 5, 1952.)

238. T. Minagawa and Tibor Radó: *On the rigidity of the torus.*

Let B_e, B_i denote the outer belt (of positive curvature) and the inner belt (of negative curvature) of the torus T, and let s be a vector function on T, of class C', which induces an infinitesimal deformation of T. By a theorem of Rembs (extended to the class C' case by the authors), the infinitesimal deformation is trivial on B_e. Thus on the boundary of B_i, which is of negative curvature in its interior, the infinitesimal deformation coincides with a trivial one. By an extension of one of the authors' previous results concerning the rigidity of surfaces of negative curvature, it is inferred that z is trivial on B_t also. It follows that the torus is infinitesimally rigid. (Received October 8, 1952.)

239*t*. J. T. Schwartz: *DeRham's theorem for arbitrary spaces.*

By considering pairs (S, T) of compact subspaces of a Euclidean space E^N and appropriate local spaces of forms near S and T, a theory of differential forms on arbitrary finite-dimensional compact spaces can be constructed. The construction leads to groups $H_*(S, T)$ which are shown to satisfy all the Eilenberg-Steenrod cohomology axioms for cohomology with real coefficients. In the case where S is a C^∞ manifold regularly imbedded in E^N, the groups $H_*(S, T)$ can easily be seen to depend only on the space of forms on S. In this way, the abstract method of Eilenberg-Steenrod yields deRham's theorem for an arbitrary compact manifold. Since dualization for groups derived from forms is especially simple, this method also gives elementary proofs of the Jordan curve theorem and other theorems of the Alexander type. (Received December 8, 1952.)

240. Oswald Wyler: *Perpendicularity in absolute geometry.*

Perpendicularity of lines and planes in a space satisfying Hilbert's axioms of incidence may be characterized by three axioms which, like the axioms of incidence, are satisfied in any open set of a Euclidean, hyperbolic, or elliptic geometry. If a space satisfying these axioms is embedded into a projective geometry by the construction of bundles, perpendicularity becomes part of a well-determined "absolute" polarity of the projective geometry. This polarity is degenerate if, and only if, two distinct proper planes have the same bundle of perpendicular lines. The theory is easily extended to spaces of any finite dimension ≥ 3. No notions of order are needed. (Received November 12, 1952.)

LOGIC AND FOUNDATIONS

241. Hermann Rubin and P. C. Suppes: *Transformations of systems of relativistic particle mechanics.*

A set of axioms is given for systems of relativistic particle mechanics (in the sense of the special theory of relativity). Transformation theorems are then established analogous to the theorems established by McKinsey and Suppes for classical particle mechanics (*Transformations of systems of classical particle mechanics*, forthcoming in Journal of Rational Mechanics and Analysis). (Received November 14, 1952.)

242*t*. A. R. Schweitzer: *On the foundations of geometry.*

The foundations of geometry are classified into four types: descriptive, metrical, projective and group-theoretic. With each type topological concepts are associated. Common to the first three types are reflexiveness, symmetry, and transitiveness of the corresponding (undefined) generating relations. The latter are symbolized by K, E, K^* expressing respectively sameness of sense, equality (leading to congruence), and cospatiality of $(n+1)$-ads of points $(n=1, 2, 3, \cdots)$. The relations E and K^* are respectively termed metrical and projective analogues of the relation K. Correspondingly, axiomatic systems generated by the relations E and K^* are metrical and projective analogues of the system generated by the relation K. Reference is made to the author's theory of geometrical relations, Amer. J. Math. vol. 31 (1909) p. 366. (Received November 12, 1952.)

243t. A. R. Schweitzer: *On the transition from Grassmann's space analysis to combinatorial topology.*

It is assumed that the simplex is a fundamental concept of the foundations of geometry, Grassmann's space analysis, and combinatorial topology. Genetically, topology involves both generalization and specialization. Subjects discussed include: 1. The simplex. 2. Sets of simplexes. 3. Homology group. 4. Correspondence. 5. Invariance. 6. Fundamental group. (Received November 12, 1952.)

244t. A. R. Schweitzer: *Topological problems in the foundations of geometry.*

In this paper topological concepts are associated with descriptive geometry. The author discusses assumptions under which the following statements are valid. I. Given a finite set of points S in a descriptive plane P, there exists in P a triangle T such that S lies "entirely within" T. In particular, S may consist of the vertices of two triangles T_1 and T_2. If the triangle T_1 lies entirely within the triangle T, then T can be subdivided into a set of triangles, including T_1, such that T_1 and T have the same or opposite senses. II. Given the triangles T_1 and T_2 in P, there exists in P a triangle T such that T lies "entirely without" T_1 and T_2. Two triangles in P of which one lies entirely without the other can be joined by a sequence of "consecutive" triangles in P and therefore have the same or opposite senses. (Two triangles in P are consecutive if they have two vertices and corresponding sides in common and the remaining vertices are opposite relatively to the common side.) The preceding is generalized to n dimensions ($n=2, 3, \cdots$). (Received November 12, 1952.)

245t. D. E. van Tijn: *An interpretation of free variables.*

Given a formal system L it is possible to define another system L' with the following properties: (1) If x is a variable in L, x^* will be a constant in L'. (2) There are variables t in L', and $x^*(t)$ is an individual term. One can then map formulas of L containing free variables into closed formulas in L' and variables of L into constants of L', and this mapping will have all the properties which are accepted as giving "meaning" to closed formulas. The resultant system L' has application in describing restricted quantification and such concepts as "a variable depends on another." (Received November 21, 1952.)

STATISTICS AND PROBABILITY

246. P. R. Rider: *The distribution of the product of ranges in samples from a rectangular population.*

The distribution of the product of ranges of independent, random samples from a continuous rectangular population is considered. For the distribution of the product of two samples a general formula is given. This formula fails to hold if the sample sizes are the same or if they differ by unity. Special consideration is given to these cases. The distribution of the product of k samples of equal size is derived. Simple formulas for the moments about the origin are given for all of these distributions. (Received October 14, 1952.)

247*t*. F. L. Spitzer: *A limit theorem for two-dimensional brownian motion.*

Let $X_i(t)$ be independent Wiener processes with $X_i(0)=0$, $i=1, 2, \cdots, n$. Then $R_n(t)=(X_1^2(t)+\cdots+X_n^2(t))^{1/2}$ denotes the distance from the origin at time t of a particle in brownian motion in an n-dimensional euclidean space. For $n\geqq 3$, it was shown by Dvoretzky and Erdös (Proceedings of the 2nd Berkeley Symposium, University of California Press, 1951) that for monotone decreasing functions $\phi(t)$ the probability $P_n=\Pr\{R_n(t)<t^{1/2}\phi(t)$ for arbitrarily large $t\}$ equals zero or one according as the infinite series $\sum_{k=1}^{\infty}\phi^{n-2}(2^k)$ converges or diverges. The result of the present paper is the analogous theorem for the two-dimensional case ($n=2$): With the above notation, P_2 equals zero or one according as the infinite series $\sum_{k=1}^{\infty}[\log\phi(2^k)]^{-1}$ converges or diverges. The method which is used to obtain this result also yields that of Dvoretzky and Erdös. (Received October 17, 1952.)

248*t*. F. L. Spitzer: *The angular component of two-dimensional brownian motion.*

Let $X(t)$ and $Y(t)$ be independent Wiener processes defining a stochastic vector process $Z(t)=X(t)+iY(t)$ of two-dimensional brownian motion. Let $R(t)=|Z(t)|$ and $\theta(t)=\operatorname{amp} Z(t)$. $\theta(t)$ ranges over the real numbers, so that it may be interperted as the angular component of brownian motion on the Riemann surface of log z. $\theta(t)$ becomes a well-defined stochastic process if the initial conditions $\theta(0)=0$ and $R(0)=R\neq 0$ are imposed. Then $\theta(t)$ is a continuous function of t, with probability one, during any finite interval $[t_1, t_2]$. By solving the diffusion equation on the Riemann surface of log z, explicit formulae are obtained for the distribution functions of $\theta(t)$. The distributions of transition and of one-sided absorption satisfy the relation $G(\alpha, t; R)=\Pr\{\theta(t)<\alpha \mid \theta(0)=0, R(0)=R\}=1-2^{-1}\Pr\{\max_{0\leqq\tau\leqq t}\theta(\tau)>\alpha \mid \theta(0)=0, R(0)=R\}$, and $\partial G(\alpha, t; R)/\partial\alpha$ is a symmetric density in α with infinite absolute mean. With proper normalization, simple asymptotic distributions are shown to exist for $\theta(t)$, viz. $\lim_{t\to 0} G(\alpha t^{1/2}/R, t; R)=(1/(2\pi)^{1/2})\int_{-\infty}^{\alpha}e^{-s^2/2}ds$ and $\lim_{t\to\infty} G(\alpha\log t^{1/2}, t; R)=(1/\pi)\int_{-\infty}^{\alpha}ds/(1+s^2)$. (Received October 17, 1952.)

TOPOLOGY

249. R. D. Anderson: *A homogeneous one-dimensional compact continuous curve.*

Let M denote the Sierpinski universal curve (a one-dimensional analogue of the Cantor middle-third set in Euclidean three space), let $A_1, \cdots, A_k$ denote k points of M, and let $B_1, \cdots, B_k$ denote k points of M; then there exists a homeomorphism f of M onto itself with for each i, $1\leqq i\leqq k$, $f(A_i)=B_i$. (Received November 14, 1952.)

250. B. J. Ball: *Some theorems concerning spirals in the plane.*

In September, 1948, R. L. Moore delivered an address on the subject *Spirals* before a meeting of Section A of the American Association for the Advancement of Science held in conjunction with the summer meetings of the American Mathematical Society and the Mathematical Association of America. Using definitions introduced by Professor Moore, it is shown that there exists a closed, bounded, totally disconnected point set M in the plane such that every arc containing it spirals down on some point. In order that M should be a subset of a nonspiraling arc, it is necessary and sufficient that it be a subset of the sum of the elements of a continuous and equicontinuous collection G of mutually exclusive nonspiraling arcs such that G with respect to its elements as points is closed, bounded, and totally disconnected. Certain related theorems are also established. (Received November 13, 1952.)

251. R. H. Bing: *Locally tame surfaces are tame.*

A set M is tame in a geometric complex K if there is a homeomorphism of K onto itself that carries M onto a subcomplex of some simplicial subdivision of K. M is locally tame in K if for each point p of M there is a neighborhood N of p in K and a homeomorphism f of $\bar{N}$ onto a geometric complex that carries $M\cdot\bar{N}$ onto a subcomplex of $f(\bar{N})$. If M is a locally tame closed subset of a triangulated 3-manifold K such that no arc contains an open subset of M, then M is tame in K. Consequently any 2-manifold with or without boundary which is locally tame in E^3 is tame there. Also, any 3-manifold with boundary can be triangulated. (Received November 13, 1952.)

252. Raoul Bott and Hans Samelson: *On the Pontryagin product.*

The space E of all paths in a space X, starting at a fixed point x_0, and its subspace Ω of closed paths permit a composition $\Omega\times E\to E$ (a loop followed by a path is a path). From this one can define an operation of the homology group $H(\Omega)$ on the groups $E_r (r\geqq 2)$ of the spectral sequence of E, which commutes with the differentials ϕ_r. Using this one shows that the Pontryagin ring of Ω, when X is union of k spheres, of dimensions n_i, with a point in common, is the free associative algebra generated by k elements of dimensions n_i-1; actually a more general theorem is proved. (Received October 31, 1952.)

253*t*. R. E. Chamberlin and J. H. Wolfe: *Homeomorphisms of differentiable manifolds.* Preliminary report.

Let M^n be a differentiable n-manifold and let f be a homeomorphism of M^n onto itself. Given $p\in M^n$, let $g_1, g_2 : E^n\to M^n$ be differentiable homeomorphisms of E^n into M^n with $p\in g_1(E^n)$ and $f(p)\in g_2(E^n)$. (E^n is euclidean n-space with a fixed coordinate system.) Put $h=\overset{-1}{g_2}fg_1$ and let $h_\epsilon(q)=\lambda(\epsilon)\int_{S(\epsilon)}hdv$. Here $\lambda(\epsilon)$ is the reciprocal of the volume of the solid n-sphere of radius ϵ and the integral is taken over the solid n-sphere of radius ϵ with center at q. A point $p\in M^n$ is called a regular point for (f, g_1, g_2) if there exist $\delta_1>0$, $\delta_2>0$ such that $h_\epsilon(q)$ is a homeomorphism in the δ_1-nbd. of $\overset{-1}{g_1}(p)$ for every $\epsilon<\delta_2$. It is shown that if p is regular for (f, g_1, g_2), then it is regular for (f, g_1', g_2') where g_1' and g_2' are differentiable homeomorphisms of E^n into M^n such that $\overset{-1}{g_1}g_1'$ and $\overset{-1}{g_2}g_2'$ are regular homeomorphisms of open subsets of E containing $(g_1')^{-1}(p)$ and $(g_2')^{-1}(f(p))$ respectively. (Reg. hom. means differentiable with nonvanishing jacobian.) Let A_f be the subset of M consisting of those points which are not regular. If f is lipschitz, it is shown that $\dim A_f\leqq n-2$ (lipschitz relative to a given differentiable metric). Other properties of A_f are also determined. (Received November 13, 1952.)

254. Eldon Dyer: *Concerning irreducibility of the sum of the elements of a continuous collection of continua.*

In 1935, B. Knaster showed (Fund. Math. vol. 25 (1935) pp. 568–577) that there exists a *continuous* collection G of mutually exclusive nondegenerate continua such that G is an arc with respect to its elements and G^* (the sum of all of the elements of G) is a compact *irreducible* metric continuum. He raised the question whether there exists a collection fulfilling these conditions together with the additional condition that all of its elements are topologically equivalent to each other. It is shown in the present paper that there does not exist a continuous collection G of mutually exclusive, nondegenerate, *decomposable*, continua such that G^* is a compact irreducible metric continuum. This is an extension of results of E. E. Moise (Bull. Amer. Math. Soc. vol. 55 (1949) pp. 810–811) and M. Hamstrom (Bull. Amer. Math. Soc. vol. 58 (1952) p. 204). (Received December 5, 1952.)

255. D. O. Ellis and H. D. Sprinkle: *Topology in B-metrized spaces.*

In a preliminary announcement (David Ellis and H. D. Sprinkle, Bull. Amer. Math. Soc. vol. 58 (1952) pp. 573–574) the Boolean metrization of zero-dimensional spaces and some other results were announced. In addition, it may be shown that: (1) *The tier function of a B-metrized space is normal* (for terminology here and below see Vaidyanathaswami, *Treatise on set topology*, Indian Math. Soc., 1947); (2) *Every B-metrized space is completely regular*; (3) *The distance function of a B-metrized space is continuous simultaneously*; (4) *Every B-metrized space has a completion of the Cantor-Hausdorff-Meray type*; (5) *There exist Boolean metrizable weak product topologies on cartesian products of Boolean metrized spaces.* (Received November 13, 1952.)

256. J. S. Griffin, Jr.: *A condition that a fibre space be an E-F bundle.*

It is proved that, if $\{X, B, \pi, \Omega, \phi_U\}$ is a fibre space in the sense of S. T. Hu (Proc. Amer. Math. Soc. vol. 1 (1950) pp. 756–762) satisfying the conditions (1) for any points x and y, $\pi^{-1}(x)$ and $\pi^{-1}(y)$ are homeomorphic, (2) for any member U of Ω and any a, b, and x such that $a \in U$ and $b \in U$ and $x \in \pi^{-1}(a)$, $\phi_U(a, \phi_U(b, x)) = x$, then there is an E-F bundle (Steenrod, *The topology of fibre bundles*, pp. 18–20) with bundle space X, base space B, and projection π; conversely, if X, B, and π are respectively the bundle space, base space, and projection of an E-F bundle, then there are Ω and ϕ_U such that $\{X, B, \pi, \phi_U\}$ is a fibre space satisfying these conditions. There are also slightly weaker, though considerably less attractive, conditions than those given. This result may be used to show, for example, that every fibre space with totally disconnected fibre and connected locally arcwise connected base space is an E-F bundle. (Received October 31, 1952.)

257t. Mary-Elizabeth Hamstrom: *Concerning certain types of webs.*

In the author's dissertation (Bull. Amer. Math. Soc. Abstract 58-6-715, to appear in Trans. Amer. Math. Soc.) a W_n set was defined as follows: If $n > 1$, a W_n set is a compact continuum M for which there exists a family F of n elements such that (1) each element of F is an upper semicontinuous collection of mutually exclusive continua which fills up M and is an arc with respect to its elements and (2) if G is a collection of continua each belonging to some, but no two to the same, collection of the family F, then the continua of the collection G have a point in common and their common part is totally disconnected. In that paper it was stated without proof (Theorems 13 and 14) that in the plane there exists a W_3 set M whose boundary is the point set $B(M)$ and which has a complementary domain whose boundary, J, contains six limit points of $B(M) - J$, but that no W_3 set has a complementary domain whose boundary contains seven such points. It is the purpose of this paper to prove these theorems. (Received November 13, 1952.)

258. Frank Harary and G. E. Uhlenbeck: *On the number of free Husimi trees.*

A "Husimi tree" is a connected graph in which no line lies on more than one cycle. A "pure" Husimi tree contains only one type of polygons; a "mixed" Husimi tree is not pure. We have extended Otter's procedure for trees (Ann. of Math. vol. 49 (1948) pp. 583–599) to the counting of the topologically different free Husimi trees in terms of the rooted ones (Bull. Amer. Math. Soc. Abstract 58-2-131). This extension makes use of the notion of the "dissimilarity characteristic" of a Husimi tree (Harary and Norman, Ann. of Math., to appear). Explicit formulas will be presented for pure Husimi trees consisting of triangles ("cacti"), rectangles, pentagons, and hexagons. The counting problems for mixed Husimi trees can be reduced to the case of pure ones. (Received November 13, 1952.)

259. Edwin Hewitt: *Representation of functions as Fourier-Stieltjes transforms.*

Let G be a locally compact Abelian group and let G^* be the group of continuous characters of G. Let (x, y) denote the value of the character $y \in G^*$ at the point $x \in G$, or, dually, the value of the character $x \in G$ at the point $y \in G^*$. Let ϕ be a bounded complex Radon measure on G. The Fourier-Stieltjes transform of ϕ is defined as the function on G^* equal to $\int_G (x, y) d\phi(x)$ for $y \in G^*$. It is shown that every bounded uniformly continuous complex function on G^* can be represented as a Fourier-Stieltjes transform if and only if G and G^* are finite. This is accomplished by showing that if every continuous almost periodic function on G^* is a Fourier-Stieltjes transform, then G^* is finite. For a certain class of groups, it is shown that every continuous almost periodic function which is a Fourier-Stieltjes transform has an absolutely convergent Fourier series. (Received November 10, 1952.)

260. F. B. Jones: *On certain well-ordered monotone collections of sets.*

The study of Souslin spaces or nonmetric normal Moore spaces (such spaces are not known to exist) leads to certain situations analogous to the following: Suppose that F is a well-ordered family of collections of mutually exclusive sets such that (1) if F' is a subsequence of F which runs through F, then $|F'| = |F|$, (2) if F' is a subsequence of F which does *not* run through F, then $|F'| < |F|$, and (3) if f is an element of F, each set in f is a subset of some set in each preceding element of F. Theorem. If there exists a cardinal number N such that for each element f of F, $|f| < N < |F|$, then there exists a monotone collection α of sets such that each element of F contains only one element of α. (Received November 10, 1952.)

261. W. S. Massey: *Products in exact couples.*

Let $\langle A, C; f, g, h\rangle$ be an exact couple of abelian groups (for the definitions and notation, see Ann. of Math. vol. 56 (1952) p. 367) which satisfies the following three conditions: (1) A multiplication is defined in C, making it a ring. (2) There are defined involutions $a \to \bar{a}$ and $c \to \bar{c}$ of A and C respectively such that $f(\bar{a}) = (\overline{fa})$, $g(\bar{a}) = (\overline{ga})$, and $h(\bar{c}) = -(\overline{hc})$. (3) For any integer $n \geq 0$ and elements $a_1, a_2 \in A$, $c_1, c_2 \in C$ such that $h(c_1) = f^n(a_1)$, $h(c_2) = f^n(a_2)$, there exists an element $a \in A$ such that $h(c_1 c_2) = f^n(a)$ and $g(a) = g(a_1) \cdot c_2 + \bar{c}_1 \cdot g(a_2)$ (here f^n denotes the nth iterate of f). It is then possible to define a multiplication in C' and involutions of A' and C' such that these three conditions also hold for the derived exact couple $\langle A', C'; f', g', h'\rangle$. The case is also considered where A is a ring, g is a ring homomorphism, and various conditions are imposed on f and k. It is shown that the purely algebraic aspects of the theory of filtered rings with a differential operator (cf. J. Leray, J. de Math. vol. 29) may be subsumed under that of exact couples satisfying the above three conditions. (Received November 12, 1952.)

262. E. A. Michael: *An existence theorem for continuous functions.*

Theorem: The following condition is necessary and sufficient for a T_1-space to be paracompact (resp. normal): If E is a Banach space (resp. separable Banach space), and $\mathfrak{F}$ a function from X into the space of nonempty, closed (resp. compact), convex subsets of E such that $\{x\in X \mid \mathfrak{F}(x)\cap U\neq\phi\}$ is open in X for every open $U\subset E$, then there exists a continuous $f:X\rightarrow E$ such that $f(x)\in\mathfrak{F}(x)$ for every x in X. Among the immediate consequences of this theorem, the author obtains the following known results. (1) (Katetov-Tong) For any T_1-space X, the following condition is equivalent to normality: If R is the real line, $g:X\rightarrow R$ upper semi-continuous, $k:X\rightarrow R$ lower semi-continuous, and $g\leqq k$, then there exists a continuous $f:X\rightarrow R$ such that $g\leqq f\leqq k$. (2) (Arens) If X is paracompact and E a Banach space, then every continuous function from a closed subset of X into E can be extended to a continuous function from X into E. (3) (Bartle-Graves) If E is a Banach space, and F a closed subspace of E, then there exists a continuous $f:E/F\rightarrow E$ such that $f(x)\in x$ for every x in X. (Received November 6, 1952.)

263. L. V. Toralballa: *Local topological rings.*

Some algebraic and topological properties of a system Σ with double composition satisfying the following conditions are considered: 1. Σ is a topological space. 2. Σ is a commutative local topological group with respect to addition. 3. Σ is closed and associative with respect to multiplication. 4. Multiplication is continuous with respect to both factors. 5. If $b+c$ is defined, then so are $a+b+a\cdot c$ and $b\cdot a+c\cdot a$ and both distributive laws hold. 6. There exists a neighborhood of zero which is homeomorphic to a neighborhood of the origin of a Euclidean space. (Received November 13, 1952.)

264. H. C. Wang: *Complex completely parallelisable manifolds.*

Let M be a complex manifold of $2n$ dimension. It is called complex completely parallelisable if there exist, over M, n everywhere linearly independent complex analytic tangent vector fields. Unlike the real analytic case, such manifolds, if closed, cannot be simply-connected. It is shown that each compact, complex completely parallelisable manifold is analytically homeomorphic with the coset space G/H of a complex Lie group over a discrete subgroup H of G. Most of the complex Lie groups do not have any discrete subgroup D such that G/D is compact. The condition for G to possess such a discrete subgroup is studied. (Received November 13, 1952.)

265. L. E. Ward, Jr.: *Partially ordered spaces.* I.

Let X be a partially ordered space, that is, a partially ordered set with topology in which the order relation, $\leqq$, is semi-continuous in the sense of Wallace [Bull. Amer. Math. Soc. vol. 51, pp. 413–416]. A subset C of X is a *chain* if it is linearly ordered with respect to $\leqq$. X has *closed bounded property* (=CBP) if, whenever A is a closed subset of X, $a\in X$, there are $q(a)$, $p(a)\in A$ such that $q(a)$ is minimal in A, $p(a)$ is maximal in A, and $q(a)\leqq a\leqq p(a)$. X has CBP if and only if every maximal chain of X is compact. Thus every compact partially ordered space has CBP. By means of an argument employing monotone nets and a result of Nachbin, some fixed point theorems are proved for order-preserving maps $f:X\rightarrow X$, where X has CBP. (Received November 12, 1952.)

J. W. T. YOUNGS,
Associate Secretary

APPENDIX

EXCERPTS FROM REPORT OF TREASURER

TO THE BOARD OF TRUSTEES OF THE
AMERICAN MATHEMATICAL SOCIETY:

Gentlemen:

I have the honor to submit herewith the report of the Treasurer for the fiscal year ended November 30, 1952, as is customary in two parts, (1) exhibits and schedules, and (2) a discussion and commentary.

On November 30, 1952, the market value of securities held for invested funds exceeded book value by $27,767, while the market value of securities held for current funds was less than book value by $440.00.

During the year the book value of securities was revalued incident to the addition to the investment pool of $50,000 derived from the sale of the Library. Hence the excess of market value over book value as of November 30, 1952 is not comparable with the figure given in last year's report. Our return this year on our invested funds has been 5.21 percent computed on average book value of investments.

Following is a summary of the changes in security holdings made during the year:

Acquired as result of stock splits

150 shares American Can Co.
3 shares Standard Oil Co. of New Jersey

During the year the accounts of the 1950 International Congress of Mathematicians were closed, and the sum of slightly over $20,000 added to the Society's surplus. By action of the Board of Trustees, the funds accumulated as a reserve for possible reprinting of out-of-print numbers of the Bulletin were transferred to surplus, since a committee appointed by the Council decided that it was unnecessary ever to reprint issues of the Bulletin. Similarly, the Transactions reprinting fund was reduced to $2,500, with a transfer of $5,094.43

to surplus, because the same committee felt that the Transactions need be kept in print only for ten years. In the judgment of the Trustees the sum of $2,500 is an adequate reserve against the contingency of reprinting numbers which appeared so recently.

The Trustees have also adopted the policy of using annually the income of all non-restricted funds. Hitherto only the Endowment Fund income was thus utilized for the general purposes of the Society. Accumulated balances of something over $2,000 were transferred to surplus and additional investment income of $4,000 for the year used for general purposes.

It is a pleasure to report that the Society has lived within its income this year. Receipts for general purposes amounted to $93,000; the general expenses of the Society were approximately $41,000, the Bulletin, Proceedings and Transactions deficit amounted to just under $36,000, to which should be added appropriations to other publications of some $6,500, or a total for general publications of $42,000. This left a surplus for the year of approximately $10,000.

An additional schedule has been added to the exhibits, namely Exhibit C which shows the cost of operating the headquarters building. It is believed that members of the Society will be generally interested in the cost of maintaining our own headquarters. The cost of operating the building has been allocated to the various activities housed in it, namely the several publications, the Accounting Department and the Office of the Executive Director. Thus the building account does not explicitly appear on any of the other schedules.

It should perhaps also be pointed out that $3,500 has been appropriated to repay the Library Proceeds Funds for the cost of moving to Providence and preparing the headquarters building for occupancy. The full cost of the relocation of the Society's office should have been recovered in five years.

While the financial problems of the Society continue to be a concern of the Board of Trustees, the return to a balanced budget for the year 1952 and, it is anticipated, for the year 1953, would indicate, in the absence of unforeseen contingencies, that for the next few years at least the Society can continue its activities on the present scale without further increase of dues. This statement, however, is contingent upon the continuation of Government support for Mathematical Reviews, and indeed its increase, for the Mathematical Reviews account incurred a deficit of $4,000 this year.

Respectfully submitted,
ALBERT E. MEDER, JR.
Treasurer

BALANCE SHEET

Assets

	November 30, 1952	November 30, 1951
CURRENT FUNDS:		
Cash	$ 19,430.02	$ 39,634.51
Account Receivable (United States Government)	4,667.27	3,137.82
Investments	67,329.96	67,329.31
	$ 91,427.25	$110,101.64
INVESTED FUNDS:		
Cash	$ 1,109.03	$ 1,427.00
Investments	288,090.07	244,972.71
	$289,199.10	$246,399.71
TOTAL ASSETS	$380,626.35	$356,501.35

Liabilities

	November 30, 1952	November 30, 1951
CURRENT FUNDS:		
Publications	($ 5,042.85)	$ 5,667.37
International Congress		41,992.58
Policy Committee	858.01	778.30
Office Transfer Fund	2,865.83	6,549.19
Combined List of Members	(4,685.55)	
Prize Funds and Accumulated Income	2,405.27	4,346.02
Profit/Loss on Sales of Securities	2,049.85	2,049.85
Miscellaneous (credit)	1,809.67	2,033.45
Account Payable (United States Government)		28.95
Surplus (See Exhibit B)	70,741.77	23,485.26
Reserve for Proceedings and Transactions deficit (See Exhibit B)	20,425.25	23,170.67
	$ 91,427.25	$110,101.64
INVESTED FUNDS:		
Endowment Fund Principal	$ 94,000.00	$ 71,000.00
Library Proceeds Fund	50,000.00	50,000.00
Other Expendable Funds	28,551.47	28,301.47
Prize Funds	6,575.00	5,381.75
Life Membership and Subscription Reserve	3,822.48	4,162.64
Mathematical Reviews	80,000.00	65,000.00
Reserve for Investment Losses	5,550.00	4,385.89
Profit/Loss on Sales of Securities	20,700.15	18,167.96
	$289,199.10	$246,399.71
TOTAL LIABILITIES	$380,626.35	$356,501.35

EXHIBIT A

GENERAL RECEIPTS AND DISBURSEMENTS

GENERAL RECEIPTS:			
Dues from Ordinary Memberships		$50,520.65	
Dues from Contributing Memberships		702.53	
Dues from Institutional Memberships		21,995.17	
Publication Charges from Non-Member Institutions		277.50	
Initiation Fees		904.87	
Income from			
Henderson Estate	$ 7,300.00		
Current Funds Investments	1,508.34		
Invested Funds	9,486.32	$18,294.66	
Miscellaneous			
Donations	34.95		
Meeting Fees	669.80	$ 704.75	
TOTAL GENERAL RECEIPTS			$93,400.13
GENERAL DISBURSEMENTS:			
General Activities			
Accounting Department	$ 8,782.08		
Executive Director	22,163.71		
New Haven Office	3,630.73		
Officers' Expense	1,295.89		
Meetings	4,683.59		
Policy Committee	168.81		
Theoretical Mechanics	100.00	$40,824.81	
Publications			
Bulletin	$15,192.75*		
American Journal	2,770.00		
Canadian Journal	1,000.00		
Pacific Journal	1,500.00		
Annals	1,175.00	$21,637.75	
TOTAL GENERAL DISBURSEMENTS			$62,462.56
EXCESS OF GENERAL RECEIPTS OVER GENERAL DISBURSEMENTS			$30,937.57

* Receipts $3,223.19, Disbursements $18,415.94.

EXHIBIT B

ANALYSIS OF SURPLUS

	Additions	Deductions	
Surplus at December 1, 1951			$23,485.26
Adjustments to Previous Year	$ 500.01	$ 1,461.97	
Excess of General Receipts over General Disbursements (see above)	30,937.57		
Bulletin Reprinting Fund	9,975.21		
Transactions Reprinting Fund	5,094.43		
Congress	20,417.59		
Transfer of Accumulated Balances in Expendable Funds	2,129.31		
Adjustments: Life Membership and Subscription Reserve, Library	340.16	250.55	
	$69,394.28	$ 1,712.52	
Net Additions			$67,681.76
Surplus before Reserves			$91,167.02
Reserves for Proceedings and Transactions Deficit			20,425.25
Surplus, November 30, 1952			$70,741.77

EXHIBIT C

COST OF OPERATING THE HEADQUARTERS BUILDING

Building Expense:	
Headquarters Utility Expense	$ 1,371.68
Rent	2,400.00
Equipment	1,374.59
Repairs	203.07
Salary of Janitor	1,226.51
Miscellaneous	8.80
Supplies	355.30
Amortization of Office Transfer	3,500.00
Total	$10,439.95

EXHIBIT D

SUMMARY OF CURRENT FUNDS

	Balance 12/1/51	Additions	Deductions	Balance 11/30/52
CURRENT FUNDS				
Publications				
Mathematical Reviews	$ 4,543.12	$ 56,969.87	$ 61,160.56	352.43
Colloquium	13,261.29	10,811.11	6,722.51	17,349.89
Mathematical Surveys	(5,652.58)	3,093.84	1,596.42	(4,155.16)
Birkhoff's Collected Papers	(1,247.52)	775.20	105.27	(577.59)
Symposia on Applied Mathematics	(878.87)	624.52	266.36	(520.71)
Memoirs	2,328.45	1,045.85	2,722.79	651.51
Russian Vocabulary	105.97	180.05	368.42	(82.40)
Translations	(370.77)	518.45	283.25	(135.57)
Bulletin Reprinting	9,154.52	820.69	9,975.21	
Transactions Reprinting	7,594.43		5,094.43	2,500.00
	$28,838.04	$ 74,839.58	$ 88,295.22	$15,382.40
Proceedings	(15,614.16)	20,386.34	18,341.03	(13,568.85)
Transactions	(7,556.51)	18,157.67	17,457.56	(6,856.40)
	($23,170.67)	$ 38,544.01	$ 35,798.59	($20,425.25)
TOTAL PUBLICATIONS	$ 5,667.37	$113,383.59	$124,093.81	($ 5,042.85)
International Congress	$41,992.58	$ 3,863.33	$ 45,855.91	
Office Transfer Fund	6,549.19	5,211.97	8,895.33	2,865.83
Policy Committee	778.30	396.70	316.99	858.01
Combined List of Members			4,685.55	(4,685.55)
Prize Funds Accumulated Income	2,216.71	288.56	100.00	2,405.27
Profit/Loss on Sale of Securities	2,049.85			2,049.85
	$53,586.63	$ 9,760.56	$ 59,853.78	$ 3,493.41
TOTAL	$59,254.00	$123,144.15	$183,947.59	($ 1,549.44)

SUMMARY OF INVESTED FUNDS

	PRINCIPAL				INCOME			
	Balance 12/1/51	Additions	Deductions	Balance 11/30/52	Balance 12/1/51	Additions	Deductions	Balance 11/30/52
INVESTED FUNDS								
Permanent Funds								
Endowment	$ 71,000.00	$23,000.00		$ 94,000.00		$ 4,019.76	$ 4,019.76*	
Expendable Funds								
Boyle		$ 250.00		$ 250.00				
Brown	$ 1,000.00			1,000.00	$ 110.81	$ 52.10	$ 162.91	
Henderson	1,000.00			1,000.00	110.81	52.10	162.91	
Hutchinson	1,000.00			1,000.00	110.81	52.10	162.91	
Merrill	650.00			650.00	87.76	33.40	121.16	
Reilly	23,651.47			23,651.47	321.29	1,217.02	1,538.31	
Whittemore	1,000.00			1,000.00	110.81	52.10	162.91	
Library Proceeds	50,000.00			50,000.00	1,277.02	2,574.30	3,851.32	
	$ 78,301.47	$ 250.00		$ 78,551.47	$2,129.31	$ 4,033.12	$ 6,162.43*	
Prize Funds								
Bocher	$ 1,188.00	$ 262.00		$ 1,450.00	$ 813.93	$ 54.77		$ 868.70
Cole	2,093.13	456.87		2,550.00	1,169.95	116.22	$ 100.00	1,186.17
Moore	2,100.62	474.38		2,575.00	232.83	117.57		350.40
	$ 5,381.75	$ 1,193.25		$ 6,575.00	$2,216.71	$ 288.56	$ 100.00	$2,405.27
Mathematical Reviews	$ 65,000.00	$15,000.00		$ 80,000.00		$ 3,584.25	$ 3,584.25†	
Reserves								
Life Membership	$ 4,065.34		$338.38	$ 3,726.96		$ 209.75	$ 209.75	
Transactions Subscription	97.30		1.78	95.52		5.34	5.34	
Investment Losses	4,385.89	$ 1,164.11		5,550.00		244.47	244.47	
Profit/Loss on Sales of Securities	18,167.96	2,532.19		20,700.15		973.88	973.88	
	$ 26,716.49	$ 3,696.30	$340.16	$ 30,072.63		$ 1,433.44	$ 1,433.44*	
TOTAL	$246,399.71	$43,139.55	$340.16	$289,199.10	$4,346.02	$13,359.13	$15,299.88	$2,405.27

* Transferred to General Receipts.
† Transferred to Mathematical Reviews Operating Account.

BOOK REVIEWS

Foundations of combinatorial topology. By L. S. Pontryagin. Trans. from the first (1947) Russian ed. by F. Bagemihl, H. Komm, and W. Seidel. Rochester, The Graylock Press, 1952. 12+99 pp. $3.00.

In the preface, the author states his purpose as follows:

"This book represents essentially a semester course in combinatorial topology which I have given several times at the Moscow National University. It contains a very rigorous but concise presentation of homology theory. The formal prerequisites are merely a few simple facts about functions of a real variable, matrices, and commutative groups. Actually, however, considerable mathematical maturity is required of the reader. An essential defect in the book is its complete omission of examples, which are so indispensable for clarifying the geometric context of combinatorial topology. In this sense a good complementary volume would be *Sketch of the fundamental notions of topology* by Alexandrov and Efremovitch,[1] in which the attention is focussed on geometric content rather than on the completeness and rigor of proofs. In spite of this shortcoming, it seems to me that the present work has certain advantages over existing voluminous treatises, especially in view of its brevity. It can be used as a reference for obtaining preliminary information required for participation in a serious seminar on combinatorial topology. It is convenient in preparing for an examination in a course, since the proofs are carried out with sufficient detail. For a more qualified reader, e.g., an aspiring mathematician, it can also serve as a source of basic information on combinatorial topology."

The reviewer considers that the author has done an excellent job of achieving his objectives within the limitations he has set down. Certainly none of the existing texts on combinatorial topology attains these objectives as well in so few pages. Experts may not always agree that he has chosen the clearest or most direct method of proof for some of his theorems, but of course this is largely a matter of taste.

The text consists of a two-page historical introduction, and three chapters. The introduction mentions the early ideas of Poincaré on homology theory, and the names of Alexander, Veblen, Lefschetz,

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[1] Apparently this book has not been translated from the Russian. Such material may be found in Chap. VI of *Anschauliche Geometrie* by D. Hilbert and S. Cohn-Vossen or in *Einfachste Grundbegriffe der Topologie* by Paul Alexandroff.

Hopf, and Alexandrov in connection with the later development of the subject. In Chapter I the notion of a simplicial complex is introduced, and the homology groups of a finite simplicial complex are defined, using an arbitrary abelian group for coefficients. In Chapter II, it is proved that the homology groups are topological invariants, i.e. independent of the choice of a simplicial decomposition. The proof of invariance involves the use of simplicial mappings, the simplicial approximation theorem, and invariance under barycentric subdivision. In Chapter III, homology theory is applied to the study of continuous mappings and fixed points. It is proved that the homomorphism induced by a continuous map is invariant under homotopies. It is also proved that if the "Lefschetz number" of a mapping of a polyhedron into itself does not vanish, then the mapping has a fixed point.

There are a couple of results included which are digressions from the main line of development. In Chapter I it is proved that any n-dimensional compact metric space can be imbedded in Euclidean $(2n+1)$-space, and in Chapter II Sperner's Lemma is proved, and then used to demonstrate that the topological dimension of an n-simplex is actually n, and the Brouwer fixed point theorem.

Several topics which other authors might consider important are completely omitted from this small book. Examples of such topics are homology theory for general spaces (e.g., the singular or Čech homology theory), relative homology groups, cohomology theory, products, and duality theorems.

W. S. MASSEY

Bessel functions. Part II. *Functions of positive integer order*. Ed. by W. G. Bickley, L. J. Comrie, J. C. P. Miller, D. H. Sadler, and A. J. Thompson. (British Association for the Advancement of Science, Mathematical Tables, vol. 10.) Cambridge University Press, 1952. 40+255 pp. $11.00.

This is the tenth and final volume of the *British Association Mathematical Tables*. The Mathematical Tables Committee of the "B.A." has a long and honorable history; a brief account is included in the final report which has been reprinted in Mathematical Tables and Other Aids to Computation vol. 3 (1949) pp. 333–340. For many years this Committee represented probably the only organized effort to plan and compute in a systematic manner mathematical tables, and its work entailed cooperation between professional computers, mathematicians, and amateurs, between paid and voluntary workers. That this intricate system worked at all might be thought a minor miracle; that it worked as well as it did is entirely due to the selfless devotion to this task of a succession of highly qualified enthusiasts. In 1948 the Committee was dissolved, and its activities were taken over by the newly-established Mathematical Tables Committee of the Royal Society which has started a new series of tables (the first of which was reviewed in this Bulletin vol. 57 (1951) pp. 325–326). Several projects of the B.A. Committee were taken over by the new organization, and among these was the completion of certain tables of Bessel functions. It was a happy idea to publish as a final volume of a famous series this "fulfilment of a long-standing promise and the completion of a task."

Volume One of the B.A. Bessel function tables appeared in 1937 and was reviewed in this Bulletin (vol. 44, p. 766). It contains functions of order zero and one. At the time of publication of the first volume it was stated that the second volume is "in an advanced state of preparation," and in the final report of the B.A. Mathematical Tables Committee the volume is stated to have passed for press. Since the computation of the present tables several important tables of Bessel functions (mainly of the first kind) appeared. Of these, Cambi's tables and the monumental Harvard tables of Bessel functions (this Bulletin vol. 55 (1949) pp. 78, 79) were compared with the tables contained in the volume under review.

In spite of the very large number of available tables of Bessel functions, the B.A. tables are likely to become the standard work for general use. In two handy volumes they contain about all the tables of Bessel functions one needs in general practice: they contain Bessel functions of the first and second kind and modified Bessel functions of the first and third kind, their reliability is notorious, and the the physical appearance of the volumes most satisfactory. For special purposes (larger number of digits, high or fractional orders, complex variable, zeros, special combinations, etc.) one can always fall back on the more specialized tables. In the Preface, W. G. Bickley remarks, without committing his Committee to any promises, that further work on Bessel functions is proceeding.

The first four of the eight tables contained in this volume give values of

$$J_n(x),\ Y_n(x) \text{ or } x^nY_n(x),\ x^{-n}I_n(x) \text{ or } e^{-x}I_n(x),\ x^nK_n(x) \text{ or } e^xK_n(x)$$

for $n=2$ (1) 20, $x=0$ (.1 or .01) 10 (.1) 20 or 25. 8 decimals are given for J_n, 8 significant figures for the other functions. Second central differences (often modified) are included. The second quartet of tables contains 10 decimal or 10 figure values of $J_n(x)$, $Y_n(x)$,

$I_n(x)$, $K_n(x)$ for $n=0$ (1) 20, $x=0$ (.1) 20 or 25. The arrangement of the tables resembles that of volume 1. Welcome innovations are catch headings in large type showing the function tabulated (J, Y, I, or K) on every page, the argument range on right-hand pages, and the range of orders on left-hand pages; and a tabular page index to the first four tables. The introductory material contains a preface, the tabular page index, description of the tables and an account of their preparation, instructions for interpolation, acknowledgments, a bibliography, a useful comparison between the notations used in the present volume and other notations of Bessel functions, and a 10 page list of definitions of and formulas relating to Bessel functions. The notations used in this book are the standard notations as in Watson's *Bessel functions*.

The arrangement, outward appearance, and printing of these tables is superb, and no higher compliment can be paid to the production than by saying that the late Dr. Comrie, had he lived to see these tables, would have been pleased.

The tables are a joint effort of several distinguished computers and mathematicians and it would seem invidious to single out any of them for praise; yet it is appropriate to express special thanks of the mathematical community to the chief editor of this volume, Professor Bickley, who in face of physical handicaps, and at a period of considerable distress, devoted much effort and loving care to this enterprise. The result is such as even he could wish.

A. ERDÉLYI

NEW JOURNAL

Journal of Rational Mechanics and Analysis. Volume 1. Bloomington, Indiana, The Graduate Institute for Applied Mathematics, Indiana University, 1952. 4+652 pp. $18.00; $6.00 to individuals engaged in research or teaching.

The aims of this journal are stated by its editors as follows. "The Journal of Rational Mechanics and Analysis nourishes mathematics with physical applications, aiming especially to close the rift between 'pure' and 'applied' mathematics and to foster the discipline of mechanics as a deductive, mathematical science in the classical tradition. Its scope comprises those parts of pure mathematics or other theoretical sciences which contribute to mechanics; among the included fields are all branches of analysis, differential geometry, analytical dynamics, elasticity, fluid dynamics, plasticity, thermodynamics, relativity, and statistical mechanics. Engineering applications, numerical work, perturbation procedures, etc., are acceptable only as incidental illustration in a paper devoted to sound mathematical theory. . . . A high expository level is desired, and papers written in an excessively condensed or crabbed style will not be printed." The expressed scope of the journal is thus far wider than its title seems to imply. On the other hand, in practice it is, at least so far, narrower than its editors claim. The twenty papers in this volume are about either physical problems or mathematical problems suggested by physical problems; but they differ markedly from the contents of most journals devoted to applied mathematics in that they contain precisely stated and serious mathematical theorems which are proved. The greater number deal in one way or another with problems in differential equations, whether or not disguised (as more than half are) as mechanics of continua. Analysis unmotivated by physical situations has not yet made its appearance in this volume.

The editors have insisted on a high stylistic and typographical level, in contrast to most present-day editors, who seem to feel that content (even if obscured by stylistic deficiencies) is more important than style. Doubtless this insistence on elegance and clarity has reduced the number of papers which could be published in this volume, but since mathematics is hardly dying of inanition at the present time (as witness the swollen girth of Mathematical Reviews), it is gratifying to see attention paid to form as well as to content.

It is hardly necessary to remark that this will be an essential journal for any mathematical library with any pretensions to completeness.

R. P. BOAS, JR.

BRIEF MENTION

Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability, Held at the Statistical Laboratory, Department of Mathematics, University of California, July 31–*August* 12, 1950. Ed. by J. Neyman. Berkeley and Los Angeles, University of California Press, 1951. 10+666 pp. $11.00.

The Proceedings of the First Symposium were published in 1947 and reviewed in this Bulletin, vol. 56, p. 267. This volume contains 46 papers by 52 authors, representing 8 categories: mathematical statistics, probability, astronomy, biometry, econometrics, physics, traffic engineering, and wave analysis.

NOTES

UNESCO Book Coupons may be sent directly to the Society in payment for its publications. A folder describing these coupons and listing the countries in which they are available may be obtained from the Society upon request.

An Institute for Pure and Applied Mathematics has been founded in Rio de Janeiro by the National Research Council of Brazil. The new institution will sponsor the publication of the principal Brazilian mathematical journal Summa Brasiliensis Mathematicae, and the noted mathematician and astronomer Dr. Lelio I. Gama has been appointed Director of the Institute, whose address is Caixa Postal 46, Rio de Janeiro, Brazil.

The National Science Foundation has announced grants in mathematics to the following institutions, to support studies by the professors indicated: Brown University, Herbert Federer, David Gale, and Bjarni Jónsson; Columbia University, Claude Chevalley, E. R. Kolchin; Cornell University, Paul Olum; State University of Iowa, H. T. Muhly; University of Pennsylvania, R. D. Schafer; University of Southern California, L. A. Henkin; University of Washington, F. H. Brownell; University of Wisconsin, R. H. Bing, L. C. Young.

Professor A. A. Albert of the University of Chicago has been elected to a corresponding membership in the Academia Brasileira de Ciências.

Dr. Leila A. Dragonette of the University of Chicago has received the 1952 Research Award given by Sigma Delta Epsilon, Graduate Women's Scientific Fraternity, for the best research paper submitted by a member.

Dr. E. C. Molina has been awarded the Cressan Medal by the Franklin Institute in recognition of his contributions to telephony.

Professor Rolf Gran-Olsson of the Norwegian Institute of Technology is on leave of absence and has been appointed to a visiting professorship at Brown University.

Professor F. W. Levi of the Tata Institute of Fundamental Research has been appointed to a professorship at the Free University of Berlin.

Dr. E. L. Arnoff of the National Advisory Committee for Aeronautics has accepted a position as research associate with the Operations Research Group, Case Institute of Technology.

Associate Professor E. W. Barankin of the University of California, Berkeley has been appointed to an associate professorship at the University of California, Los Angeles.

Dr. D. R. Belcher of the American Telephone and Telegraph Company has retired.

Mr. James Bercos of the Naval Ordnance Plant, Macon, Georgia, has accepted a position as personnel statistician with the Lockheed Aircraft Company, Marietta, Georgia.

Miss Marion Clarke of the University of Nebraska has accepted a position as research mathematician with the Ballistic Research Laboratories, Aberdeen Proving Ground, Maryland.

Mr. G. R. Costello of the National Advisory Committee for Aeronautics has accepted a position as aerophysics project engineer with Chance-Vought Aircraft, Dallas, Texas.

Mr. G. P. Dinneen of the Goodyear Aircraft Corporation has accepted a position as staff member with the Lincoln Laboratories, Massachusetts Institute of Technology.

Mr. R. P. Eddy of the Naval Ordnance Laboratory, Silver Spring, Maryland, has accepted a position as mathematician with the David Taylor Model Basin, Washington, D. C.

Dr. B. G. Farley of the Bell Telephone Laboratories has accepted a position with the Digital Computer Laboratory, Massachusetts Institute of Technology.

Professor A. H. Fox of Union College is on leave of absence and has been appointed a consultant at the Oak Ridge National Laboratory.

Mr. R. D. Glauz of Brown University has been appointed a staff member at the Los Alamos Scientific Laboratory.

Mr. J. V. Harrison of the Eckhart-Mauchly Computer Corporation has accepted a position as head of the Computation Laboratory, Operations Research Office, Chevy Chase, Maryland.

Professor I. O. Horsfall of the University of Utah has retired with the titles professor emeritus and director emeritus of the Extension Division.

Professor W. G. Hubert of City College, New York City, has retired with the title professor emeritus.

Dr. M. P. Jarnagin of the University of Maryland has accepted a position as mathematician with the Naval Aviation Ordnance Test Station, Chincoteague, Virginia.

Assistant Professor E. R. Keown of the Florida State University has been appointed to an assistant professorship at the Agricultural and Mechanical College of Texas.

Dean H. R. Kingston of the University of Western Ontario has retired.

Mr. C. C. Miesse of the Battelle Memorial Institute has accepted a position as mathematical physicist with the Aerojet Engineering Corporation, Azusa, California.

Associate Professor A. N. Milgram of Syracuse University has been appointed to an associate professorship at the University of Minnesota.

Professor Knox Millsaps of the Alabama Polytechnic Institute has accepted a position as assistant chief scientist with the Flight Research Laboratory, Wright Air Development Center, Dayton, Ohio.

Professor E. E. Moots of Cornell College has retired with the title professor emeritus.

Professor H. M. Sheffer of Harvard University has retired with the title professor emeritus.

Dean Emeritus T. M. Simpson of the University of Florida has been appointed a consultant in general education, at Henderson State Teachers College, Arkadelphia, Arkansas.

Mr. H. W. Stephens of Columbia University has been appointed to an assistant professorship at Ball State Teachers College.

Associate Professor E. W. Titt of the University of Texas is on leave of absence and has been appointed to a visiting associate professorship at Indiana University.

Associate Professor J. I. Tracey of Yale University has retired with the title professor emeritus.

The following promotions are announced:

A. B. Cunningham, West Virginia University, to an associate professorship.

A. M. Feyerham, Iowa State College of Agriculture and Mechanic Arts, to an assistant professorship.

Irving Kaplansky, University of Chicago, to an associate professorship.

K. B. Leisenring, University of Michigan, to an assistant professorship.

E. A. Lew, Metropolitan Life Insurance Company, to an associate actuary and statistician.

The following appointments to instructorships are announced: Dartmouth College: Mr. E. L. Whitney; University of Maryland: Dr. H. S. Collins; Ohio State University: Dr. R. C. Fisher.

President Emeritus H. N. Davis of the Stevens Institute of Technology died on November 3, 1952 at the age of seventy-one years. He had been a member of the Society for fifty years.

Professor M. W. Dehn of Black Mountain College died on June 27, 1952, at the age of seventy-four years.

Professor Emeritus E. V. Huntington of Harvard University died on November 25, 1952 at the age of seventy-nine years. He had been a member of the Society for fifty-one years.

Dr. F. R. Moulton died on December 7, 1952, at the age of eighty years. He was a member of the Society for forty-eight years.

Assistant Professor J. D. Newburgh of the Tulane University of Louisiana died on January 3, 1953 at the age of thirty-two years.

Professor Leigh Page of Yale University died on September 14, 1952 at the age of sixty-eight years. He had been a member of the Society for thirty-two years.

Professor T. R. Running of the University of Michigan died on October 10, 1952 at the age of eighty-six years. He had been a member of the Society for forty-nine years.

The note concerning Harold W. Kuhn which appeared on page 686 of the November 1952 issue of the Bulletin should read as follows: Dr. Harold W. Kuhn of Princeton University has been appointed to an assistant professorship at Bryn Mawr College.

The note concerning the promotion of G. G. Lorentz on page 690 of the November 1952 issue of the Bulletin is in error. Professor Lorentz has been an assistant professor since June 1951.

NEW PUBLICATIONS

Altshiller-Court, N. *College geometry. An introduction to the modern geometry of the triangle and the circle.* 2d ed. New York, Barnes and Noble, 1952. 19+313 pp. $4.00.

Amaldi, U. See Levi-Cività, T.

Baer, R. *Linear algebra and projective geometry.* New York, Academic Press, 1952. 8+318 pp. $6.50.

Breidenbach, W. *Das Delische Problem.* (*Die Verdoppelung des Wurfels.*) 2d ed. Leipzig, Teubner, 1952. 59 pp. 2.30 DM.

Cartan, E. *Oeuvres complètes.* Part I. *Groupes de Lie.* Paris, Gauthier-Villars, 1952. Vol. I, 32+568 pp.; vol. II, 8+788 pp. 5,500 fr.

Davenport, H. *The higher arithmetic. An introduction to the theory of numbers.* New York, Longmans, Green, 1952. 8+172 pp. $2.25.

Delone, B. N. *Kratkiĭ kurs matematičeskih mašin.* Part 1. *Malye sčetnye mašiny i matematičeskie pribory.* Moscow, Gosudarstvennoe Izdatel'stvo Tehniko-Teoretičeskoi Literatury, 1952. 135 pp. 3.60 rubles.

Denjoy, A. *L'énumération transfinie.* Book II. *L'arithmétisation du transfini.* Part 1. *Les permutations spéciales.* Paris, Gauthier-Villars, 1952. 230 pp. 3000 fr. Part 2. *Les suites canoniques.* 178 pp. 2,500 fr.

Dubošin, G. N. *Osnovy teorii ustoĭčivosti dviženiya.* Moscow, Izdatel'stvo Moskov. Univ., 1952. 318 pp. 8.30 rubles.

Dynkin, E. B., and Uspenskiĭ, V. A. *Matematičeskie besedy. Zadači o mnogocvetnoĭ raskraske. Zadači iz teorii čisel. Slučaĭnye bluždaniya.* Moscow, Gosudarstvennoe Izdatel'stvo Tehniko-Teoretičeskoi Literatury, 1952. 288 pp. 5.65 rubles.

Fröberg, C. E. *Hexadecimal conversion tables.* Lund, Gleerup, 1952. 20 pp. 3kr.

Gnedenko, B. V., and Hinčin, A. Y. *Èlementarnoe vvedenie v teoriyu veroyatnosteĭ.* 3d ed. Moscow, Gosudarstvennoe Izdatel'stvo Tehniko-Teoretičeskoi Literatury, 1952. 144 pp. 2.20 rubles.

Hartree, D. R. *Numerical analysis.* Oxford, Clarendon, 1952. 14+287 pp. $6.00.

Hinčin, A. Y. See Gnedenko, B. V.

Jones, C. W. *A short table for the Bessel functions* $I_{n+1/3}(x)$, $(2/\pi)K_{n+1/3}(x)$. Prepared on behalf of the Mathematical Tables Committee of the Royal Society. Cambridge University Press, 1952. 20 pp. $1.25.

Kammerer, A. *Les propriétés mécaniques des solides réels et la théorie de l'élasticité.* (Actualités Scientifiques et Industrielles, no. 1161.) Paris, Hermann, 1951. 132 pp. 1125 fr.

Kleene, S. C. *Introduction to metamathematics.* New York, Van Nostrand, 1952. 10+550 pp. $8.75.

Kraitchik, M. *Introduction à la théorie des nombres.* Paris, Gauthier-Villars, 1952. 7+202 pp.

von Krbek, F. *Eingefangenes Unendlich. Bekenntnis zur Geschichte der Mathematik.* Leipzig, Geest and Portig, 1952. 7+331 pp.

Krull, W. *Elementare und klassische Algebra vom modernen Standpunkt.* 2d ed. (Sammlung Göschen, vol. 930.) Berlin, de Gruyter, 1952. 136 pp. 2.40 DM.

Leman, A. *Vom periodischen Dezimalbruch zur Zahlentheorie.* 3d ed. Ed. by B. Schoeneberg. Leipzig, Teubner, 1952. 60 pp. 2.10 DM.

Levi-Cività, T., and Amaldi, U. *Lezioni di meccanica razionale.* Vol. II. *Dinamica dei sistemi con un numero finito di gradi di libertà.* New ed. Bologna, Zanichelli, 1951, 1952. Part 1, 9+510 pp., 4,000 lire; part 2, 9+673 pp. 5,000 lire.

Maeda, F. *Rensoku Kikagaku.* [*Continuous geometry.*] Tokyo, Iwanami Shoten, 1952. 2+3+225 pp. 600 yen.

Mandelbrojt, S. *Séries adhérentes, régularisation des suites, applications.* Paris, Gauthier-Villars, 1952. 14+277 pp.

Mercier, A. *Leçons sur les principes de l'électrodynamique classique.* Neuchatel, Griffon, 1952. 74 pp.

Mihlin, S. G. *Problema minimuma kvadratičnogo funkcionala.* Moscow, Gosudarstvennoe Izdatel'stvo Tehniko-Teoretičeskoi Literatury, 1952. 216 pp. 6.15 rubles.

Neiss, F. *Einführung in die Zahlentheorie.* Leipzig, Hirzel, 1952. 8+113 pp.

Pack, D. C. *Hodograph methods in gas dynamics.* Mimeographed Lecture Notes. Ed. by H. Roth. (Institute for Fluid Dynamics and Applied Mathematics, University of Maryland, Lecture Series, no. 17.) College Park, University of Maryland, 1952. 4+59 pp.

Poirier, R. *Logique et modalité du point de vue organique et physique.* (Actualités Scientifiques et Industrielles, no. 1163.) Paris, Hermann, 1952. 113 pp. 600 fr.

Quenouille, M. H. *Associated measurements.* New York, Academic Press, 1952. 10+242 pp. $5.80.

Reeb, G. See Wu, W. T.

Rosser, J. B., and Turquette, A. R. *Many-valued logics.* Amsterdam, North-Holland, 1951. 7+124 pp. $3.25.

Roth, H. See Pack, D. C.

Schoenberg, B. See Leman, A.

A selection of tables for use in calculations of compressible airflow. Prepared on behalf of the Aeronautical Research Council by the Compressible Flow Tables Panel. Oxford, Clarendon, 1952. 8+143 pp. $8.00.

Sommerfeld, A. *Mechanics.* (Lectures on theoretical physics, Vol. I.) Trans. from the 4th German ed. by M. O. Stern. New York, Academic Press, 1952. 14+289 pp., 1 plate. $6.50.

Stern, M. O. See Sommerfeld, A.

Tippett, L. H. C. *The methods of statistics.* 4th ed. New York, Wiley, 1952. 395 pp. $6.00.

Turquette, A. R. See Rosser, J. B.

Uspenskiĭ, V. A. See Dynkin, E. B.

Wilder, R. L. *Introduction to the foundations of mathematics.* New York, Wiley, 1952. 14+305 pp. $5.75.

Wu, W. T., and Reeb, G. *Sur les espaces fibrés et les variétés feuilletées.* (Actualités Scientifiques et Industrielles, no. 1183.) Paris, Hermann, 1952. 153 pp.

ON SMOOTHING OPERATIONS AND THEIR GENERATING FUNCTIONS[1]

I. J. SCHOENBERG

INTRODUCTION

In this paper we are mainly concerned with two kinds of linear transformations, the sequence convolution transformation

$$y_n = \sum_{\nu=-\infty}^{\infty} a_{n-\nu} x_\nu \tag{1}$$

and the integral convolution transformation

$$g(x) = \int_{-\infty}^{\infty} \Lambda(x - t) f(t) dt, \tag{2}$$

where the sequence $\{a_n\}$ and the function $\Lambda(x)$ are thought of as given. In §2 we also consider the ordinary linear transformation

$$y_i = \sum_{k=1}^{n} a_{ik} x_k \qquad (i = 1, \cdots, m). \tag{3}$$

The loosely connected topics to be discussed concerning these transformations are perhaps best brought together under the general subject of smoothing operations.

In §1 we are concerned with transformations (1) of the kind used for the purpose of smoothing numerical data. Erastus L. De Forest proposed long ago the problem of describing the asymptotic behavior of the coefficients of high-order iterates of (1) (see Wolfenden [15]). This question, as well as the question of when a formula (1) may rightly be called a smoothing formula, was answered by the author in [38] and [31]. It is shown in §1 that the author's criterion for a smoothing formula is essentially of the nature of a stability condition of the kind required of difference methods for the numerical integration of partial differential equations. From this point of view De Forest's problem amounts to constructing by difference methods the fundamental solutions of certain parabolic differential equations. These remarks are merely special cases of Fritz John's recent work

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An address delivered before the Fresno meeting of the Society on May 3, 1952 by invitation of the Committee to Select Hour Speakers for Far Western Sectional Meetings; received by the editors June 30, 1952.

[1] This work was performed on a National Bureau of Standards contract with the University of California, Los Angeles and was sponsored (in part) by the Office of Scientific Research, USAF.

[49] on the integration of parabolic differential equations.

The remainder of the paper discusses those transformations (1) and (2) which are variation-diminishing. These special classes of smoothing operations, while perhaps too narrow for practical smoothing purposes (because, in actuarial terms, they preserve only linear functions), seem to deserve particular attention because of the interesting problems which arise if we attempt to characterize their respective generating functions

$$\sum_{-\infty}^{\infty} a_n z^n \quad \text{and} \quad \int_{-\infty}^{\infty} e^{-xs}\Lambda(x)dx.$$

§§3 and 4 are devoted to these problems concerning (2) and (1), respectively. A discussion of (2), initiated by Pólya in 1915, has recently received considerable attention by Widder, Hirschman, and the author. The transformation (1), studied by M. Aissen, Anne Whitney and the author, is the subject of very significant as yet unpublished contributions at the hands of Albert Edrei (§4). Edrei's results are used to establish (Article 4.3) a conjecture of Schoenberg [31, Part B, pp. 131–132].

Our discussion in §2 of the transformation (3) is mainly intended to provide a background for the transcendental cases of (1) and (2). Attention is called to a recent theorem of R. Gantmakher and M. Krein which is of interest in the present connection and is also helpful in a discussion of curves which are convex in higher-dimensional spaces.[1]

1. THE STABILITY OF SMOOTHING FORMULAE

1.1. What is a smoothing formula? Let

$$y_m = \sum_{\nu=-\infty}^{\infty} a_{m-\nu} x_\nu \tag{1.1}$$

be a given "moving average" formula which is such that the function

$$F(z) = \sum_{-\infty}^{\infty} a_\nu z^\nu \qquad (r < |z| < r^{-1}), \tag{1.2}$$

which may be called its ***generating function***, is regular in an open ring containing the circle $|z| = 1$. Let us assume, as is generally the case in practice, that the given sequence of "weights" a_m is symmetric about a_0:

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[1] This remark explains the presence of several geometrical papers in our bibliography. As seen from this brief description of contents, the paper is concerned with the most general smoothing operations on the one hand (§1), with a very restricted class on the other (§§3, 4). For the middle-range of operations which are useful in the practice of smoothing data we refer to the excellent expositions by Whittaker-Robinson [17], and Wolfenden [15]. For the related subject of osculatory interpolation see Greville [29] and Schoenberg [31].

$$a_m = a_{-m} \qquad (m = 1, 2, \cdots). \tag{1.3}$$

It was pointed out in [31] that the generating function (1.2) or, equivalently, the *characteristic function*

$$\phi(u) = F(e^{iu}) = a_0 + 2a_1 \cos u + 2a_2 \cos 2u + \cdots \tag{1.4}$$

may be used to advantage in describing some of the useful attributes of a moving average formula (1.1). Thus the formula (1.1) will leave unchanged, or invariant, sequences $\{x_m\}$ which are equidistant ordinates of polynomials of degree $2k-1$, but of no higher degree, provided $\phi(u)$ has an expansion in powers of u which is of the form

$$\phi(u) = 1 - \lambda u^{2k} + \cdots \qquad (\lambda \neq 0). \tag{1.5}$$

Good arguments were presented in [31] and still better ones in [38] to the effect that a moving average formula (1.1) could be called a *smoothing formula* only if (perhaps also if) its characteristic function satisfies the condition

$$|\phi(u)| < 1 \qquad \text{if } 0 < u < 2\pi. \tag{1.6}$$

This condition implies that the coefficient λ of (1.5) is positive.[3]

1.2. **Behavior of the iterates of (1.1): De Forest's problem.** A conclusive argument in support of the necessity of our condition (1.6) is furnished by the solution of the following problem first stated and attacked by Erastus L. De Forest.[4] If we subject the given sequence $\{x_m\}$ n times in succession to the same transformation (1.1), we obtain a linear transformation

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[3] During his connection in the last war with the Data Analysis Section of the Boeing Aircraft Company, Dr. C. Lanczos has reached the same criterion (1.6) from the following point of view: If the formula (1.1) is applied to the simple vibration of frequency u, $x_\nu = e^{i\nu u}$ $(-\infty < \nu < \infty)$, we obtain the transformed sequence $y_m = e^{imu}\phi(-u)$ $(-\infty < m < \infty)$, which is again a simple vibration of frequency u, of amplitude $|\phi(-u)|$. Requiring that the amplitude of every simple vibration with frequency in the range $0 < u < 2\pi$ be *diminished* in the process, Dr. Lanczos reached our condition (1.6). This interesting remark shows that the values of $\phi(u)$ play the role of characteristic values of the matrix of the linear operation (1.1). The condition (1.6) is now seen to be related to the conditions found by R. Oldenburger and A. Dresden (for references see [47]) which insure that the nth power of an ordinary matrix converges to a limit as n tends to infinity.

[4] Erastus L. De Forest (1834–1888) contributed numerous original and fundamental papers on various aspects of the problem of fitting and smoothing irregular series by means of polynomials. His pioneer work remained practically unknown, however, until 1924, when its importance in relation to the graduation processes of actuaries was described by Hugh H. Wolfenden in his paper [15]. A further summary of De Forest's contributions can be found in H. H. Wolfenden, *The fundamental principles of mathematical statistics*, Toronto, Macmillan, 1942.

$$y_m^{(n)} = \sum_{\nu=-\infty}^{\infty} a_{m-\nu}^{(n)} x_\nu \tag{1.7}$$

which is the n-fold iterate of (1.1). *What is the asymptotic behavior of the coefficients of* (1.7) *as* $n \to \infty$? This question was answered for the case when all coefficients of (1.1) are non-negative, hence necessarily that $k=1$ in (1.5), by De Forest and by G. B. Dantzig [26].[5] A general solution is as follows:

Let (1.1) be such that (1.3), (1.6), and (1.5) are satisfied, hence $\lambda > 0$. Let

$$G_k(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-v^{2k}} \cos vx\,dv, \tag{1.8}$$

which is the normal function

$$G_1(x) = \frac{1}{2\pi^{1/2}} e^{-x^2/4} \tag{1.9}$$

if $k=1$, otherwise ($k=2, 3, \cdots$) an entire even function having infinitely many zeros, all real. *The coefficients of* (1.7) *satisfy the asymptotic relations*

$$a_\nu^{(n)} = (\lambda n)^{-1/2k} G_k(\nu(\lambda n)^{-1/2k}) + o(n^{-1/2k}), \qquad as\ n \to \infty, \tag{1.10}$$

where the "little o" symbol holds uniformly for all integers ν (see Schoenberg [38]). It was also shown there by examples that (1.10) no longer holds if the equality sign is allowed in (1.6), and that the coefficient $a_0^{(n)}$ diverges exponentially to $+\infty$, as $n=2m$ tends to infinity through even values, if the inequality (1.6) is reversed anywhere in the range $(0, 2\pi)$.

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[5] The following additional references, which I owe to T. S. Motzkin, came too late to my attention for inclusion in our bibliography: The case of non-negative weights ($k=1$) in the more general form where the convoluted formulae need not be identical was treated by Eugen Slutzky in a Russian paper of 1927. See its English translation *The summation of random causes as the source of cyclic processes*, Econometrica vol 5 (1937) pp. 105–146, in particular pages 134–138. Concerning the same problem see also R. v. Mises, *Wahrscheinlichkeitsrechnung und ihre Anwendungen in der Statistik und theoretischen Physik*, Leipzig and Wien, 1931, and R. v. Mises, *Generalizzazione di un teorema sulla probabilità della somma di un numero illimitato di variabili casuali*, Giornale dell' Istituto Italiano degli Attuari vol. 5 (1934) pp. 483–495.

A corresponding discussion of smoothing of periodic data is related to the so-called finite Fourier series (see Schoenberg [47]). Some unpublished work by Fritz John deals with the analogue of (1.10) for the periodic case.

1.3. **A differential equation.** The connection with differential equations is briefly as follows. Observe that (1.8) implies

$$U(x, t) = t^{-1/2k}G_k(xt^{-1/2k}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-tv^{2k}+ixv}dv \qquad (t > 0). \tag{1.11}$$

The function under the integral sign is immediately seen to satisfy, for all v, the differential equation

$$\frac{\partial u}{\partial t} = (-1)^{k+1}\frac{\partial^{2k}u}{\partial x^{2k}}, \tag{1.12}$$

which reduces to the familiar heat-equation if $k=1$. It follows that also $U(x, t)$, defined by (1.11), is a solution of (1.12) in the upper half-plane $t>0$. On the other hand, applying to (1.8) Fourier's inversion formula and setting $v=0$, we find that

$$\int_{-\infty}^{\infty} G_k(x)dx = 1.$$

These remarks imply the following: If $f(x)$ is continuous and $o(|x|^{-2})$, say, as $|x|\to\infty$, then

$$u(x, t) = t^{-1/2k}\int_{-\infty}^{\infty} G_k((x - v)t^{-1/2k})f(v)dv \qquad (t > 0) \tag{1.13}$$

is a solution of the differential equation (1.12) satisfying the boundary condition

$$\lim_{t\to+0} u(x, t) = f(x).$$

This particular solution $u(x, t)$ may now also be constructed as follows: Draw in the (x, t)-plane the lattice of points

$$(\nu\Delta x, n\Delta t) \qquad (\nu = 0, \pm 1, \cdots; n = 0, 1, \cdots).$$

Define in it a lattice-function $u_{\nu,n}$ by starting with

$$u_{\nu,0} = f(\nu\Delta x)$$

and computing the values along each horizontal line from those on the line below by means of (1.1). This evidently amounts to *iterating* (1.1), so that we have

$$(1.14) \qquad u_{\nu,n} = \sum_{\mu=-\infty}^{\infty} a_{\nu-\mu}^{(n)} f(\mu\Delta x).$$

For any given x and $t>0$, (1.14) goes over into (1.13) under the following circumstances: We first connect the mesh-sizes Δx and Δt by the relation

$$(1.15) \qquad \Delta t = \lambda(\Delta x)^{2k}.$$

If the integers ν and n are such that

$$\nu\Delta x \to x \quad \textit{and} \quad n\Delta t \to t, \quad \textit{as} \quad \Delta x \to +0,$$

then

$$u_{\nu,n} \to u(x, t).$$

This follows readily from (1.14) and (1.13), in view of the asymptotic relations (1.10): (1.14) differs from a Cauchy-Riemann sum for the integral (1.13) by a quantity which tends to zero because of the uniformity in ν of the error term of (1.10).

It is interesting to note that it matters little which formula (1.1) we use in this construction, as long as it preserves polynomials of degree $2k-1$ and of no higher degree, i.e. satisfies (1.5), and above all that it satisfy the *stability condition* (1.6), the term "stability" meaning here stability on iteration. For the general theory of Fritz John of which the above situation (for $k=1$) is merely a very special example, see [49].

2. VARIATION-DIMINISHING LINEAR TRANSFORMATIONS

2.1. Fundamental criteria. Let

$$(2.1) \qquad y_i = \sum_{k=1}^{n} a_{ik} x_k \qquad (i = 1, \cdots, m)$$

be a linear transformation where coefficients and variables are all real. Let $v(x)$ denote the number of changes of sign in the sequence x_k and $v(y)$ the similar number for y_i. F. Klein discussed the problem of comparing the upper bounds for the number of real roots of an algebraic equation in a given interval, as furnished by the Descartes rule of signs and similar theorems. These bounds are usually equal to the number of changes of sign of sequences arising from the coefficients of the given equation by linear transformations. For this reason, Klein's problem led Schoenberg [19; 21] to study transformations (2.1) which have the property that

$$(2.2) \qquad v(y) \leqq v(x),$$

for all values of the variables x_k. Such transformations were called *variation-diminishing* by Pólya.

The matrix $A = \|a_{ik}\|$ is called *totally positive*, provided all its minors of all orders are non-negative [19, p. 322]. A useful criterion for such matrices, based on a lemma of M. Fekete [4, pp. 4–6], was recently given by Anne Whitney [58]. Totally positive square-shaped matrices were thoroughly investigated by Gantmakher-Krein [24; 44] in connection with boundary-value problems arising in vibration problems. Schoenberg [19] showed that (1.1) is variation-diminishing if A is totally positive. Assuming that the matrix A is of rank n, he also proved the following: (1.1) is variation-diminishing if and only if A enjoys the property that two minors of A, of equal orders, are never of opposite signs. If the rank r of A is unrestricted, Motzkin [22] proved that (1.1) is variation-diminishing if and only if two minors of A, of the same order s, are never of opposite sign if $s < r$, while if $s = r$ they should never be of opposite signs if they belong to the same set of r columns of A. For a proof of this theorem, differing from Motzkin's, see Schoenberg-Whitney [51]. There also cyclic variation-diminishing transformations are characterized: (1.1) is said to be *cyclic variation-diminishing* provided it is such that the inequality

$$v_c(y) \leqq v_c(x) \tag{2.3}$$

always holds, where $v_c(x)$ is the number of changes of sign in the sequence $x_1, \cdots, x_n$ counted after these variables are placed cyclically along a circle, with a similar definition for $v_c(y)$.

2.2. A theorem of Gantmakher-Krein and convex curves. Let us assume that the rank of the matrix A, of (2.1), is equal to n. Choose n among the linear forms y_i which are linearly independent. As their values may be chosen at wish, for instance alternating in sign, it is clear that $\sup_x v(y) \geqq n-1$. The theorem of Gantmakher-Krein answers the question as to when we have the equality sign in this relation and may be stated as follows:

The system (1.1), *of rank equal to n, is such that we always have*

$$v(y) \leqq n - 1 \tag{2.4}$$

if and only if all nonvanishing minors of order n, of A, are of the same sign. See Gantmakher-Krein [44, Theorem 3 on p. 297], and also Schoenberg-Whitney [51, Theorem 1 on p. 142].[6]

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[6] Anne Whitney and the author were unaware of the book by Gantmakher and Krein when [51] was published. However, the priority clearly belongs to the Russian authors.

This theorem seems fundamental for two reasons: Firstly, it implies easily the results of Schoenberg and Motzkin concerning variation-diminishing transformations and plays a similar role in a discussion of the related problem (2.3). Secondly, it seems indispensable in a discussion of curves which are convex in higher-dimensional spaces. The use of the term "convex" in this connection requires some explanation. We mention first the concept of an arc in E_m of (linear) order m, due to C. Juel, intensively studied and generalized by O. Haupt, A. Marchaud [18], P. Scherk [25], and others: A Jordan arc in E_m is of order m provided it is intersected by every hyperplane in at most m points. A related but distinct notion is as follows: A continuous arc

$$\Gamma:\ x_i = x_i(t) \qquad (i = 1, \cdots, m;\ \alpha \leqq t \leqq \beta) \tag{2.5}$$

in the euclidean space E_m is said to be *convex in* E_m if it crosses every hyperplane of E_m at most m times. If we also require that Γ should not be contained in a lower-dimensional flat space, then we say that Γ is *convex on* E_m (see Schoenberg [56].)

The following criterion is an almost immediate corollary of the theorem of Gantmakher-Krein: *The arc* Γ, *defined by* (2.5), *is convex on* E_m *if and only if the determinants*

$$\begin{vmatrix} 1 & x_1(t_0) & \cdots & x_m(t_0) \\ 1 & x_1(t_1) & \cdots & x_m(t_1) \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ 1 & x_1(t_m) & \cdots & x_m(t_m) \end{vmatrix} \qquad (\alpha \leqq t_0 < t_1 < \cdots < t_m \leqq \beta) \tag{2.6}$$

do not all vanish and the nonvanishing among them are all of the same sign. The case when $m=2$ seems intuitively obvious, for the theorem says that the plane arc $P=P(t)$ is convex on E_2 provided the nondegenerate ones among the triangles $P(t_0)P(t_1)P(t_2)$ have all the same orientation. All convex curves are rectifiable. If Γ is convex *on* E_m and m is *odd*, then the arc Γ can never close. In even-dimensional spaces E_{2n} we may well have closed curves convex on E_{2n} as seen by the example of the special curve

$$\begin{aligned} C_0:\quad & x_1 = \cos t,\ x_3 = \frac{1}{2}\cos 2t,\ \cdots,\ x_{2n-1} = \frac{1}{n}\cos nt, \\ & x_2 = \sin t,\ x_4 = \frac{1}{2}\sin 2t,\ \cdots,\ x_{2n} \phantom{{}_{-1}} = \frac{1}{n}\sin nt \end{aligned} \tag{2.7}$$

$$(0 \leqq t \leqq 2\pi).$$

That C_0 spans E_{2n} is clear and equally so that C_0 is convex in E_{2n}: If $l(x_i)$ is a linear function and if we substitute the x_i from (2.7), then $l(x_i)$ becomes a real trigonometric polynomial of order n which is known to change sign within a period at most $2n$ times.

Let

$$C: x_i = x_i(t) \qquad (i = 1, \cdots, 2n; 0 \leqq t \leqq 2\pi), \tag{2.8}$$

be a closed curve in E_{2n} defined by continuous functions of period 2π. C is found to be convex on E_{2n} provided the above restriction on the signs of the determinants (2.6) (with $m = 2n$) holds for parameter values restricted by the inequalities $t_0 < t_1 < \cdots < t_n < t_0 + 2\pi$. This criterion leads naturally to a useful expression for the $2n$-dimensional volume $V = V(K)$ of the convex hull $K = K(C)$ of the curve C. Indeed, assuming the $x_i(t)$ to be absolutely continuous, as is the case if t is proportional to the arc-length, then V can be expressed by the following Lebesgue integral

$$V = \frac{\epsilon}{n!(2n)!} \int_0^{2\pi} \cdots \int_0^{2\pi} \det \left\| x_i(t_1),\ x_i'(t_1),\ x_i(t_2),\ x_i'(t_2), \cdots, \right.$$
$$\left. x_i(t_n),\ x_i'(t_n) \right\| dt_1 \cdots dt_n,$$

where $\epsilon = 1$, or -1, depending on the common sign of the determinants (2.6). This expression for V, which reduces to the classical area-integral if $n = 1$, allows one to generalize, by A. Hurwitz's method of Fourier series [3], the classical isoperimetric inequality to closed curves in E_{2n}: *If the closed curve C is convex in E_{2n}, of length L, then the following inequality holds*:

$$L^{2n} \geqq (2\pi n)^n n!(2n)! V, \tag{2.9}$$

with the equality sign if and only if C is similar to the curve C_0, defined by (2.7), *or to its reflexion* (see [56]). If $n = 1$, (2.9) reduces to the well known inequality $L^2 \geqq 4\pi V$ for convex curves in E_2, V being the area enclosed by C, and where equality holds only if C is a circle.

It is to be expected that this new approach to curves convex in E_m will be fruitful and work in collaboration with T. S. Motzkin is in progress. In concluding it should be pointed out that the notion of convexity of curves is also related to the Chebyshev systems of functions (Kellogg [9; 10; 11], S. Bernstein [16], Laasonen [42]) and to the systems satisfying Descartes' rule of signs (Pólya-Szegö [14]). Especially close is Pólya's discussion [13] of linear homogeneous differential equations. Indeed, the notion of a curve convex on E_m being affine invariant, such differential equations would seem to be the appropriate tool.

3. VARIATION-DIMINISHING INTEGRAL TRANSFORMATIONS

3.1. Pólya's work of 1915. An entire function $\Psi(s)$ may obviously be represented as a limit of a sequence of polynomials, uniformly convergent in every bounded domain of the complex plane: The sections of its expansion in powers of s have this property. Laguerre [1] and Pólya [5] have investigated the following problem: Which entire $\Psi(s)$ ($\not\equiv 0$) are limits of a sequence of real polynomials having only real zeros? They found this to be the case if and only if $\Psi(s)$ is of the form

$$\Psi(s) = Ce^{-\gamma s^2+\delta s}s^m\prod_1^\infty (1+\delta_\nu s)e^{-\delta_\nu s} \tag{3.1}$$

$$(C \gtrless 0,\ \gamma \geqq 0,\ \delta,\ \delta_\nu \text{ real},\ \sum \delta_\nu^2 < \infty).$$

They also showed that the subclass of those functions $\Phi(s)$ which are limits of polynomials with only real zeros and non-negative coefficients is characterized by the representation

$$\Phi(s) = Ce^{\gamma s}s^m\prod_1^\infty (1+\delta_\nu s) \quad (C > 0,\ \gamma \geqq 0,\ \delta_\nu \geqq 0,\ \sum \delta_\nu < \infty). \tag{3.2}$$

See also Szász [28]. For an extension of these results to entire functions of n variables see Motzkin-Schoenberg [50].

Pólya and Schur [7] have described these classes in terms of their Taylor expansions: A formal real power series

$$a_0 + (a_1/1!)\,s + (a_2/2!)\,s^2 + \cdots$$

is the expansion of a function of the form (3.1) if and only if the polynomials

$$P_n(x) = a_0 + \binom{n}{1}a_1x + \cdots + a_nx^n \qquad (n = 1, 2, \cdots) \tag{3.3}$$

have only real zeros; also that it is the expansion of a function of the form (3.2) if and only if the polynomials (3.3) have only real zeros and non-negative coefficients.

In [8], Pólya considered the subclass

$$\Psi(s) = Ce^{-\gamma s^2+\delta s}\prod_1^\infty (1+\delta_\nu s)e^{-\delta_\nu s} \tag{3.4}$$

$$(C > 0,\ \gamma \geqq 0,\ \delta,\ \delta_\nu \text{ real},\ 0 < \gamma + \sum \delta_\nu^2 < \infty),$$

of those functions of the class (3.1), with $\Psi(0) > 0$, which are not exponentials $C \exp \{\delta s\}$, and investigated the power series expansions of their reciprocals

$$\frac{1}{\Psi(s)} = \sum_0^\infty \frac{(-1)^\nu \mu_\nu}{\nu!} s^\nu. \tag{3.5}$$

He discovered the following properties:

1. The Hankel determinants

$$\begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ \mu_n & \mu_{n+1} & \cdots & \mu_{2n} \end{vmatrix} \quad \text{are all positive } (n = 1, 2, \cdots). \tag{3.6}$$

2. If $f(x)$ is a real polynomial, then

$$g(x) = \sum_0^\infty \frac{(-1)^\nu \mu_\nu}{\nu!} f^{(\nu)}(x) = \frac{1}{\Psi(D)} f(x) \tag{3.7}$$

is a polynomial having no more real zeros than $f(x)$. Conversely, if a real formal power series $\sum(-1)^\nu \mu_\nu s^\nu/\nu!$ $(\mu_0 > 0)$ is such that $g(x)$, defined by (3.7), has never more zeros than the arbitrary polynomial $f(x)$, then it is the expansion (3.5) of the reciprocal of a function $\Psi(s)$ of the form (3.4).

In 1920 Hamburger [12], using his then recent solution of the moment problem bearing his name, concluded from (3.6) that the coefficients μ_ν, of (3.5), are the moments of a nondecreasing function $\psi(x)$ and that (3.5) may be represented as a Laplace-Stieltjes integral

$$\frac{1}{\Psi(s)} = \int_{-\infty}^{\infty} e^{-sx} d\psi(x)$$

within the vertical strip of regularity of the left-hand function, containing the origin of the s-plane. More recently Hirschman-Widder [40] and Schoenberg [34; 55] have established a representation of (3.5) in terms of a bilateral Laplace integral

$$\frac{1}{\Psi(s)} = \int_{-\infty}^{\infty} e^{-sx} \Lambda(x) dx. \tag{3.8}$$

Here $\Lambda(x)$ is a frequency function,[7] i.e. $\Lambda(x) \geq 0$, which may be represented by the familiar inversion formula

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[7] Widder [35] interprets $\Lambda(x-y)$ as the Green's function of a differential system for the whole real axis. For a thorough investigation of differential systems in a *finite* range whose Green's functions are totally positive (Kellogg kernels) see Gantmakher-Krein [44].

$$\Lambda(x) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{e^{xs}}{\Psi(s)}\,ds \qquad (-\infty < x < \infty), \tag{3.9}$$

where the integral is defined as usual as a principal value.

Since

$$\mu_\nu = \int_{-\infty}^{\infty} x^\nu \Lambda(x)\,dx,$$

we may now write the differential operation (3.7) as an integral transformation. Indeed, if $f(x)$ is a polynomial, we find

$$\begin{aligned}\int_{-\infty}^{\infty} \Lambda(t) f(x-t)\,dt &= \int_{-\infty}^{\infty} \Lambda(t)\left\{\sum_{0}^{\infty} \frac{1}{\nu!} f^{(\nu)}(x)(-t)^\nu\right\} dt \\ &= \sum_{0}^{\infty} \frac{(-1)^\nu \mu_\nu}{\nu!} f^{(\nu)}(x) = g(x)\end{aligned}$$

or

$$g(x) = \int_{-\infty}^{\infty} \Lambda(x-t) f(t)\,dt. \tag{3.10}$$

Hence for polynomials, (3.10) being equivalent to (3.7), this integral transformation is inverted by

$$f(x) = \Psi(D) g(x), \tag{3.11}$$

where the operator $\Psi(D)$ is to be applied after its expansion in powers of D. This theory goes back to Pólya's paper [8] of 1915, except for the integral representations (3.8) and (3.10), the transformation (3.10), for polynomials, appearing in Pólya's paper in terms of the equivalent differential operation (3.7).

3.2. **Pólya frequency functions.** The determining functions $\Lambda(x)$, of (3.8), may be described in terms of the following definition: A frequency function $\Lambda(x)$, i.e. a measurable non-negative function such that

$$0 < \int_{-\infty}^{\infty} \Lambda(x)\,dx < \infty,$$

is said to be a *Pólya frequency function* provided that for any two sets of numbers

$$x_1 < x_2 < \cdots < x_n, \quad t_1 < t_2 < \cdots < t_n \qquad (n = 1, 2, \cdots) \tag{3.12}$$

we have the inequality

$$\det \|\Lambda(x_i - t_j)\|_{1,n} \geqq 0. \tag{3.13}$$

In terms of this definition, the following theorem holds: If $\Psi(s)$ is of the form (3.4), then its reciprocal may be represented in the form (3.8) within the vertical strip of regularity of $1/\Psi$ which contains the origin, and where $\Lambda(x)$ is a Pólya frequency function. Conversely, given a Pólya frequency function $\Lambda(x)$, then its Laplace transform (3.8) converges within such a vertical strip and represents the reciprocal of an entire function of the form (3.4). See Schoenberg [**34**; **55**]; also Wintner [**23**, Theorem 1].

Similar results concerning the ordinary Laplace transform are connected with entire functions of the form

$$\Phi(s) = Ce^{\gamma s}\prod_{1}^{\infty}(1 + \delta_\nu s) \tag{3.14}$$

$$(C > 0,\ \gamma \geqq 0,\ \delta_\nu \geqq 0,\ 0 < \gamma + \textstyle\sum \delta_\nu < \infty).$$

These functions forming a subclass of the class of functions (3.4), the previous theorem applies to their reciprocals. However, the function $\Psi(s)$ of (3.8) is of the special form (3.14) if and only if the Pólya frequency function $\Lambda(x)$, of (3.8), vanishes for all negative values of x (see [**34**; **55**]).

A related question is the following: The meromorphic function on the left-hand side of (3.8) has in general several vertical strips of regularity separated by vertical lines through its poles. What is the nature of the Laplace integral representations of $1/\Psi$ in the strips which do not contain the origin? An answer requires the following definition: A real measurable function $\Lambda(x)$ $(-\infty < x < \infty)$ is said to be *totally positive* provided it satisfies the following three conditions: 1. (3.12) implies (3.13), 2. $\Lambda(x)$ is different from zero for at least two distinct values of x, 3. $\Lambda(x) \not\equiv \exp\{ax+b\}$. It is clear that the Pólya frequency functions are totally positive. Moreover, it can be shown that every totally positive function turns into a Pólya frequency function, upon multiplication by appropriate exponentials. As a result the above stated correspondence between functions $\Psi(s)$ and Pólya frequency functions now extends to a one-to-one correspondence between functions $\Psi(s)$ and the totally positive functions $\Lambda(x)$, by the relation (3.8) with the sign changed if necessary, in vertical strips of regularity for $1/\Psi$ which are no longer required to contain the origin. In particular, a given $1/\Psi(s)$ allows a representation (3.8) in each of its strips of regularity with a different totally positive $\Lambda(x)$ corresponding to each of the strips (see [**34**; **55**]).

As particular examples of Pólya frequency functions we mention

$$\Lambda_1(x) = e^{-x^2}, \qquad \Lambda_2(x) = e^{-x-e^{-x}}, \qquad \Lambda_3(x) = \frac{1}{\cosh x}, \tag{3.15}$$

whose respective transforms $\pi^{1/2}e^{s^2/4}$, $\Gamma(s+1)$, $\pi/\cos(\pi s/2)$ are seen to be reciprocals of functions of the form (3.4). As a nontrivial example of Pólya frequency functions vanishing for $x<0$ we mention

$$\Lambda_4(x) = \begin{cases} \sum_{-\infty}^{\infty} (-1)^{\nu+1}\nu^2 e^{-x\nu^2} & \text{if } x > 0, \\ 0 & \text{if } x \leqq 0, \end{cases}$$

whose transform $\prod_1^\infty (1+s^2/\nu^2)^{-1}$ is indeed the reciprocal of a function of the form (3.14). Its cumulative distribution function

$$\int_{-\infty}^{x} \Lambda_4(x)dx = \begin{cases} \sum_{-\infty}^{\infty} (-1)^{\nu} e^{-x\nu^2} & \text{if } x > 0, \\ 0 & \text{if } x \leqq 0 \end{cases}$$

is the subject of the Kolmogorov-Smirnov limit theorem (see Feller [36]).

Given a Pólya frequency function $\Lambda(x)$ in terms of its transform (3.8) and two sets of numbers (3.12), Schoenberg-Whitney [43; 57] have answered the question as to when the determinant (3.13) is actually positive. For a discussion of the differentiability properties of Pólya frequency functions see [55, Corollary 2], Widder [35, Theorems 7 and 12], Hirschman-Widder [40]. For a new characterization of Pólya frequency functions as the only possible limits of so-called spline frequency functions see Curry-Schoenberg [33]. Concerning the interesting property of Pólya frequency functions of being "bell-shaped" see Hirschman-Widder [40], Schoenberg [34], and a related note [45] by Hirschman.

3.3. **Smoothing properties of Pólya frequency functions.** The remarkable smoothing property of the transformation (3.7), or (3.10), to the effect that the polynomial $g(x)$ has no more real zeros than $f(x)$, suggested the problem of finding the most general transformation of the form (3.10) which is *variation-diminishing*. By this we mean, obviously enough, the following: Let $v(f)$ (the number of changes of sign of $f(x)$) denote the supremum of the number of changes of sign in the sequence $f(x_1), \cdots, f(x_n)$, where $-\infty < x_1 < \cdots < x_n < +\infty$, $n=2, 3, \cdots$. Let $\Lambda(x)$ be summable, not

$=0$ p.p., and such that, for every bounded $f(x)$, (3.10) should imply the inequality

$$v(g) \leqq v(f).$$

It turns out that this is the case if and only if $\Lambda(x)$ or $-\Lambda(x)$ is a Pólya frequency function. The variation-diminishing property of (3.10) therefore characterizes the Pólya frequency functions. For a somewhat stronger result in terms of Stieltjes integrals see Schoenberg [46]. The author's old theorem [19] to the effect that the linear transformation (2.1) is variation-diminishing if the matrix A is totally positive is the origin of the peculiar property (3.13) of Pólya frequency functions. Indeed, (3.10) may be regarded as a linear transformation defined by the matrix $\Lambda(x-t)$, while (3.13) expresses its total positivity.

3.4. **Inversion and representation problems.** The prototype of the functional transformations to be now discussed is the Laplace transformation

$$\gamma(s) = \int_0^\infty e^{-su}\phi(u)du. \tag{3.16}$$

Well known results are its inversion by means of the Post-Widder inversion operator (Widder [32, Chap. VII]) and the Bernstein-Widder theorem which characterizes the functions representable in the form

$$\gamma(s) = \int_0^\infty e^{-su}d\alpha(u) \qquad (d\alpha(u) \geqq 0)$$

by the condition of complete monotoneity [32, Chap. VII, §14]. Following Widder we now set $u=\exp(t)$, $s=\exp(-x)$, whereupon (3.16) changes into

$$g(x) = \int_{-\infty}^\infty \Lambda_1(x-t)f(t)dt, \tag{3.17}$$

where $g(x)=\gamma(e^{-x})e^{-x}$, $f(t)=\phi(e^t)$, while $\Lambda_1(x)=\exp(-x-e^{-x})$ is one of the Pólya frequency functions (3.15). The Laplace transformation (3.16) is thus seen to be equivalent to a special instance of the convolution transformation (3.10).[8] For wide classes of Pólya frequency functions $\Lambda(x)$, Hirschman and Widder have extended the above results concerning the Laplace transformation to the general transformation

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[8] This explains, in view of Article 3.3, the variation-diminishing properties of the Laplace transformation discovered by Laguerre and Pólya. See Pólya-Szegö [14, Problem 80, p. 50].

$$g(x) = \int_{-\infty}^{\infty} \Lambda(x - t)f(t)dt, \tag{3.18}$$

obtaining an inversion theory (Widder [35; 52; 53], Hirschman-Widder [37; 40]) and a representation theory (Hirschmann-Widder [41]). Their results include as special cases earlier work by R. P. Boas and H. Pollard on the so-called Stieltjes transformation (which again may be reduced to (3.18) if $\Lambda(x) = \Lambda_2(x) = 1/\cosh x$) and its iterates. In case $f(x)$ is a polynomial we have seen that (3.18) is inverted by the differential operation (3.11). However, for arbitrary $f(x)$ the relation (3.11) is meaningless.

In order to illustrate the Hirschman-Widder inversion theory for (3.18) and to see how it goes beyond Pólya's formal inversion by (3.11), let us consider briefly the following special case discussed by Widder, *The inversion of a generalized Laplace transform*, Proc. Nat. Acad. Sci. U.S.A. vol. 33 (1947) pp. 295–297: Let $\Lambda(x)$ be a Pólya frequency function defined by the relations (3.8) and (3.4), with $\gamma = 0$ and all $\delta_\nu \neq 0$. If the given function $f(x)$ is continuous and summable ($f \in C \cdot L$), then (3.18) is inverted by the relation

$$f(x) = \lim_{n \to \infty} \prod_{\nu=1}^{n} (1 + \delta_\nu D) g\left(x - \sum_{1}^{n} \delta_\nu\right) \qquad (-\infty < x < \infty). \tag{3.19}$$

A proof follows readily from the relation

$$(1 + s\delta)e^{-\delta s}/\Psi(s) = \int_{-\infty}^{\infty} e^{-sx}(1 + \delta D)\Lambda(x - \delta)dx, \tag{3.20}$$

where δ is a real constant, and which follows from (3.8) by integration by parts. Applying the transformation (3.20) to (3.8), n times in succession with $\delta = \delta_1, \delta_2, \cdots, \delta_n$, we obtain the relation

$$\prod_{\nu=n+1}^{\infty} \frac{e^{\delta_\nu s}}{1 + \delta_\nu s} = \int_{-\infty}^{\infty} e^{-sx}\Lambda_n(x)dx, \tag{3.21}$$

where $\Lambda_n(x)$ is evidently a Pólya frequency function explicitly given by

$$\Lambda_n(x) = \prod_{\nu=1}^{n} (1 + \delta_\nu D)\Lambda\left(x - \sum_{1}^{n} \delta_\nu\right). \tag{3.22}$$

If we now operate with

$$\prod_{\nu=1}^{n} (1 + \delta_\nu D)e^{-\delta_\nu D}$$

on both sides of the relation (3.18), we obtain, by (3.22),

$$f_n(x) = \prod_{1}^{n} (1 + \delta_\nu D) g\left(x - \sum_{1}^{n} \delta_\nu\right) = \int_{-\infty}^{\infty} \Lambda_n(x - t) f(t) dt,$$

and we are to show that

$$f_n(x) = \int_{-\infty}^{\infty} \Lambda_n(x - t) f(t) dt \to f(x), \qquad \text{as } n \to \infty. \tag{3.23}$$

This follows from the fact that (3.23) is a singular integral satisfying the classical conditions of Jordan:

$$\int_{-\infty}^{\infty} \Lambda_n(x) dx = 1 \qquad \text{for all } n, \tag{i}$$

$$\lim_{n\to\infty} \int_{|x|\geq\delta} \Lambda_n(x) dx = 0, \qquad \text{for every } \delta > 0. \tag{ii}$$

Indeed, on expanding both sides of (3.21) in powers of s and comparing coefficients we find

$$\int_{-\infty}^{\infty} \Lambda_n(x) dx = 1, \qquad \int_{-\infty}^{\infty} x^2 \Lambda_n(x) dx = \sum_{n+1}^{\infty} \delta_\nu^2 = \sigma_n^2.$$

The fact that the variance σ_n^2 converges to zero, as $n \to \infty$, implies the property (ii) by a familiar Chebyshev argument. See Widder, loc. cit., where it is shown that the Post-Widder inversion operator for the Laplace transformation (3.16) is a special case of (3.19).

See Widder [52; 53] for the particularly interesting case of $\Lambda(x) = \Lambda_1(x) = \exp(-x^2)$ when (3.18) becomes the Weierstrass transform.

4. VARIATION DIMINISHING SEQUENCE TRANSFORMATIONS

4.1. Totally positive sequences. The sequence transformation

$$y_n = \sum_{\nu=-\infty}^{\infty} a_{n-\nu} x_\nu \tag{4.1}$$

gives rise to a theory in some respects analogous to the theory of §3. Assuming that $\sum |a_\nu| < \infty$, we say that the transformation (4.1) is variation-diminishing, provided that for every bounded sequence $\{x_n\}$, (4.1) implies the inequality

$$(4.2) \qquad v(y) \leqq v(x).$$

Here, as before, $v(x)$ and $v(y)$ denote the numbers of changes of sign in their respective sequences and may well be infinite. Since $x_0 = 1$, $x_n = 0$ (if $n \neq 0$) implies that $y_n = a_n$, for all n, we see that (4.2) implies that $v(a_n) = 0$. We may therefore assume all a_n non-negative.

A sequence $\{a_n\}$ is called *totally positive* provided that the matrix

$$(4.3) \qquad A = \|a_{i-j}\| \qquad (-\infty < i, j < \infty),$$

is totally positive, i.e. has only non-negative minors, and that $\{a_n\}$ is *not* a geometric progression of the form $a_n = Cr^n (C \geqq 0, r > 0)$. We say that the sequence $\{a_n\}$ is *totally positive normalized* provided $\{a_n\}$ is totally positive and the series $\sum a_n$ converges. The totally positive normalized sequences are the discrete analogues of the Pólya frequency functions of §3. A restricted kind of total positivity (k-times positive sequences) was discussed as early as 1912 by M. Fekete [4].

We saw in §2 that the transformation (2.1) is variation-diminishing if its matrix $\|a_{ik}\|$ is totally positive. In our case of the convolution transformation (4.1) this condition is not only sufficient but also necessary as stated by the following theorem: *The transformation* (4.1) *is variation-diminishing if and only if the sequence* $\{a_n\}$ *is totally positive normalized* (see [38]):

4.2. **Generation of totally positive sequences: Edrei's theorem.** In view of the last theorem, the problem of constructing variation-diminishing transformations (4.1) is equivalent to the problem of constructing sequences $\{a_n\}$ which are totally positive and normalized. A first step in this direction was made in [38] where the following was proved: If

$$(4.4) \qquad F(z) = Ce^{az+bz^{-1}}z^m \frac{\prod_1^\infty (1 + \alpha_\nu z) \prod_1^\infty (1 + \beta_\nu z^{-1})}{\prod_1^\infty (1 - \gamma_\nu z) \prod_1^\infty (1 - \delta_\nu z^{-1})}$$

is a meromorphic function in the ring $0 < |z| < \infty$, such that its constants satisfy the conditions

$$(4.5) \qquad \begin{gathered} C > 0,\ a \geqq 0,\ b \geqq 0,\ m \text{ integer},\ \alpha_\nu \geqq 0,\ \beta_\nu \geqq 0, \\ 0 \leqq \gamma_\nu < 1,\ 0 \leqq \delta_\nu < 1,\ \sum (\alpha_\nu + \beta_\nu + \gamma_\nu + \delta_\nu) < \infty, \end{gathered}$$

and if we expand $F(z)$ in a Laurent series

$$(4.6)\qquad F(z)=\sum_{-\infty}^{\infty} a_n z^n \qquad (r<|z|<r^{-1})$$

in the ring containing the unit circle inside, then the sequence $\{a_n\}$ so generated is totally positive normalized.[9]

This was easily established, the real difficulties arising as soon as we raise the following inverse problem: Given a totally positive normalized sequence $\{a_n\}$, it is not hard to show that the corresponding Laurent series (4.6) converges in an open ring containing the unit circle and defining an analytic function $F(z)$. *Is it true, then, that* $F(z)$ *is meromorphic in* $0<|z|<\infty$ *and of the form* (4.4), (4.5)?

Let us look at some examples. The following remark is obvious: If $\Lambda(x)$ is a Pólya frequency function (Article 3.2) then $a_n=\Lambda(n)$ $(-\infty<n<\infty)$ is a totally positive sequence which is normalized; for, indeed, $\|a_{i-j}\|=\|\Lambda(i-j)\|$ is a "submatrix" of $\Lambda(x-t)$. Using the functions (3.15) we now obtain sequences $\{a_n\}$ for exploratory purposes. Choosing $\Lambda(x)=\exp(-hx^2)$, where h is a positive constant and setting $q=\exp(-h)$, we obtain the sequence $a_n=\Lambda(n)=q^{n^2}$ which is totally positive and normalized. That the corresponding generating function (4.6) is of the form (4.4), (4.5) is shown by the famous Jacobian identity

$$(4.7)\qquad \sum_{-\infty}^{\infty} q^{n^2}z^n = C_1\prod_{n=0}^{\infty}(1+q^{2n+1}z)(1+q^{2n+1}z^{-1}) \qquad (0<|z|<\infty),$$

where C_1 does not depend on z. For the same reason (see (3.15)) also the sequence

$$a_n=\Lambda_3(hn)=1/\cosh(hn)$$

is totally positive normalized. The following identity which is equivalent to the Fourier expansion of the elliptic function dn$u$:

$$\sum_{-\infty}^{\infty}\frac{1}{\cosh(hn)}z^n = C_2\,\frac{\prod_{0}^{\infty}(1+q^{2n+1}z)(1+q^{2n+1}z^{-1})}{\prod_{0}^{\infty}(1-q^{2n+1}z)(1-q^{2n+1}z^{-1})} \qquad (q<|z|<q^{-1})$$

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[9] In Rademacher-Schoenberg [**30**, pp. 156–159], the function $-1/J_0((13z)^{1/2})$ is expanded, as a numerical illustration, in a Laurent series $\sum \omega_n z^n$ in a ring between its first and second pole. The extremely smooth behavior of the coefficients ω_n there obtained is explained by the fact that the sequence $\{\omega_n\}$ is totally positive normalized, in view of the result just stated.

shows that its generating function is of the form (4.4), (4.5).

It was conjectured in [38] that the answer to the question is affirmative. This was recently established by A. Edrei [62] who proved the following theorem: *If* $\{a_n\}$ *is a sequence which is totally positive and normalized, then its generating function* (4.6) *is necessarily of the form* (4.4), (4.5).[10]

Three noteworthy special cases, stated with their converses, are as follows:

1. If the power series

$$\sum_{0}^{\infty} a_n z^n \qquad (a_0 = 1)$$

converges for all z, then it represents a function of the form

$$e^{\gamma z}\prod_{1}^{\infty}(1+\delta_\nu z) \qquad (\gamma \geqq 0,\ \delta_\nu \geqq 0,\ \sum \delta_\nu < \infty) \tag{4.8}$$

if and only if the matrix

$$A = \begin{Vmatrix} a_0 & 0 & 0 & \cdots \\ a_1 & a_0 & 0 & \cdots \\ a_2 & a_1 & a_0 & \cdots \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \end{Vmatrix}$$

is totally positive.

2. If the Laurent series

$$\sum_{-\infty}^{\infty} a_n z^n$$

converges for all $z \neq 0$, then it represents a function of the form

$$Ce^{az+bz^{-1}}z^m\prod_{1}^{\infty}(1+\alpha_\nu z)\prod_{1}^{\infty}(1+\beta_\nu z^{-1}) \tag{4.9}$$

$$(C \geqq 0,\ a \geqq 0,\ b \geqq 0,\ m \text{ integer},\ \alpha_\nu \geqq 0,\ \beta_\nu \geqq 0,\ \sum(\alpha_\nu+\beta_\nu) < \infty)$$

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[10] Notice the interesting symmetry of (4.4) with respect to reciprocation: Also $1/F(-z)$ is of the form (4.4). An easy corollary of Edrei's theorem is as follows: If the identity $(\sum_{-\infty}^{\infty} a_n z^n)(\sum_{-\infty}^{\infty} (-1)^n b_n z^n) = 1$ holds in a ring $r_1 < |z| < r_2$, and if $\{a_n\}$ is totally positive, then either $\{b_n\}$ or else $\{-b_n\}$ is a totally positive sequence. As it happens, this proposition is actually a lemma in Edrei's proof of his theorem.

It should also be remarked that in analogy with the situation of §3, the Laurent expansions of (4.4) in every ring of regularity of $F(z)$ will generate, up to a common negative sign, totally positive sequences.

if and only if the sequence $\{a_n\}$ is totally positive.

3. If the symmetric Laurent series

$$\sum_{-\infty}^{\infty} a_n z^n \qquad (a_n = a_{-n})$$

converges for all $z \neq 0$, then it represents a function of the form

$$Ce^{a(z+z^{-1})} \prod_{1}^{\infty} (1 + \alpha_\nu z)(1 + \alpha_\nu z^{-1}) \tag{4.10}$$

$$\left(C \geqq 0, a \geqq 0, 1 \geqq \alpha_1 \geqq \alpha_2 \geqq \cdots \geqq \alpha_n \geqq \cdots \geqq 0, \sum \alpha_\nu < \infty\right)$$

if and only if the sequence $\{a_n\}$ is totally positive.

For a different characterization of the power series expansion of entire functions of the form (4.8) see J. Grommer [6, pp. 157–158]. A third characterization of the same expansion in terms of the zeros of the polynomials (3.3) was stated in Article 3.1. Edrei's general theorem should be regarded as a notable contribution to the general type of problem initiated by Hadamard in his classical dissertation [2]. Of course, the condition of Edrei's theorem that the sequence $\{a_n\}$ be totally positive is very strong, but then (4.4), (4.5) show the precise nature of the information on $F(z)$ which the theorem furnishes. Moreover, the condition is not only sufficient but also necessary.

Let us briefly review the work leading up to Edrei's theorem. Aissen, Schoenberg, and Whitney [59] investigated totally positive sequences $\{a_n\}$ with the property that $a_n = 0$ if $n < 0$, $a_0 = 1$, and showed that their generating function is necessarily of the form

$$\sum_{0}^{\infty} a_n z^n = e^{g(z)} \frac{\prod_{1}^{\infty} (1 + \alpha_\nu z)}{\prod_{1}^{\infty} (1 - \gamma_\nu z)},$$

where $\alpha_\nu \geqq 0$, $\gamma_\nu \geqq 0$, $\sum(\alpha_\nu + \gamma_\nu) < \infty$, and where $g(z)$ is entire and such that the exponential factor $\exp\{g(z)\}$, if expanded in powers of z, will also generate a totally positive sequence. Combining this result with a refinement of Picard's theorem due to R. Nevanlinna, Edrei [60] showed that necessarily $g(z) = \gamma z$ $(\gamma \geqq 0)$, thereby proving his theorem for the case when (4.6) reduces to a power series. A different approach to this case was given by Edrei in [61]. For the general case see Edrei [62].

4.3. **An application of Edrei's theorem.** With the problem to be now discussed we return to the stability of smoothing formulae which is the theme of §1. In 1946 the author constructed certain classes of analytic approximation formulae to given equidistant ordinates $\{y_n\}$ $(-\infty < n < \infty)$, which we need not describe here in detail; they are of the "cardinal" type

$$f(x) = \sum_{\nu=-\infty}^{\infty} y_\nu L(x - \nu),$$

$L(x)$ being a certain even entire function, accurate tables of values for $L(x)$ and its derivatives being available. Moreover, if $y_\nu = P(\nu)$, where $P(x)$ is a polynomial of degree $2k-1$ (k being a certain fixed number), then $f(x) \equiv P(x)$. If we now let $x = n$ be an integer, we are faced with the natural question whether the sequence transformation

$$f(n) = \sum_{\nu=-\infty}^{\infty} L(n - \nu) y_\nu$$

satisfies our stability (or smoothing) condition (1.6) (see Schoenberg [31, Part B, pp. 131–133]). This question was there shown to depend on a conjecture which the author was unable to prove at the time. The conjecture there formulated is an easy corollary of a theorem to be stated and proved below by means of Edrei's theorem.

We start with the "rectangular" frequency function

$$M_1(x) = \begin{cases} 1 & \text{if } |x| \leq 1/2, \\ 0 & \text{if } |x| > 1/2 \end{cases} \tag{4.11}$$

having the transform

$$\int_{-\infty}^{\infty} e^{-xs} M_1(x)dx = \frac{2 \sinh (s/2)}{s} \tag{4.12}$$

and the normal frequency function

$$M_0(x, t) = \frac{1}{(\pi t)^{1/2}} e^{-x^2/t} \qquad (t > 0) \tag{4.13}$$

of transform

$$\int_{-\infty}^{\infty} e^{-xs} M_0(x, t)dx = e^{ts^2/4}. \tag{4.14}$$

If we convolute $M_1(x)$ with itself $k-1$ times and convolute the result with $M_0(x, t)$, we obtain a distribution function $M_k(x, t)$ whose transform, by (4.12) and (4.14), is evidently

$$\int_{-\infty}^{\infty} e^{-sx} M_k(x, t)\,dx = e^{ts^2/4}\left(\frac{2 \sinh s/2}{s}\right)^k. \tag{4.15}$$

The theorem to be proved is as follows: *The Laurent series*

$$F(z) = \sum_{-\infty}^{\infty} M_k(n, t) z^n \qquad (0 < |z| < \infty) \tag{4.16}$$

admits a product representation

$$F(z) = C \cdot \prod_{\nu=1}^{\infty} (1 + \alpha_\nu z)(1 + \alpha_\nu z^{-1}) \tag{4.17}$$

where the $\alpha_\nu = \alpha_\nu(t)$ *are real and satisfy the conditions* $0 < \alpha_\nu < 1$, $\sum \alpha_\nu < \infty$, *while* $C = \prod(1+\alpha_\nu)^{-2}$.

REMARKS.[11] 1. The conclusion (4.17) "implies" that the sequence $a_n = M_k(n, t)$ is totally positive. This, naturally, will turn out to be the case. However, if $k > 0$, the transform (4.15), of $M_k(x, t)$, has zeros. It is therefore not of the form (3.8) and we conclude that $M_k(x, t)$ is not a Pólya frequency function. We can therefore not apply the convenient argument used in Article 4.2 to show that the sequence $M_k(n, t)$ is totally positive.

2. The argument is of course valid if $k = 0$ since $M_0(x, t)$ is the normal function (4.13). In this case our conclusion (4.17) is evidently true since it reduces to Jacobi's identity (4.7). If $k > 0$, the representation (4.17) may be regarded as a generalization of Jacobi's relation.

3. Quite recently, J. Berghuis [54] succeeded in proving (4.17) for the two values $k = 1$ and $k = 2$ by standard methods of the theory of functions.

A general proof of (4.17) is as follows: We define the successive integrals of (4.13) by the relations

$$g_0(x) = M_0(x, t)$$

and

$$g_\nu(x) = \int_{-\infty}^{x} g_{\nu-1}(x)\,dx \qquad (\nu = 1, 2, \cdots).$$

The Laplace transform of $g_k(x)$ is found from (4.14), by successive integrations by parts, to be

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[11] The conjecture stated in [31, Part B, pp. 131–132] is as follows: *The reciprocal of the function* $F(e^{iu})$ *admits a representation* (*) $1/F(e^{iu}) = \sum_0^\infty C_m(t)(2 \sin u/2)^{2m}$ *whose coefficients are positive and which converges for all real* u. This is implied by (4.17) as follows: $F(e^{iu}) = C\prod(1+\alpha_\nu^2+2\alpha_\nu \cos u) = C\prod\{(1+\alpha_\nu)^2-4\alpha_\nu \sin^2 u/2\}$ and (*) follows by reciprocation and expansion in powers of $\sin^2 u/2$.

$$\int_{-\infty}^{\infty} e^{-xs} g_k(x)\,dx = \frac{1}{s^k}\, e^{ts^2/4} \qquad (\mathrm{R}s > 0), \tag{4.18}$$

whence

$$\int_{-\infty}^{\infty} e^{-xs} g_k(x+h)\,dx = \frac{1}{s^k}\, e^{ts^2/4} e^{hs} \qquad (\mathrm{R}s > 0). \tag{4.19}$$

From these relations we draw two conclusions: 1. Replacing s by $s+1$ in (4.18) we obtain

$$\int_{-\infty}^{\infty} e^{-xs} e^{-x} g_k(x)\,dx = \frac{1}{(s+1)^k}\, e^{(ts^2+2ts+t)/4} \quad (\mathrm{R}s > -1).$$

The right-hand side being a reciprocal $1/\Psi(s)$, where $\Psi(s)$ is of the form (3.4), we conclude (Article 3.2) that $g_k(x)e^{-x}$ is a Pólya frequency function and hence:

(4.20) *The function $g_k(x)$ is totally positive.*

2. Writing $h=1/2$, $h=-1/2$ in (4.19) and subtracting one of the equations so obtained from the other, we find

$$\int_{-\infty}^{\infty} e^{-xs} \delta g_k(x)\,dx = \frac{1}{s^k}\, e^{ts^2/4} 2 \sinh s/2,$$

where δ is the symbol of central differencing with unit step. On repeating the operation altogether k times we find that

$$\int_{-\infty}^{\infty} e^{-xs} \delta^k g_k(x)\,dx = \frac{1}{s^k}\, e^{ts^2/4} (2 \sinh s/2)^k.$$

The transform so obtained agrees with (4.15) and we obtain the relation[12]

$$M_k(x, t) = \delta^k g_k(x). \tag{4.21}$$

For integral $x=n$ we obtain from (4.21)

$$M_k(n, t) = \delta^k g_k(n) = \Delta^k g_k(n - k/2),$$

a relation which is equivalent to the identity

$$\sum_{-\infty}^{\infty} M_k(n, t) z^n = (1-z)^k \sum_{n=-\infty}^{\infty} g_k\left(n + \frac{k}{2}\right) z^n \qquad (0 < |z| < 1).$$

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[12] See [31, Part A, p. 85] for a different derivation of the relation (4.21) which is used there to construct tables of values of $M_k(x,t)$.

Replacing the left-hand side by $F(z)$, in view of (4.16), and replacing z by $z/2$ we obtain

$$\sum_{-\infty}^{\infty} \frac{1}{2^n} g_k\left(n + \frac{k}{2}\right) z^n = \frac{F(z/2)}{(1 - z/2)^k} \qquad (0 < |z| < 2). \tag{4.22}$$

From (4.20) we conclude that $\{g_k(n+k/2)\}$ is a totally positive sequence. Therefore $\{2^{-n}g_k(n+k/2)\}$ is a totally positive sequence which is also normalized. By Edrei's theorem the left-hand side of (4.22) can be continued into the domain $0<|z|<\infty$ as a function of the form (4.4), (4.5). On the other hand, the right-hand side of (4.22) is another such representation valid throughout the region $0<|z|<\infty$, having as only singularity the pole $z=2$ of exact order k. On comparing these two analytic continuations of the left-hand side of (4.22) we conclude that $F(z/2)$ is of the form (4.9) and therefore so is $F(z)$. However, the Laurent expansion (4.16) of $F(z)$ being symmetric, we conclude that $F(z)$ allows a representation of the form (4.10), hence

$$F(z) = Ce^{a(z+z^{-1})} \prod_{1}^{\infty} (1 + \alpha_\nu z)(1 + \alpha_\nu z^{-1}), \tag{4.23}$$

where $C>0$, $a\geqq 0$, $1\geqq\alpha_1\geqq\alpha_2 \cdots \geqq 0$, $\sum\alpha_\nu<\infty$. A proof of (4.17) will be complete as soon as we show that the constants satisfy the following conditions:

1. $a=0$.
2. $\alpha_1<1$.
3. $\alpha_\nu>0$ for $\nu=1, 2, \cdots$.

Proof of 1. If we denote by $M_k(x)$ the convolution of k functions all identical with $M_1(x)$, defined by (4.11), then

$$M_k(x, t) = \frac{1}{(\pi t)^{1/2}} \int_{-\infty}^{\infty} e^{-(x-u)^2 t^{-1}} M_k(u)du.$$

However, $M_k(u)=0$ if $|u|>k/2$, hence if $x>k/2$ we find

$$\begin{aligned} M_k(x, t) &= \frac{1}{(\pi t)^{1/2}} \int_{-k/2}^{k/2} e^{-(x-u)^2 t^{-1}} M_k(u)du \\ &< \frac{1}{(\pi t)^{1/2}} \int_{-k/2}^{k/2} M_k(u)du \cdot e^{-(x-k/2)^2 t^{-1}} \end{aligned}$$

or

$$M_k(x, t) < \frac{1}{(\pi t)^{1/2}} e^{-(x-k/2)^2 t^{-1}}, \qquad \text{if } x > k/2.$$

For appropriate positive constants C_1 and γ ($\gamma < t^{-1}$) we therefore have $M_k(n, t) < C_1 e^{-\gamma n^2}$, for all n, and (4.16) implies the estimate

$$F(z) < 2C_1 \sum_0^\infty e^{-\gamma n^2} z^n, \qquad \text{if } z > 1.$$

However, if $z > 1$, all factors of (4.23) are $\geqq 1$ so that a fortiori

$$e^{az} < 2C_1 \sum_0^\infty e^{-\gamma n} z^n, \qquad \text{if } z > 1.$$

Since on the right side we have an entire function of order $\rho = 0$, we must indeed have that $a = 0$.

Proof of 2. It suffices to show that $F(-1) > 0$. This last inequality can be derived as follows: Setting $s = iu$ in (4.15) we obtain

$$\int_{-\infty}^{\infty} e^{-ixu} M_k(x, t)\, dx = e^{-tu^2/4} \left(\frac{2 \sin u/2}{u} \right)^k = \psi_k(u)$$

and by Poisson's summation formula we obtain the relation

$$\sum_{n=-\infty}^{\infty} M_k(n, t) e^{inu} = \sum_{\nu=-\infty}^{\infty} \psi_k(u + 2\pi\nu).$$

Setting $u = \pi$, we obtain

$$F(-1) = \left(\frac{2}{\pi} \right)^k \sum_{\nu=-\infty}^{\infty} (-1)^{\nu k} \frac{1}{(2\nu + 1)^k} e^{-t\pi\,(2\nu+1)\,/4}.$$

This is visibly positive if k is even. For odd values of k we write it in the form

$$\left(\frac{\pi}{2} \right)^k F(-1) = \left(\sum_{\nu=0}^{\infty} + \sum_{\nu=-1}^{-\infty} \right) (-1)^\nu \frac{1}{(2\nu + 1)^k} e^{-t\pi^2(2\nu+1)^2/4}$$

and find that each of the two ordinary infinite series indicated has terms decreasing in absolute values and alternating sign. Thus the sums of these series are positive and again $F(-1) > 0$.

Proof of 3. If only a finite number of α_ν in (4.17) were positive, then $F(z)$ would be rational, hence $z = 0$ would be a polar singularity for the right-hand side of (4.22). However, the Laurent expansion on the left-hand side of (4.22) shows that $z = 0$ is actually an essential singularity. This contradiction completes a proof of the factorization (4.17).

A concluding remark on this subject is as follows. In [31, p. 79], it was pointed out that

$$f(x) = \sum_{\nu=-\infty}^{\infty} y_\nu M_k(x - \nu, t)$$

is an analytic approximation to the given sequence of ordinates $\{y_n\}$ such that

$$f(n) = \sum_{-\infty}^{\infty} y_\nu M_k(n - \nu, t)$$

is a smoothing operation in the sense of §1. We can now make the stronger assertion that this last operation is variation-diminishing. Indeed, as already pointed out, the sequence $a_n = M_k(n, t)$ is totally positive, a fact equivalent to our assertion (see Article 4.1).

4.4. **A few unsolved problems.** 1. Our first question concerns the proof of Edrei's theorem. The ties between the totally positive functions (§3) and totally positive sequences (§4) are apparently not as close as to allow a proof of Edrei's theorem along the lines used in characterizing Laplace transforms of Pólya frequency functions. Is this really in the nature of things or could an "elementary" proof of Edrei's theorem be devised, at least for the case when (4.6) is a power series?

2. It was pointed out in Article 4.2 that if $\Lambda(x)$ is a Pólya frequency function, then

$$a_n = \Lambda(n) \qquad (-\infty < n < \infty) \tag{4.24}$$

is a totally positive normalized sequence. However, not all such sequences can be obtained in this way. Indeed, let $\{a_n\}$ be a totally positive sequence such that its generating Laurent series converges in $0 < |z| < \infty$, and hence is of the form (4.9). However, let it be different from the series (4.7). Such a sequence can *not* be obtained by (4.24), by interpolating a Pólya frequency function $\Lambda(x)$. The reason for this is very simple: Assuming (4.24) to hold, $\Lambda(x)$ would have a Laplace transform (3.8) which is entire. However, (3.8) and (3.4) show that the only Pólya frequency function having an entire transform is the normal frequency function. This situation raises the question: *What is the nature of totally positive sequences obtained by* (4.24) *from Pólya frequency functions?*

3. Let us apply the integral transformation (3.10) to a bounded function $f(x)$ of period 2π. Setting

$$\chi(x) = \sum_{n=-\infty}^{\infty} \Lambda(x + 2\pi n), \tag{4.25}$$

the transformation (3.10) becomes

$$g(x) = \int_0^{2\pi} \chi(x - t)f(t)dt, \tag{4.26}$$

which shows that $g(x)$ is continuous and of period 2π. Assuming $\Lambda(x)$ to be a Pólya frequency function, (4.26) implies that $v(g) \leqq v(f)$ (see Article 3.3); however, this relation is now meaningless since $v(g)$ and $v(f)$ either vanish or else are infinite. It is not difficult to see that (4.26) now enjoys the variation-diminishing property in the cyclic sense, i.e. it implies

$$v_c(g) \leqq v_c(f), \tag{4.27}$$

where these numbers of changes of sign are counted on the circle, or within a period. The special case of (4.26) when

$$\Lambda_\tau(x) = \frac{1}{(\pi\tau)^{1/2}} e^{-x^2/\tau} \qquad (\tau > 0), \tag{4.28}$$

was already noticed by Pólya [20], not in its integral form (4.26) but rather in terms of the equivalent convolution of the respective Fourier series;[13] $\chi(x)$ then becomes the Fourier kernel for a ring known from heat-conduction.

The following question arises: *What is the nature of the periodic functions $\chi(x)$ with the property that the transformation* (4.26) *is cyclically variation-diminishing, i.e. implies* (4.27)? Our formula (4.25) furnishes such functions but does not produce all of them. Indeed

$$\chi(x) = a + \cos t \qquad (a \text{ constant} \geqq 1)$$

is readily shown to have the property (4.27), without, however, being of the form (4.25).

Concluding, let us briefly mention a connection between the transformations (4.26) and the convex curves of Article 2.2. Let us assume

$$\int_0^{2\pi} \chi(t)dt = 1, \tag{4.29}$$

and let C, defined by (2.8), be convex in E_{2n}. *Then the new curve*

$$\bar{C}:\quad x_i = \bar{x}_i(x) = \int_0^{2\pi} \chi(x - t)x_i(t)dt \tag{4.30}$$

$$(i = 1, \cdots, 2n;\ 0 \leqq x \leqq 2\pi)$$

---

[13] For yet another example of a special function $\chi(x)$ see Pólya-Wiener [27, Lemma 1].

is also convex in E_{2n}. Indeed, (4.29) and (4.30) imply the relation

$$A_0 + \sum_1^{2n} A_i\bar{x}_i(x) = \int_0^{2\pi} \chi(x - t)\left(A_0 + \sum_1^{2n} A_i x_i(t)\right) dt,$$

and by (4.27):

$$v_c(A_0 + \sum A_i\bar{x}_i(t)) \leq v_c(A_0 + \sum A_i x_i(t)).$$

This relation means that $\bar{C}$ crosses a hyperplane not more often than C does, hence at most $2n$ times.

If we choose in particular the Fourier kernel $\chi_\tau(x)$ arising from (4.25) and $\Lambda(x) = \Lambda_\tau(x)$, defined by (4.28), then we see that the $2n$ functions (4.30) are analytic (even entire) functions of the parameter x. As $\tau \to +0$, we evidently have $\bar{x}_i(x) \to x_i(x)$. This remark proves the following: *A curve convex in E_{2n} is the limit of a family of curves convex in E_{2n} which are analytic.*

Added in proof, April 1953. Professor W. Fenchel kindly pointed out to me that also Gantmakher and Krein have been essentially anticipated in the discovery of their theorem of Article 2.2 by the late J. Hjelmslev in his paper *Introduction à la théorie des suites monotones*, Oversigt over det Kgl. Danske Videnskabernes Selskabs Forhandlinger, 1914, pp. 1–74. Indeed Hjelmslev gives analytic conditions in order that the arc Γ in E_m, defined by (2.5), be of order m. The precise statement of his result (loc. cit. p. 62) is as follows: *If no subarc of Γ is in E_{m-1}, then Γ is of order m if and only if all determinants* (2.6) *are positive, or all these determinants are negative.* Arcs satisfying these determinant conditions are called *monotone arcs* by Hjelmslev.

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THE FEBRUARY MEETING IN NEW YORK

The four hundred eighty-ninth meeting of the American Mathematical Society was held at Hunter College on Saturday, February 28. There were almost 300 persons in attendance, including the following 246 members of the Society:

C. R. Adams, M. I. Aissen, R. G. Albert, A. G. Anderson, R. L. Anderson, R. G. Archibald, P. N. Armstrong, N. J. Artin, Silvio Aurora, W. G. Bade, Joshua Barlaz, J. H. Barrett, L. K. Barrett, R. G. Bartle, J. D. Baum, Anatole Beck, E. G. Begle, Lipman Bers, Nicholas Bilotta, D. W. Blackett, E. K. Blum, Samuel Borofsky, S. G. Bourne, J. W. Bower, C. B. Boyer, J. W. Brace, A. D. Bradley, Roslyn Braverman, F. E. Browder, A. B. Brown, C. T. Bumer, J. H. Bushey, Jewell H. Bushey, F. P. Callahan, P. W. Carruth, Joshua Chover, W. L. Chow, L. W. Cohen, Richard Courant, L. M, Court, V. F. Cowling, M. L. Curtis, P. C. Curtis, Jr., J. H. Curtiss, M. D. Darkow, D. A. Darling, R. B. Davis, J. C. E. Dekker, V. J. Doberly, Jesse Douglas, James Dugundji, Nelson Dunford, Jaques Dutka, J. E. Eaton, Samuel Eilenberg, C. C. Elgot, Bernard Epstein, M. P. Epstein, Paul Erdös, M. E. Estill, Trevor Evans, W. H. Fagerstrom, J. M. Feld, Chester Feldman, F. G. Fender, W. E. Ferguson, Irwin Fischer, R. M. Foster, Gerald Freilich, Orrin Frink, M. P. Gaffney, G. N. Garrison, Murray Gerstenhaber, H. A. Giddings, B. P. Gill, Wallace Givens, Sidney Glusman, Samuel Goldberg, S. I. Goldberg, Daniel Gorenstein, Morikuni Goto, H. S. Grant, L. W. Green, F. L. Griffin, H. M. Griffin, Emil Grosswald, Laura Guggenbuhl, Felix Haas, Sister M. Raphael Hafner, Carl Hammer, Alvin Hausner, G. A. Hedlund, Alex Heller, R. T. Herbst, Aaron Herschfeld, David Hertzig, L. S. Hill, I. I. Hirschman, F. E. P. Hirzebruch, S. P. Hoffman, Banesh Hoffmann, Alfred Horn, Alfred Huber, Ralph Hull, T. R. Humphreys, D. H. Hyers, B. M. Ingersoll, H. G. Jacob, R. C. James, Shizuo Kakutani, Aida Kalish, Yukiyosi Kawada, M. E. Kellar, J. F. Kiefer, H. S. Kieval, M. S. Klamkin, George Klein, I. I. Kolodner, B. O. Koopman, Saul Kravetz, J. B. Kruskal, Jr., M. D. Kruskal, R. R. Kuebler, A. W. Landers, M. K. Landers, P. D. Lax, Solomon Leader, Solomon Lefschetz, Benjamin Lepson, J. J. Levin, D. J. Lewis, S. D. Liao, H. M. Lieberstein, M. A. Lipschutz, E. R. Lorch, D. B. Lowdenslager, Eugene Lukacs, R. C. Lyndon, B. H. McCandless, L. A. MacColl, H. M. MacNeille, Wilhelm Magnus, Irwin Mann, A. J. Maria, M. H. Maria, Imanuel Marx, A. P. Mattuck, A. E. Meder, Jr., L. F. Meyers, K. S. Miller, W. H. Mills, Don Mittleman, J. C. Moore, Morris Morduchow. G. D. Mostow, Simon Mowshowitz, E. R. Mullins, Jr., D. S. Nathan, C. A. Nelson, L. W. Neustadt, Morris Newman, Katsumi Nomizu, A. B. Novikoff, C. O. Oakley, R. E. O'Donnell, A. F. O'Neill, J. C. Oxtoby, J. S. Oxtoby, T. A. Paley, F. P. Pederssen, Everett Pitcher, H. O. Pollak, Walter Prenowitz, M. H. Protter, Howard Raiffa, D. B. Ray, G. E. Raynor, Helene Reschovsky, H. G. Rice, Moses Richardson, C. E. Rickart, E. K. Ritter, I. F. Ritter, Robin Robinson, Selby Robinson, David Rosen, M. A. Rosenlicht, J. E. Rosenthal, J. P. Russell, Charles Salkind, Hans Samelson, J. E. Sammet, Jacob Samoloff, J. E. Sanders, Arthur Sard, Shigeo Sasaki, R. D. Schafer, J. A. Schatz, Samuel Schecter, Pincus Schub, Abraham Schwartz, C. H. W. Sedgewick, R. R. Seeber, Jr., R. J. Semple, H. N. Shapiro, H. S. Shapiro, V. L. Shapiro, James Singer, P. A. Smith, J. J. Sopka, W. K. Spears, E. P. Starke, J. S. Stubbe, M. M. Sullivan, G. L. Thompson, D. L. Thomsen, Jr., M. L. Tomber, P. M. Treuenfels, A. W. Tucker, H. G. Tucker, R. J. Turyn, D. H. Wagner, Sylvan

Wallach, M. T. Wechsler, J. V. Wehausen, J. H. Weiner, Louis Weisner, John Wermer, M. E. White, A. L. Whiteman, P. M. Whitman, Albert Wilansky, J. E. Wilkins, Jr., A. B. Willcox, K. G. Wolfson, V. M. Wolontis, M. A. Wurster, Fumio Yagi, Hidehiko Yamabe, Michael Yanowitch, Bertram Yood, D. M. Young, E. H. Zarantonello, J. A. Zilber, H. J. Zimmerberg, Leo Zippin.

An address, *On the foundations of algebroid geometry*, was presented at the General Session by Professor W. L. Chow of The Johns Hopkins University by invitation of the Committee to Select Hour Speakers for Eastern Sectional Meetings. Professor C. R. Adams presided.

Sessions for contributed papers were held in the morning and in the afternoon. Professors R. M. Foster and Louis Weisner presided in the morning and Professors Paul Erdős and Everett Pitcher in the afternoon.

Coffee was served by students and members of the Department of Mathematics after the General Session.

Abstracts of the papers presented follow, those with "*t*" after the abstract number having been presented by title. Of the papers having joint authorship paper 286 was presented by Miss Scheerer, paper 295 by Dr. Lukacs, and paper 304 by Professor Young. Dr. Kalaba was introduced by Dr. Robert Davies and Mr. Darsow by Professor Irving Kaplansky.

ALGEBRA AND THEORY OF NUMBERS

266*t*. Volodymyr Bohun-Chudyniv: *On a method of determining orthogonal square matrices of* $(2^{1+\lambda}-1)2^K(\lambda \geqq 1,\ K \geqq 1;\ \lambda = 0,\ K > 2)$ *order, composed of differing integers.*

In the works of: (1) J. J. Sylvester, *Thoughts on inverse orthogonal matrices* etc., Philosophical Magazine (4) vol. 34 (1867) pp. 461–475; (2) A. C. Paley, *On orthogonal matrices*, Journal of Mathematics and Physics vol. 12 (1933) pp. 311–320, are given methods of determining orthogonal square matrices of 2^K $(K \geqq 2)$ order and in Paley's work also ones of other orders (see table I, p. 317 of his work), but elements of these matrices can only be ± 1. In author's paper *On orthogonal and non-orthogonal closed systems of K-nions and their application* (Bull. Amer. Math. Soc. Abstract 58-6-562) is given a method of determining closed systems of K-nions and a method of constructing with their help orthogonal matrices of $(2^{1+\lambda}-1)$ $(\lambda > 1)$ and 2^K $(K > 2)$ orders composed of differing integers. In the present paper the author determines the method of constructing orthogonal square matrices of $(2^{1+\lambda}-1)2^K$ $(\lambda \geqq 1,\ K \geqq 1;\ \lambda = 0,\ K > 2)$ order, elements of which are also differing integers. Matrices of this type we determine with the help of 2 lemmas and schemes of K-nions, constructed by the author for determining orthogonal square matrices of $(2^{1+\lambda}-1)$ and 2^K orders. A particular case of the first lemma is the first lemma of Paley. The largest number of elements in these matrices can equal $3 \cdot (2^{1+\lambda}-1)^2$ for the first group of matrices and 3 2^{2K} for the second. The Sylvester-Paley orthogonal square matrices are included in author's matrices as a particular case. (Received January 14, 1953.)

267*t*. Leonard Carlitz: *A note on partitions in $GF[x, q]$.*

Problems of the following kind are discussed. For $A \in GF[x, q]$, $A \neq 0$, let $p(A)$ denote the number of solutions of $A = U_1 + U_2 + \cdots$, $\deg A = \deg U_1 > \deg U_2 > \cdots$; let $p_k(A)$ denote the number of solutions in which there are precisely k summands. Then it is shown that if $\deg A = m$, then $p(A) = \prod_{i=0}^{m-1} (1 + (q-1)q^i)$, while $p_k(A) = q^{(k-1)(k-2)/2}\left[{m \atop k-1}\right]$. (Received January 12, 1953.)

268*t*. Leonard Carlitz: *Note on some partition identities.*

The paper contains elementary proofs of some identities proved by Newman, *Remarks on some modular identities*, Trans. Amer. Math. Soc. vol. 37 (1952) pp. 313–320, as well as of some similar formulas. (Received January 12, 1953.)

269*t*. Leonard Carlitz: *The first factor of the class number of a cyclic field.*

Let $p - 1 = ab$, where b is odd and >1. Let $\zeta = e^{2\pi i/p}$, and let $K \subset R(\zeta)$ denote the cyclic field of degree a over the rational field; let h_a denote the first factor of the class number of K. It is shown that $h_a \equiv 2^{-(a-2)/2} \prod B_{ubp^n+1} \pmod{p^n}$, where $u = 1, 3, \cdots, a-1$. The method is that used by Vandiver, *On the first factor of the class number of a cyclotomic field*, Bull. Amer. Math. Soc. vol. 25 (1918–19) pp. 458–461. (Received January 12, 1953.)

270*t*. Eckford Cohen: *The finite Goldbach problem in algebraic number fields.*

In a previous paper (Bull. Amer. Math. Soc. Abstract 58-4-327) the author gave a criterion for the representability of the elements of the ring $R(m)$ of residue classes modulo an integer $m > 1$, as sums of prime elements of $R(m)$. This result is extended to an arbitrary finite extension F of the rationals. Let $A \neq 1$ be an integral nonzero ideal of F and suppose that A has m distinct prime ideal factors, of which h are of norm >2 and k are of norm $=2$, $(h + k = m)$. If A has but a single prime divisor Q of norm 2 $(k = 1)$, denote by μ the maximum power to which Q divides A. The following theorem is proved: *If $R(A)$ denotes the ring of residue classes* (mod A), *then there exists an n such that all elements of $R(A)$ are expressible as a sum of n primes of $R(A)$ if and only if $m > 1$, $h > 0$. For such A, the minimum value M of n is given by $M = 2$ if $k = 0$ and $h \geq 2$; by $M = 3$ if $k = 1$ and $h \geq 2$, if $k = 1$, $h = 1$, and $\mu = 1$, or if $k = 2$ and $h \geq 1$; by $M = 4$ if $k = 1$, $h = 1$, and $\mu > 1$; and by $M = k$ if $k \geq 3$ and $h \geq 1$.* Other results of a related nature are also proved. (Received February 12, 1953.)

271*t*. Eckford Cohen: *The number of solutions of quartic congruences in two unknowns.*

Using an approach employed previously in the case of cubic congruences, the author obtains explicit formulas for the number of solutions of the quartic congruences (I) $ax^4 + by^4 \equiv n \pmod{p^\lambda}$ and (II) $ax^4 + by^2 \equiv n \pmod{p^\lambda}$, where p is a prime $\equiv 1 \pmod 4$, $(a, p) = (b, p) = 1$, and n is arbitrary. The formulas obtained for (I) and (II) are particularly simple in case $t \not\equiv 0 \pmod{p^\lambda}$, t being defined by $n = p^t\xi$, $(\xi, p) = 1$. On the basis of these results, solvability criteria for (I) and (II) are derived. For example, it is shown, *in case $p \equiv 1$* (mod 8), *$p \neq 17$, that* (I) *is insolvable if and only if $t \not\equiv 0$* (mod 4), *$t < \lambda$, and a and b have a different quartic character* (mod p), *and in case $p \equiv 5$* (mod 8), *$p > 29$, that* (I) *is insolvable if and only if $t \not\equiv 0$* (mod 4), *$t < \lambda$, and a and b either have the same quartic character or have a different quadratic character* (mod p). Formulas for the number of primitive solutions of (I) and (II) in the case of a prime modulus ($\lambda=1$) are also found. (Received February 12, 1953.)

272. V. J. Doberly: *Doberly's series and products.*

Through disintegrating the product of any number (n) of successive terms of arithmetical progression a, $a+k$, $a+2k, \cdots$, $[a+(n-1)k]$ into an algebraical series, all terms of which are proved to be expressible as $(n-2)!\cdot(a+k)\sum_{m=1}^{m=n-1} a(a+k)(a+2k)\cdots[a+(m-1)k]k^{n-m-1}/(n-1)!$, the author has found many new types of final series, all easily yielding to summation through a simple algebraical expression consisting of one single term only. Typical among the simplest of the series permitting such summation of any number of their terms are numerical successions similar to $1\cdot 2\cdot 3+2\ 3\ 4+3\cdot 4\cdot 5+\cdots$ and $3+3\ 4/1!+3\cdot 4\cdot 5/2!+3\cdot 4\cdot 5\ 6/3!+\cdots$ as well as the progression of a given definite number of terms with successively decreasing powers in one of their factors as in the series $3\cdot 2^3+3\cdot 5\cdot 2^2/1!+3\cdot 5\cdot 7\cdot 2^1/2!+3\cdot 5\cdot 7\cdot 9\cdot 2^0/3!$, where $n=4$. (Received January 9, 1953.)

273. Alfred Horn: *Doubly stochastic matrices and the diagonal of an orthogonal matrix.*

If $x_1\geqq\cdots\geqq x_n$, $y_1\geqq\cdots\geqq y_n$, and if $\sum_{i=1}^{k} x_i\leqq\sum_{i=1}^{k} y_i$, $1\leqq k\leqq n-1$, and $\sum_{i=1}^{n} x_i=\sum_{i=1}^{n} y_i$, then there exists a real orthogonal matrix $(a_{i,j})$ such that $x_i=\sum_{j=1}^{n} a_{i,j}^2 y_j$. This generalizes a well known theorem of Hardy, Littlewood, and Pólya. As an application it is shown that a vector x is the diagonal of a real orthogonal matrix with determinant $+1$ if and only if it lies in the convex hull of those points $(\pm 1, \cdots, \pm 1)$ of which an even number of coordinates are -1. It follows that a vector x can be the diagonal of a real orthogonal matrix if and only if $\sum_{j\neq i}|x_j|\leqq n-2+|x_i|$ and $|x_i|\leqq 1$, $1\leqq i\leqq n$. If $x_i\geqq 0$, these last conditions are also necessary and sufficient for the diagonal of a doubly stochastic matrix. (Received January 15, 1953.)

274t. J. R. Isbell: *Birkhoff's problem* 111.

A doubly stochastic infinite matrix is a limit of linear combinations of permutation matrices, error being measured by the least upper bound of sums in rows and columns of absolute errors, if and only if for every $\epsilon>0$ there is n such that in every row and column the sum of the largest n elements is $>1-\epsilon$. In other norms there are unsatisfactory trivialities. The fundamental tool is König's selection theorem. This is a substantially complete solution of the last problem in Birkhoff's well known list. (Received January 15, 1953.)

275t. J. R. Isbell: *On finitary games.*

The fundamental results of the theory of non-cooperative games are extended to two classes of gamelike processes. A finite Kuhn diagram a play of which may intersect one information set up to n times is fully equivalent, if players are permitted to randomize at each move, to a continuous game on a product of cells with payoff a polynomial of the nth degree; all polynomials in appropriate variables are such payoffs. A stochastic and competitive process with payoff only at absorbing states and all transition probability distributions atomic has stationary equilibrium points; if it is two-person zero-sum, the payoff in the stationary strategies is linear fractional and the sets of good strategies are convex. A game of the latter type has an essential subgame with a closed solution satisfying the dimensionality relation, and its solution is equivalent to the solution of a system of matrix games; but inessential pure strategies may figure in good strategies. (Received January 15, 1953.)

276. H. G. Jacob, Jr.: *A theorem on Kronecker products.*

By defining the notion of rank and coherence for elements of a Kronecker product of vector spaces of arbitrary dimensions the results of L. K. Hua (Sci. Rep. Nat. Tsing Hua Univ. Ser. A vol. 5 (1948) pp. 150–181) may be generalized. Let $\mathfrak{X}_i$, $\mathfrak{Y}_i$, $i=1, 2$, be right and left vector spaces respectively over the division rings Δ_i. Let $\mathfrak{X}_i \times_i \mathfrak{Y}_i$ denote the Kronecker products of these spaces and $\mathfrak{Y}_i \otimes_i \mathfrak{X}_i$ denote the products of the same spaces regarded as right and left vector spaces respectively over Γ_i, the anti-isomorphic images of Δ_i. Let $T = \sum_{j=1}^{n} x_j \times_1 y_j \in \mathfrak{X}_1 \times_1 \mathfrak{Y}_1$ and $S = \sum_{j=1}^{n} y_j \otimes_1 x_j \in \mathfrak{Y}_1 \otimes_1 \mathfrak{X}_1$. Theorem: A necessary and sufficient condition that the mapping $T \rightarrow T^\tau$ of $\mathfrak{X}_1 \times_1 \mathfrak{Y}_1$ onto $\mathfrak{X}_2 \times_2 \mathfrak{Y}_2$ be one-to-one, coherence invariant, and possesses a coherence invariant inverse is that either (1) $T^\tau = \sum_{j=1}^{n} x_jP \times_2 y_jQ + R$, where P and Q are semi-linear transformations of $\mathfrak{X}_1$ onto $\mathfrak{X}_2$ and $\mathfrak{Y}_1$ onto $\mathfrak{Y}_2$ respectively, with the same isomorphisms of Δ_1 onto Δ_2, and $R \in \mathfrak{X}_2 \times_2 \mathfrak{Y}_2$; or (2) $T^\tau = S^\tau$, where $T \rightarrow S$ is the natural mapping of $\mathfrak{X}_1 \times_1 \mathfrak{Y}_1$ onto $\mathfrak{Y}_1 \otimes_1 \mathfrak{X}_1$ and $S^\tau = \sum_{j=1}^{n} y_jQ \times_2 x_jP + R$ where Q and P are semilinear transformations of $\mathfrak{Y}_1$ onto $\mathfrak{X}_2$ and $\mathfrak{X}_1$ onto $\mathfrak{Y}_2$ respectively, with same isomorphism, and R is as described above. (Received January 15, 1953.)

277. D. J. Lewis: *Singular quartic forms.*

In the study of forms over a $\mathfrak{p}$-adic field K, it is well known that if a form has a nonsingular zero modulo $\mathfrak{p}$ in K, it has a nontrivial zero in K. Thus it is a natural problem to study forms which have only singular zeros modulo $\mathfrak{p}$ in K; or equivalently, forms over a finite field k which have only singular zeros in k. The author has previously (Ann. of Math. vol. 56 (1952) pp. 473–478) exhibited and made use of properties of such forms of degree three. In this paper it is shown that if a quartic form over $k = GF(q)$ has only singular zeros in k, then if $2 \nmid q$ and q is sufficiently large, the form must be of one of the following types: (i) $\epsilon(Q_1^2 - vQ_2^2)$, (ii) $\epsilon(L_1^2 - vL_2^2)(L_3^2 - uL_4^2)$, or (iii) $G(L_1, L_2, \cdots, L_s)$, where $\epsilon = \pm 1$, v and u are either zero or non-squares of k, Q_i and L_i are respectively quadratic and linear forms over k, and $G(x_1, x_2, \cdots, x_s)$ is a quartic form having only the trivial zero in k, hence $s \leq 4$. An essential step in the proof utilizes a generalization of a theorem of Carlitz (Duke Math. J. vol. 19 (1952) pp. 471–474). (Received January 12, 1953.)

278t. R. C. Lyndon: *Identities in finite algebras.*

A nonassociative multiplication is defined over a finite set S, with the property that there exists no finite set of identities for S from which all others are derivable. (Received January 19, 1953.)

279. A. B. Willcox: *Some structure theorems for a class of Banach algebras.*

Let R be a B-algebra with unit and whose Stone-Jacobson structure space, $S(R)$, is Hausdorff. If R is commutative this is equivalent to R being regular in the sense of Šilov [Travaux de l'Inst. Math. Stekloff, 1947]. For B-algebras with H.s.s. this paper presents some theorems on representation of closed ideals as intersections of primitive ideals or of closed primary ideals. Some of these are generalizations of theorems due to Šilov in the regular-commutative case. Also generalized from Šilov's paper are the basic theorems on separation of closed disjoint sets in $S(R)$ and on existence of minimal closed primary ideals $\overline{J(P)}$, P any primitive ideal. Using the ideals $\overline{J(P)}$ the definition of "type C" is the obvious generalization of Silov's definition, and R is of type C_0 if it is of type C and if the functions $\|x(P)\|$ are continuous on $S(R)$ for all $x \in R$. If R, of type C, has H.s.s. then R is a sub-direct sum of the primary B-algebras $R/\overline{J(P)}$, and the sum has certain continuity properties if R is of type C_0. The reverse process of constructing B-algebras of type C or C_0 from arbitrary primary B-algebras is possible under some continuity restrictions on the sums used, and this process is, in a sense, unique if the elements of the sum distinguish between the elements of the Hausdorff space of indices. (Received January 9, 1953.)

280. K. G. Wolfson: *An ideal-theoretic characterization of the ring of all linear transformations.*

Let K be an abstract ring, and call a left ideal of K a left annulet if it is the totality of left annihilators of a subset of K. Then K is isomorphic to a ring $T(F, A)$ of all linear transformations of the vector space A over the division ring F if and only if (1) K_0, the socle of K, is not a zero-ring, and is contained in every nonzero two-sided ideal of K. (2) Any left ideal of K which is annihilated on the right only by zero contains K_0. (3) The sum of two left annulets is a left annulet. (4) K contains an identity element. The method of proof depends on the existence of a Galois-type correspondence between the lattice of annulets of a ring of linear transformations, and the lattice of subspaces of the underlying vector space. A characterization of all two-sided ideals of $T(F, A)$ is also given. (Received January 13, 1953.)

ANALYSIS

281t. Joseph Andrushkiw: *A note on power series whose partial sums have only real zeros.*

Let (a_n), $n=0, 1, 2, \cdots$, $a_0=1$, be a sequence of real numbers such that $f_n(z) = 1+\sum_{k=1}^{n} a_k z^k$, $n>N$, is a polynomial whose zeros are either all positive or all negative. Since the derivative $f_n^{(n-2)}(z)$ has also only real zeros, it follows that $|a_n|^{1/n} < 2^{(1-n)/2}a_1$, $n>N$, and hence that the power series $f(z)=1+\sum_{k=1}^{\infty} a_k z^k$ is convergent for all values of z. The polynomial $f_n(z)$, $n>N$, possessing at most double zeros (Andrushkiw, Bull. Amer. Math. Soc. Abstract 58-2-151), one can apply the theorem of Grommer (J. Reine Angew. Math. vol. 144 (1914) pp. 114–165) in modified form (Andrushkiw, Bull. Amer. Math. Soc. Abstract 58-3-289) and conclude that $f(z)$ is an integral function with only positive (negative) zeros and of genus 0, except for the factor e^{cz}, $c \leqq 0$ $(c \geqq 0)$. If $f_n(z)$, $n>N$, has only real (but not necessarily of the same sign) zeros, it can be shown that $|a_n|^{1/n} < n^{-1/2}(a_1^2-2a_2)^{1/2}$, $n>N$. In a similar way it follows that $f(z)$ is an integral function with only real zeros and of genus 1, except for the factor e^{cz^2} $(c \leqq 0)$. (Received January 6, 1953.)

282. E. K. Blum: *The fundamental group of the principal component of a commutative Banach algebra.*

Let B denote an arbitrary commutative Banach algebra over the complex numbers. It is assumed that B contains a unit element, e, with norm equal to 1. If b is an element in B such that b^{-1} exists $(bb^{-1}=b^{-1}b=e)$, then b is "regular." Let G denote the set of regular elements. G is an open set, hence is a union of maximal open connected sets, its components. Let G_1 be the component which contains e. Now, the function $\exp(x) = e+\sum_{n=1}^{\infty} x^n/n!$ is defined for all x in B and has the usual properties of the classical exponential function. If $\prod_1(G_1)$ denotes the fundamental group of G_1, the main result of this paper may be stated as follows: *Theorem.* Let P be the set of all x in B such that $\exp(x)=e$. Then P is an additive group which is isomorphic to

$\prod_1(G_1)$. Two proofs are given. The first is based on Schreier's theory of the universal covering group. The second depends only on results from the theory of Banach algebras. (Received January 12, 1953.)

283. F. E. Browder: *The existence of a fundamental solution in the large for the general linear elliptic system of differential equations with analytic coefficients.*

Let D be a bounded domain in E^n, r and m positive integers, $C_0^{2m}(D)$ the family of $2m$-times continuously differentiable r-vector functions with compact support in D. The system of r differential operators of order $2m$, $K_i = \sum_{k_1 \cdots k_{2m}, j} a_{k_1 \cdots k_{2m};i,j}(x)\partial^{2m}/\partial x_{k_1} \cdots \partial x_{k_{2m}} + R_i$, is said to be linear elliptic system on D if each of the R_i is a linear differential operator on D of order less than $2m$ acting on $u_1, \cdots u_r$ and if in addition $\sum_{k_1 \cdots k_{2m}} a_{k_1 \cdots k_{2m};i,j}(x)\xi_{k_1} \cdots \xi_{k_{2m}}$ is a nonsingular $(r \times r)$-matrix for $x \in D$ and any real n-vector (ξ). By a fundamental solution matrix for such a system is meant an $(r \times r)$-matrix $e_{ij}(x, s)$ whose entries are defined for $x, s \in D$ and $x \neq s$ and such that $u_i(x) = \sum_j \int_D e_{ij}(s)\overline{K}_j(u)ds$ for $(u) \in C_0^{2m}(D)$ ($\overline{K}$ the adjoint system to K). It is shown that if the coefficients of K are analytic on D, then there always exists a fundamental solution for K on D, each of whose entries is analytic in x and in s for $x, s \in D$ and $x \neq s$. The proof uses a device of Holmgren, usually employed in showing the uniqueness of the solution of the Cauchy problem. (Received January 14, 1953.)

284t. P. L. Butzer and Waclaw Kozakiewicz: *A theorem on the generalised derivatives.*

Let the function $f(x)$ be defined in the open interval (a, b), then the "right" difference of order $s+1$ is defined inductively by the relations $\Delta_h^1 f(x) = f(x+h) - f(x)$, $\Delta_h^{s+1} f(x) = \Delta_h^s f(x+h) - \Delta_h^s f(x)$, $h > 0$. Among the results obtained is the following: If $f(x)$ is continuous in (a, b) and if there exists a sequence $\{h_n\}$ of positive numbers converging to zero and a function $s(x)$ Lebesgue integrable in every closed subinterval of (a, b) such that $\lim_{n\to\infty} s_n(x) = \lim_{n\to\infty} [\Delta_{h_n}^{l+1} f(x)]/h_n^{l+1} = 0$ almost everywhere in (a, b) and $|s_n(x)| \leqq s(x)$, $n = 1, 2, 3, \cdots$, for $a < x < x + (l+1)h_n < b$, then $f(x)$ is a polynomial of degree l in (a, b). This result contains as a particular case one due to Th. Anghelutza (Bull. Sci. Math. (2) vol. 63 (1939) pp. 239–246). A similar theorem can be stated for the "central" difference of order $s+1$ defined inductively by $\nabla_{2h}^1 f(x) = f(x+h) - f(x-h)$, $\nabla_{2h}^{s+1} f(x) = \nabla_{2h}^s f(x+h) - \nabla_{2h}^s f(x-h)$, $h > 0$. The method of proof consists of an extensive use of integral operators. (Received January 6, 1953.)

285t. W. F. Darsow: *On positive definite functions and pure states.*

For a locally compact group G with its Haar measure, let Q be the set of positive definite functions on G that can be uniformly approximated on compact sets by functions of the form $f * \tilde{f}$ where f is a continuous function on G vanishing outside some compact set and $\|f\|_2 = 1$; and let A be the uniform closure of the set of all bounded linear operators L_f on L^2 with f in L^1 where $L_f g = f * g$. It is shown that there exists a convex-linear isomorphism between the set of states of A and Q. Furthermore, a discrete, countable group is exhibited for which the constant function 1 does not belong to Q. This answers in the negative problem 5 of R. Godement (Trans. Amer. Math. Soc. vol. 63 (1948) pp. 1–84). As a consequence it follows that the irreducible, unitary representations arising from pure states in the manner of I. E. Segal (Bull. Amer. Math. Soc. vol. 53 (1947) pp. 73–88) do not in general exhaust all the (strongly continuous) irreducible, unitary representations. (Received January 6, 1953.)

286. Bernard Epstein and Anne Scheerer: *Brownian motion and the Green's function. The plane case.*

M. Kac has recently established (Proceedings of the Second Berkeley Symposium) the existence of the exterior Green's function of a domain Ω in three-dimensional space by means of the theory of Brownian motion (Wiener measure). While the method is easily extended to higher dimensions, serious difficulties arise in the two-dimensional case, due to the divergence of certain integrals. By introducing a parameter ϵ which renders the integrals convergent, and carrying out certain limiting operations, Kac's procedure can be modified so as to establish the existence of the exterior Green's function (with pole at infinity) of a bounded plane domain whose boundary satisfies certain mild conditions. The chief difficulty that has to be overcome is that the probabilistic interpretation of certain functions has to be abandoned, thus necessitating a proof by analytic methods of certain inequalities which are obvious by probability arguments in the three-dimensional case. (Received January 15, 1953.)

287*t*. Herbert Federer: *An integralgeometric theorem.*

Let H^k be the k-dimensional Hausdorff measure over Euclidean n-space and let μ be a Haar measure over the group of isometries of n-space. Suppose A and B are analytic subsets of n-space, A is (H^k, k)-rectifiable, B is countably l-rectifiable [for definitions of these terms see H. Federer, *The* (ϕ, k)*-rectifiable subsets of n-space*, Trans. Amer. Math. Soc. vol. 62 (1947) p. 126], and suppose $m = k + l - n \geq 0$. Then $A \cap f(B)$ is (H^m, m)-rectifiable for μ almost all isometries f, and $\int H^m[A \cap f(B)] d\mu f = c \cdot H^k(A) \cdot H^l(B)$ where c depends only on k, l, n, and μ. [The proofs given in the paper use a particular Haar measure μ and yield the corresponding value of c explicitly in terms of k, l, and n.] (Received January 15, 1953.)

288. Murray Gerstenhaber: *Singularities on the varieties of modules of Riemann surfaces.*

The conjectures which Teichmueller made public in 1939 concerning the problem of the modules enable one to conclude that if an algebraic variety (over the complex numbers) can be constructed, the points of which, after the deletion of a subvariety, are in one-to-one correspondence with the conformal equivalence classes of Riemann surfaces of genus g, then this variety must have singular points whenever g is greater than one. In particular, that point of the variety which corresponds to the unique hyperelliptic Riemann surface of genus g which possesses a conformal self-transformation of order $2g+1$ will be singular; the variety will in fact not be locally euclidean at that point. This singularity will be isolated if and only if $2g+1$ is prime. Until Teichmueller's conjectures are proved, these conclusions are also only conjectures, but in the case of genus two they have been verified independently by algebraic methods by Max Rosenlicht and by transcendental methods by the author. It is further indicated by the Teichmueller conjectures that, in the case of genus 2, the singular point described in the foregoing is unique. (Received January 12, 1953.)

289*t*. Seymour Ginsburg: *Incomparable order types.*

Pairs of simply ordered sets, whose symmetric difference contains just two elements and whose order types are incomparable, are studied. The principal result obtained is the following. Let p and q, $p<q$, be two fixed points in the set $A \cup \{p, q\}$. If the set $\{x \mid p < x < q, x \in A\}$ is infinite, then the order types of $A \cup \{p\}$ and $A \cup \{q\}$ are incomparable. The converse is also true, namely, if B and C have incomparable order types, and if the symmetric difference of B and C contains just two elements, p and q, where $p<q$ in $D=B\cup C$, then (1) p and q are both fixed points in D; (2) $B=A\cup\{y\}$ and $C=A\cup\{z\}$, where y is one of the points p or q, and z is the other point; and (3) the set $\{x\mid p<x<q, x\in D\}$ is infinite. (Received January 12, 1953.)

290. Felix Haas: *The Poincaré-Bendixon theorem for closed two-dimensional manifolds different from the torus.*

In the following M is a two-dimensional, closed, orientable, manifold. V is a vector field on M which satisfies a Lipschitz condition and has at most a denumerable number of singular points. C^+ is a characteristic of V. $\overline{C}$, the set of ω-limit points of C^+, is assumed to be free of singular points. Then the following theorem holds: Unless M is a torus and V is free of singular points, $\overline{C}$ is a closed curve; C^+ either equals $\overline{C}$ or spirals toward $\overline{C}$ from one side. In the proof it is first shown that a closed curve S belongs to $\overline{C}$. It is then shown that if $\overline{C}$ is nowhere dense on M, then S is a characteristic. By a previous result of the author this implies that S is a characteristic unless M is a torus and V is free of singular points. Next it is proved that under the same hypothesis $\overline{C}$ contains no points which do not belong to S. And finally, it is shown that C^+ either equals $\overline{C}$ or spirals toward $\overline{C}$ from one side. (Received January 14, 1953.)

291. Alfred Huber: *A theorem of Phragmén-Lindelöf type.*

For any real constant $k<1$ we consider solutions $u(x_1, x_2, \cdots, x_n)$ of $x_n\Delta u+ku_{x_n}=0$, defined in the half-space $H[x_n>0]$ and satisfying at the finite boundary B of H the condition $\lim\sup_{P\to Q} u(P)\leqq 0$ $(P\in H,\ Q\in B)$. Let S_r denote the half-sphere $H\cap[\sum_{i=1}^{n} x_i^2=r^2]$. The results are: (1) The limit $\alpha=\lim_{r\to\infty} m(r)/r^{1-k}$, where $m(r)=\sup_{P\in S_r} u(P)$, always exists (finite or infinite). (2) $\alpha\geqq 0$. (3) Throughout H the inequality $u\leqq\alpha x_n^{1-k}$ holds; if the equality is attained in at least one point of H, then $u\equiv\alpha x_n^{1-k}$. The theorem is an extension of the Phragmen-Lindelöf theorem for harmonic functions $(k=0)$ in the formulation of M. Heins (Trans. Amer. Math. Soc. vol. 60 (1946) pp. 238–244). The work was done under the Office of Naval Research Contract No. N7onr-39705. (Received January 14, 1953.)

292. Benjamin Lepson: *On the variations of a function of bounded variation.*

It is well known that a necessary and sufficient condition that a real-valued function of a real variable be of bounded variation over a finite interval is that it can be expressed as the difference of two nondecreasing functions. The sufficiency is obvious, while the necessity is usually proved by decomposing the function into its positive and negative variations (aside from an additive constant). This may be called the *canonical decomposition* of the function. However, it is readily seen that, given a pair of nondecreasing functions, it is not always possible to find a function of bounded variation for which they form such a decomposition. It is the purpose of this paper to find conditions on the given pair of functions so that this is possible. (Received January 15, 1953.)

293*t*. A. E. Livingston: *The Lebesgue constants for Euler* (e, t), $0<t<1$, *summation of Fourier series.*

The Lebesgue constant of order n for the Euler (e, t), $0<t<1$, means of a trigonometric Fourier series is $L_n(t)=(2/\pi)\int_0^\pi\left|\operatorname{Im}\left[e^{inx/2}(1-t+te^{ix})^n\right]\right|/(2\sin x/2)dx$. (The summability method (e, t) is ordinarily denoted by $(E, (1-t)/t)$.) The author submitted an abstract at the New Haven meeting of October 25, 1952, in which he stated that $L_n(t)=(2/\pi^2)\log n+O(1)$ for each t as $n\to\infty$. He has now completed this result to $L_n(t)=(2/\pi^2)\log(2nt/(1-t))-(2/\pi^2)C+(2/\pi)\int_0^1 x^{-1}\sin x\,dx-(2/\pi)\int_1^\infty x^{-1}\{2/\pi-|\sin x|\}dx+\epsilon_n(t)$, where C is the Euler-Mascheroni constant and $\epsilon_n(t)\to 0$ for each t as $n\to\infty$. (Received February 25, 1953.)

294*t*. A. E. Livingston: *The zeros of a certain class of indefinite integrals.*

Let $F(x)=\int_x^\infty f(t)g(t)dt$, where $f(t)\geqq 0$ on $[0, 1)$, $f(t)\neq 0$ on any $[a, b)\subset[0, 1)$, $f(t)\in L(0, 1)$, $f(t+n)=(-1)^n f(t)$, for $n=1, 2, \cdots$, $g(t)>0$ for $t\geqq 0$, $g(t)\downarrow 0$, $f(t)g(t)\in L(0, 1)$, $g(n+1)/g(n)\to 1$ as $n\to\infty$, and $g(t)$ has a point of decrease in each of an infinite set of intervals $(n, n+1)$, n integral. Let z_n denote the zero of $F(x)$ in $(n, n+1)$, $n=0, 1, \cdots$. If N is an infinite set of distinct non-negative integers and $\alpha\in[0, 1]$, then the author shows that a n.a.s.c. for $(-1)^n F(n)/g(n)\to\int_0^\alpha f(t)dt$ as $n\to\infty$, $n\in N$, is that $z_n-n\to\alpha$ as $n\to\infty$, $n\in N$. If, in addition, $g(t)-g(t+1)\in\downarrow$ and if $C\in(0, 1)$ is defined by $2\int_0^C f(t)dt=\int_0^1 f(t)dt$, then it is shown that $(-1)^n F(n)/g(n)\to\int_0^C f(t)dt$ as $n\to\infty$ and that $z_n-n\geqq C$, $n=0, 1, \cdots$. (Received February 24, 1953.)

295. Eugene Lukacs and Otto Szász: *Non-negative trigonometric polynomials and certain rational characteristic functions.*

Let $0<b_1<b_2<\cdots<b_n$ be n integers and $0<d_1<d_2<\cdots<d_m$ be m $(m\leqq n)$ real numbers (not necessarily integers). Denote by $g(\theta)$ the Vandermonde determinant formed from the $b_1^2, b_2^2, \cdots, b_n^2$ with the first row replaced by $1-\lambda_j\cos b_j\theta$ where $\lambda_j=\prod_{i=1}^m(1-b_j^2/d_i^2)$ for $j=1, 2, \cdots, n$. The question whether $g(\theta)$ is non-negative for all values of θ is closely connected with the problem whether certain rational functions are characteristic functions. Four configurations of the $b_1, \cdots, b_n$ and $d_1, \cdots, d_m$ are are studied which lead to non-negative trigonometric polynomials. (Received January 8, 1953.)

296. M. H. Protter: *A boundary value problem for the wave equation.*

Let x_0, y_0, z_0 be fixed quantities such that $x_0^2+y_0^2<z_0^2$ and suppose D is the domain in (x, y, z) space bounded by the surfaces S_1: $x^2+y^2=(z-z_0)^2$, S_2: $(x-x_0)^2+(y-y_0)^2=z^2$, and S_3: $z=0$. Let $f(x, y)$, $g(x, y)$ be given functions having continuous first derivatives. A solution to the wave equation $u_{xx}+u_{yy}=u_{zz}$ is obtained in D satisfying the boundary conditions $u=f(x, y)$ on S_3, $u=g(x, y)$ on S_2. The proof uses the generalization of the Riemann method due to Martin as well as Asgeirsson's mean value theorem. The result can be extended to the wave equation in any number of variables. As a corollary certain theorems on mean values of a function defined in a circle may be obtained. (Received October 22, 1952.)

297. I. F. Ritter: *A parametric representation of the eigenvectors of a matrix.*

The eigenvalue problem for a matrix A of order n with numerical elements can be formulated by the identity (1) $(pA-I)\mathfrak{y}(p)=f(p)\mathfrak{c}$ in the parameter p, where $f(\lambda^{-1})\lambda^n$ is the characteristic function of A and $\mathfrak{c}$ a numerical vector. If $f(p)$ and a vector $\mathfrak{y}(p)$, satisfying (1), are obtained in polynomial form, each root p_i of $f=0$ furnishes an eigenvector $\mathfrak{y}(p_i)$ of A. A practical method, which remains feasible with increasing n, is described for computing, before f is found by (1), a solution $\mathfrak{y}(p)$ in a form which differs significantly from the columns of adj $(pA-I)$. Beginning with a vector $\vec{r}^1$ with n arbitrary parameters as components, a sequence $\vec{r}^j(p)$ is set up by eliminating from $pA\vec{r}^{j-1}(p)$ one of the parameters to form $\vec{r}^j(p)$. For nonderogatory A, with trivial exceptions, $\vec{r}^n(p)$, containing one arbitrary parameter, can be reached and taken as solution $\vec{y}(p)$. If A is derogatory, the sequence cannot be continued beyond $\vec{s}=\vec{r}^k(p)$, $k<n$. Then one component of the vector $\vec{t}(p)=(pA-I)\vec{s}(p)$ represents the reduced characteristic function $g(\lambda^{-1})\lambda^k$ of A. After solving $g=0$ (much easier to solve than $f=0$) for the eigenvalues p_i of A, either all (except one) of the parameters in $\vec{s}(p_i)$ can be determined from $p_iA\vec{s}(p_i)=\vec{s}(p_i)$, or enough of them remain arbitrary so that $\vec{s}(p_i)$ represents the invariant subspace belonging to the eigenvalue p_i. (Received January 14, 1953.)

298*t*. M. S. Robertson: *Schlicht solutions of the Sturm-Liouville differential equation*. Preliminary report.

Let $zp(z)$ be regular for $|z|<1$ and $\Re\{z^2p(z)\}\leq(x_1^2/4)|z|$ for $|z|<1$, where X_1 is the smallest positive zero of the Bessel function $J_0(x)$. Then the unique solution $W=W(z)$ of the differential equation $W''+p(z)W=0$ for which $W(0)=0$, $W'(0)=1$, is schlicht and star-like in $|z|<1$. The constant $x_1^2/4=1.4460\cdots$ is a best possible one. A similar theorem holds if $\Re\{z^2p(z)\}\leq(\pi^2/4)|z|^2$ in $|z|<1$, and again the constant $(\pi^2/4)$ is a best possible one. If $z^2p(z)$ is regular in $|z|<1$, and if $\Re\{z^2p(z)\}\leq K$, where $K=S_1^2/2+1/8$ and S_1 is the smallest positive zero $S=S_1$ of the function $\phi(S)=\int_0^\infty f(u)\cos Su\,du$, $f(u)=(3+\cosh u)^{-1/2}$, then the unique solution of the form $W=W(z)=z^\alpha\sum_{n=0}^\infty a_nz^n$, $a_0=1$, of $W''+p(z)W=0$, corresponding to the root α of the indicial equation whose real part is the larger, is such that $\{W(z)\}^{1/\alpha}=z+\cdots$ is schlicht for $|z|<1$. The constant K cannot be replaced by a larger one. These theorems are special cases of a more general parent theorem which will be announced later on completion of the present investigation. (Received January 12, 1953.)

299*t*. L. B. Robinson: *On a complete system of tensors.*

Given the system of equations $-y_i\partial f/\partial y_i-y_i'\,\partial f/\partial y_i'-y_i''\,\partial f/\partial y_i''-I_i\partial f/\partial I_i=0$, $i\neq j$; i, $j=1, 2, 3$, $-y_1\partial f/\partial y_1-y_1'\,\partial f/\partial y_1'-y_1''\,\partial f/\partial y_1''+I_2\partial f/\partial I_2+I_3\partial f/\partial I_3+I_4\partial f/\partial I_4=0$, $-y_2\partial f/\partial y_2-y_2'\,\partial f/y_2'-y_2''\,\partial f/\partial y_2''+I_1\partial f/\partial I_1+I_3\partial f/\partial I_3+I_4\partial f/\partial I_4=0$, $-y_3\partial f/\partial y_3-y_3'\,\partial f/\partial y_3'-y_3''\,\partial f/\partial y_3''+I_1\partial f/\partial I_1+I_2\partial f/\partial I_2+I_4\partial f/\partial I_4=0$, $-\Theta\partial f/\partial\Theta-y_1'\,\partial f/\partial y_1'-y_2'\,\partial f/\partial y_2'-y_3'\,\partial f/\partial y_3'+I_1\partial f/\partial I_1+I_2\partial f/\partial I_2+I_3\partial f/\partial I_3=0$. Solving these equations helps one to compute a complete system of tensors. (Received January 5, 1953.)

300. Maxwell Rosenlicht: *Simple differentials of second kind on Hodge manifolds.*

It is proved that on a Hodge manifold (in particular, on a nonsingular algebraic variety), the number of simple differentials of second kind that are linearly independent modulo exact differentials is precisely the first Betti number. The theory of abelian varieties is used to effect a simple reduction of the proof of the general theorem to the case of an algebraic curve. (Received January 12, 1953.)

301. V. L. Shapiro: *Circular summability C of double trigonometric series.*

$F(x, y)$, defined in a neighborhood of (x_0, y_0) and integrable on the circumference of every circle contained in this neighborhood with (x_0, y_0) as center, is said to have a generalized rth Laplacian at (x_0, y_0) equal to a_r if $(1/2\pi)\int_0^{2\pi}F(x_0+t\cos\theta, y_0+t\sin\theta)d\theta$

$=a_0+a_1t^2/[2!]^2+\cdots+a_rt^{2r}/[2^rr!]^2+o(t^{2r})$. By means of this definition a necessary and sufficient condition that there exist a Cesaro mean which sums a given series is obtained. In particular the following theorem is proved: Let $T=\sum c_{mn}e^{i(mx+ny)}$ be a double trigonometric series with coefficients $c_{mn}=o((m^2+n^2)^\gamma)$, $\gamma\geqq-1$. A necessary and sufficient condition that the series should be circularly summable C at the point (x_0, y_0) to the sum s is that there exists an integer $r>\gamma+1$ such that if $F(x, y)=c_{00}(x+y)^{2r}/2^r[(2r)!]+\sum_{|M|\neq 0}(-1)^rc_{mn}e^{i(mx+ny)}/(m^2+n^2)^r$, then the generalized rth Laplacian of F exists at the point (x_0, y_0) and is equal to s. (Received January 2, 1953.)

302*t*. J. M. Stark: *Distortion of analytic angle in pseudo-conformal transformation.*

Using Bergman's method of the minimum integral (Mémorial des Sciences Mathématiques, vol. 106, p. 40), the author gives bounds for the distortion of the analytic angle A (Ibid., p. 9) between two directions at a point $P\in B$ under pseudo-conformal transformations of four-dimensional domain B. Such bounds result from bounds for the ratios between the euclidean $E(A)$ and non-euclidean $N(A; B)$ measures of A. $N(A; B)$ is obtained using Bergman's metric defined in B and invariant with respect to pseudo-conformal transformations (Ibid., p. 51). $E(A)/N(A; B)$ is shown to depend upon euclidean measures of B obtained using different weighting functions. E.g., $c(m/M)^{16}\leqq\sin E(A)/\sin N(A; B)\leqq d(M/m)^{16}$ where m is the distance of P from the boundary b of B, M is the maximum distance of a point (Z_1, Z_2) from b, c and d are simple functions of $|Z_1|/M$, $|Z_2|/M$, W/M^4, W/m^4 where W is a constant $\geqq$ euclidean volume of B. The author obtains other bounds for distortion using the first few functions of an orthonormal set in B and various euclidean measures of B with conveniently chosen weighting functions. (Received February 2, 1953.)

303*t*. J. M. Stark: *Distortion theorems for pseudo-conformal transformations.*

The author derives relations connecting the minima λ_B of the integral $\int_B|f|^2d\omega$, $d\omega=dx_1dy_1dx_2dy_2$, $z_k=x_k+iy_k$, $k=1, 2$, for functions $f=f(z_1, z_2)\in L^2(B)$ and satisfying at a point $t=(t_1, t_2)\in B$ certain auxiliary conditions. E.g., if (a) $f(t)=1$, (b) $f(t)=0$, (c) $(\partial f/\partial z_1)_t=1$, (d) $\int_Bfd\omega=0$ are auxiliary conditions, let λ_B^1, λ_B^{*1}, λ_B^{01}, λ_{BB}^{01} be the minima of $\int_B|f|^2d\omega$ under conditions $\{(a)\}$, $\{(c)\}$, $\{(b)\ (c)\}$, $\{(b)\ (c)\ (d)\}$, respectively. Then $(1/\lambda_B^{01})=(1/\lambda_{BB}^{01})+(\lambda_B^1/\mathrm{Vol}\ B)\{1/\lambda_B^{*1}-1/\lambda_{BB}^{01}\}$ where Vol B is the euclidean volume of B. Using Bergman's method of the minimum integral (Mémorial des Sciences Mathématiques, 106, p. 40 and vol. 108, p. 48) and assuming known some quantities characteristic of B, such as an upper bound for Vol B, euclidean measures of B with certain weighting functions, etc., inequalities are obtained for various λ_B which are then used in distortion theorems. Bounds for distortion of arc length in an arbitrary direction as well as bounds which depend to some extent upon the direction are derived. (Received February 2, 1953.)

304. J. L. Walsh and D. M. Young, Jr.: *Degree of convergence of solution of difference equation to solution of Dirichlet problem.*

For the unit square with square grid of mesh $1/A$, in any closed interior region the convergence of the solution of the difference equation to the solution of the Dirichlet problem is of the respective orders $1/A^2$, $1/A$, $1/A^\alpha$, provided the boundary values either have a derivative of bounded variation, or are themselves of bounded variation, or satisfy a Lipschitz condition of order α, $0<\alpha\leqq 1$. For boundary values piecewise constant the convergence is of the order $1/A^2$ or $1/A$ according as all discontinuities are or are not points of the grid. If only continuity is assumed for the boundary values, the order of the convergence may be arbitrarily slow with respect to $1/A$. (Received January 12, 1953.)

305*t*. John Wermer: *On a class of bounded linear operators.* Preliminary report.

Let B be a Banach space, S a bounded operator on B with inverse, D that component of the resolvent set of S which contains the origin. Choose x in B, set C equal to the smallest closed subspace of B invariant under S and containing x, and suppose $S^{-1}x$ not in C. Let T denote the restriction of S to C. *Theorem*: *For each f in C there is a unique function $F(z)$ analytic in D such that for $n=0, 1, \cdots : f=F(0)x+F'(0)Tx+(F''(0)/2!)T^2x+\cdots+(F^{(n)}(0)/n!)T^nx+T^{n+1}y_n$, y_n in C.* Let R denote the ring of operators U, defined on C, with $UT=TU$. *Theorem*: *If for no $y\neq 0$ in C, $T^{-n}y$ is in C for all n, then R is isomorphic to a ring of functions analytic on D. In particular, if E is in R, $E^2=E$, then $E=I$ or $E=0$.* Certain special operators are discussed in detail as examples of the two theorems. (Received January 15, 1953.)

306. John Wermer: *On algebras of continuous functions.*

Let C denote the algebra of functions continuous on the unit circle. With the norm: $\|f\|=\sup_{|\lambda|=1}|f(\lambda)|$, C is a Banach algebra. Let A denote the set of all f in C which are boundary values of functions analytic in $|z|<1$ and continuous in $|z|\leqq 1$. A is then a closed subalgebra of C, and by known results A consists precisely of those f in C for which $\int_{|\lambda|=1}f(\lambda)\lambda^n d\lambda=0$, $n\geqq 0$. The following theorem is proved: If A' is any closed subalgebra of C which includes A, then either $A'=A$ or $A'=C$. (Received January 9, 1953.)

307. Albert Wilansky: *Row-finite and row-infinite summability matrices.*

In Proc. Amer. Math. Soc. vol. 3 (1952) p. 389, the author proved that a reversible matrix allowed an equipotent consistent row-infinite matrix, thus in particular, a stronger, consistent row-infinite matrix, and conjectured that the latter is true with normal ($=$reversible-triangular) instead of row-infinite. The conjecture is correct for A if (1) there exists R_n with $\lim_n \sum_{k>R_n} a_{nk}x_k=0$ for all x summable A, (2) $\sum_k |a_{nk}/t_k|<\infty$ for each n, where t_n (known to exist, loc. cit. p. 390) satisfies $\sup|t_nx_n|<\infty$ for each x summable A. *Theorem*: There is a normal matrix consistent and equipotent with any reversible row-finite matrix. The theorem is trivial if "triangular" is substituted for "normal." (Received January 7, 1953.)

APPLIED MATHEMATICS

308. Wallace Givens: *Numerical computation of the characteristic values of a real symmetric matrix.*

The problem considered is that of finding the characteristic values of a real symmetric matrix $\bar{A}$ by a method suitable for use with a high speed automatic sequenced digital computer. Let $\bar{A}$ have digital elements and norm slightly less than one; $\bar{A}$ is not required to be definite nor need its roots be separated. By a sequence of $(n-1)(n-2)/2$ rotations in the coordinate planes x_2x_3, x_2x_4, $\cdots$, x_2x_n, x_3x_4, $\cdots$,

$x_3x_4, \cdots, x_{n-1}x_n$, the quadratic form $x'Ax$ is carried into $x'\bar{S}x$ with $\bar{S}=(\bar{s}_{ij})$ and $\bar{s}_{ij}=0$ for $|i-j|>1$. After this first stage of "concentrating the data," the roots are found by determining the signature of $\bar{S}-\bar{\mu}I_n$ for suitable values of $\bar{\mu}$. This general mathematical procedure is described in a paper to appear in the Proceedings of an NBS Symposium held in Los Angeles, August 1951 [NBS Appl. Math. Ser., vol. 29]. The present paper is a detailed error analysis, using methods stemming from those of von Neumann and Goldstine [Bull. Amer. Math. Soc. vol. 53 (1947) pp. 1021–1099], which proves the stability of the computation and establishes satisfactory bounds for the error (produced by round-off) in any root. For the second stage the bound is particularly small and is independent of the order of the matrix. (Received January 14, 1953.)

309. Imanuel Marx: *Recurrence relations for prolate spheroidal wave functions.*

Solution of the three-dimensional wave equation by separation in prolate spheroidal coordinates ξ, η, ϕ leads to the angular wave functions $S_{mn}(\eta)$ and the radial wave functions $R_{mn}(\xi)$, defined over the ranges $-1\leqq\eta\leqq 1$ and $1\leqq\xi<\infty$, respectively. Only functions of the first kind, which have no logarithmic terms, are considered. The functions $\psi_{mn}=S_{mn}R_{mn}\cos m\phi$ and $\omega_{mn}=S_{mn}R_{mn}\sin m\phi$ solve the wave equation, and an arbitrary regular solution may be expanded in a termwise integrable double series of these functions. The derivatives $\partial\psi_{mn}/\partial x$, $\partial\psi_{mn}/\partial y$, $\partial\psi_{mn}/\partial z$ are likewise solutions. The trigonometric as well as the wave functions have integral orthogonality over their respective intervals. Multiplication of $\partial\psi_{mn}/\partial z$ by $\cos p\phi$ and integration with respect to ϕ, followed by multiplication by S_{pq} and integration with respect to η, gives a linear relation among R_{mn}, R'_{mn}, and R_{mq}, $q\neq n$. Multiplication by R_{pq} and integration with respect to ξ similarly gives a linear relation among S_{mn}, S'_{mn}, and S_{mq}. The same method applied to $\partial\psi_{mn}/\partial x$ and $\partial\psi_{mn}/\partial y$ gives relations among R_{mn}, R'_{mn}, and $R_{m\pm 1,q}$, or among S_{mn}, S'_{mn}, and $S_{m\pm 1,q}$. (Received January 12, 1953.)

310*t*. L. E. Payne and H. F. Weinberger: *Upper and lower bounds for polarization.*

The following problem is considered. $\Delta\phi=0$ in the exterior of a closed surface S. ϕ assumes the boundary values x_i (a rectangular coordinate) on S and vanishes at infinity. The Dirichlet integral of ϕ is called the polarization P of S. Certain theorems concerning the value of P_{ii} are proved. Among these are: (a) $P_{11}+P_{22}+P_{33}\geqq 6V_s$, where V_s is the volume of S. This inequality was conjectured by Schiffer and Szegö (Trans. Amer. Math. Soc. vol. 67 (1949)) and becomes an equality when S is a sphere. (b) If S is cut in two by a plane perpendicular to the x_i-axis and the two half-bodies so resulting are reflected in the plane to form two symmetric bodies A and B, then $P_{ii}(S)\leqq[P_{ii}(A)+P_{ii}(B)]/2$ and $P_{jj}(S)\geqq[P_{jj}(A)+P_{jj}(B)]/2$ for $j\neq i$. (c) If S is symmetric about a plane perpendicular to the x_i-axis, then $P_{ii}\geqq(4\pi)^{2/5}(15I_{ii})^{3/5}-V_s$ where I_{ii} is the moment of inertia of S about the plane of symmetry. (d) If S is symmetric about a plane perpendicular to the x_i-axis, $P_{ii}\leqq\left[\int_0^\infty dv/F(v)\right]^{-1}-V_s$ where $F(v)=M+M_1v+M_2v^2+M_3v^3+(4\pi/3)v^4$ and M_i are the analogues in 5 dimensions of the Minkowski number of the convex hull of S. This work was sponsored by the Office of Naval Research. (Received January 16, 1953.)

311. E. H. Zarantonello: *Singularities on free boundaries.*

The free boundaries of a plane, steady irrotational flow of an ideal fluid may carry isolated singularities. Well known are the singularities generating jets, wakes, and planning surfaces (cf. Milne-Thomson, *Theoretical hydrodynamics*, Chap. 12). An investigation of all possible singularities, under the only assumptions of a bounded velocity and a locally schlicht flow, leads essentially to these three types only. For this purpose the flow is locally mapped onto a domain of a parameter plane t with the free boundary corresponding to a segment of the real axis and the singularity to $t=0$. Then, at the origin, the complex potential w and the complex velocity ζ can have only the following expansions: $w=at^{-2}+bt^{-1}+c\ln t+R_1(t)$, $\zeta(t)=t^{im}\exp\{iR_2(t)\}$, where $R_1(t)$ and $R_2(t)$ are power series of t, real on the real axis. $m=0$ corresponds to the above-mentioned singularities and $m\neq 0$ yields four new types in which one free streamline curls around another of different velocity. These results are obtained by a combination of Koebe's distortion theorem and a result due to Stone characterizing poles from the behavior of the functions with regard to the distance to a straight line through the pole (M. H. Stone, *On a theorem of Pólya*, J. Indian Math. Soc. N.S. vol. 12 (1948) pp. 1–7). This work has been sponsored by the Office of Naval Research. (Received January 15, 1953.)

GEOMETRY

312. L. W. Green: *Surfaces without conjugate points.*

Let M be a two-dimensional simply-connected manifold, complete under a Riemannian metric of class C^3. Assume that the Gaussian curvature of M is bounded below. A pole of M is a point Q such that any two complete geodesic rays with initial point Q intersect only at Q. It is proved that, if P is interior to an open set of poles of M, then any two geodesic rays with initial point P must diverge. In particular, this holds for any intersecting geodesic rays if every point of M is a pole, i.e., if no geodesic of M has a pair of mutually conjugate points. This implies the existence of topological transitivity of the geodesic flow on a class of surfaces for which M is the universal covering surface, including those treated by Morse and Hedlund [Trans. Amer. Math. Soc. vol. 51 (1942) pp. 362–386]. If, in addition, it is assumed that there are no points focal to any geodesic of M, it is shown that the Gaussian curvature between two "parallel" geodesics must vanish. This implies that a surface homeomorphic to a cylinder which has bounded opening and no focal points must have zero curvature. (Received January 15, 1953.)

LOGIC AND FOUNDATIONS

313t. E. L. Post: *Solvability, definability, provability; history of an error.*

The concept general recursive function leads to a thesis (Kleene, Trans. Amer. Math. Soc. (1943) p. 60) having as a consequence the existence of absolutely unsolvable problems. Also in 1943, the author (Amer. J. Math., footnote 6) proposed the problem of absolutely undecidable propositions. The author later realized that a necessary condition for provability would suffice; still later, that to solvability and provability should be added definability. In 1947, in a lost letter to Tarski via Church, the author proposed a formulation of "Primitive Inductive-Reflective Proof" involving, besides the elementary, only mathematical induction and "Godelization." But a representation theory for Gödel's system P (1931) failed to materialize due to P's Axiom of Reducibility. Indeed, Kleene's Example (Proceedings of the International Congress of Mathematicians, 1950, vol. II, p. 683) offers hope of an impossibility proof. The obvious specificity of Mostowski's "definable set" concept (Fund. Math. (1947)) suggested a like development for the language (set of W. F. F.) of system P, and, constructivity being lost, further led to that generalization which constitutes the language of the P^{α} of the author's present research (see following abstract). Just prior to the last idea it was seen that the correct order of the earlier triplet is, of course, solvability, definability, provability. (Received February 9, 1953.)

314t. E. L. Post: *A necessary condition for definability for transfinite von Neumann-Gödel set theory sets, with an application to the problem of the existence of a definable well-ordering of the continuum.* Preliminary report.

In $P^{(\alpha)}$ (see previous abstract), α is an arbitrary ordinal. The language (sic) of $P^{(\alpha)}$ is that of the simple theory of types semantically treated. An "α-formula" is defined as an $x_i^{(\alpha)}$ $(x_j^{(\alpha)})$ range, all sets of type less than α, and any formula obtainable from α-formulas by a negation, disjunction or quantification. The present thesis is that every definable set is given by some α-formula. The well-ordered series of ordinals leads to a well-ordering A of the continuum—represented by a set of positive integers in classic manner. A Gödelian interval $0 \leq \xi < 1$, ξ sequence of definitions proves that A is an "α-set" and, subject to the filling in of five validation lacunae and checking, by a calculus of validation, the other formulas, leads to the theorem (the corresponding metatheorem is obvious): either (the defined) A well-orders the continuum-interval $0 \leq \xi < 1$ or there is no definable well-ordering of the continuum. Further verifications of the thesis separate themselves into theorems: A. That certain (a) generalizations, (b) modifications, of $P^{(\alpha)}$ lead only to α-sets. B. That a certain diagonal set of a denumerable set under certain (a) general, (b) particular conditions is again an α-set. (Received February 9, 1953.)

STATISTICS AND PROBABILITY

315. A. G. Anderson: *The prediction of quantitative characteristics in polygenic systems.* I.

A method is devised whereby the expected mean value of a given quantitative characteristic of a hybridized strain descended in any manner from members of some original set of inbred strains can be predicted. This is done by showing the system having as elements the representations of various stocks in terms of their proportions of gene-pairs involving two allelic genes which contribute equally and additively to the quantitative characteristic under consideration to be isomorphic to a subset of the normalized elements having non-negative coefficients in a nonassociative baric algebra of order three. The expected proportions of loci of the three possible types are thus obtainable as functions of the proportions in the parent stocks, and it is shown that the proportions of loci carrying pairs of the effective gene can be estimated for the members of a set of inbred stocks if the values of the quantitative characteristic are known for the stocks themselves and the F_1 hybrids resulting from crosses among them. The crossing of related stocks is discussed and it is made clear that such crosses do not lead to the most desirable values of the quantitative characteristic in question. (Received January 16, 1953.)

316t. K. L. Chung: *Note on unimodal distribution.*

A distribution function $F(x)$ is called unimodal with vertex s if $F(x)$ is convex for $x < s$, and concave for $x > s$. A. I. Lapin claimed in his thesis (1947, in Russian and unavailable in the U.S.) that the convolution of two distribution functions both unimodal with vertex 0 is again so. Using this theorem Gnedenko (Ukrain. Mat. Zurnal vol. 1 (1949) pp. 3–8) proved that every d.f. belonging to the class L is unimodal. Lapin's theorem (and proof) is trivially incorrect if one considers the uniform distribution in the interval $(-1/3, 2/3)$. Even without specifying the vertex of the convolution the theorem is still incorrect. Example. $F(x)=0$ if $x<-c$, $=1/2+x/2c$ if $-c\leqq x<0$, $=1/2+x$ if $0\leqq x<1/2$, $=1$ if $1/2\leqq x$. For any $c<1/4$ the convolution of F with itself is not unimodal, in fact its derivative has two relative maxima. Thus Gnedenko's theorem remains unproven. (Received January 28, 1953.)

317. R. E. Kalaba: *A random walk interpretation for the Poisson, Pólya, and Neyman distributions.*

A generalization of the well known Poisson random walk process is considered, in that steps of arbitrary integral length and direction along a line at each instant are permitted. The probability of being at a position n steps from the initial point after time t is obtained, as are certain general properties of the distribution. It is shown that the "contagious" distributions of Pólya and Neyman (as well as the Poisson and other distributions) are obtained as special cases, thus bringing out sharply points of difference and similarity among these distributions; by way of illustration, an application to fluctuation phenomena of high energy particles is pointed out. The point of view taken in the paper enables one to generalize results contained in the less well known papers of Lüders, Cernuschi, and Castagnetto, and to correlate them with Neyman's, Pólya's, and especially Feller's papers on contagion. (Received January 7, 1953.)

318t. G. Kallianpur and Herbert Robbins: *On the equidistribution in probability of sums of independent random variables.*

Let $X_1, X_2, \cdots$ be independent random variables with a common distribution function $F(x)$, let $S_n=X_1+\cdots+X_n$, and let $U_n(h_1, h_2)=\sum_{j=1}^{n} h_1(S_j)/\sum_{j=1}^{n} h_2(S_j)$. Under mild restrictions on $F(x)$ (for example, if (i) $F'(x)$ exists and belongs to L^p $(-\infty, \infty)$ for some $1<p\leqq 2$, and (ii) either the mean of $F(x)$ is 0 and the second moment finite, or $\Pr[n^{-1/\alpha}S_n\leqq x]\rightarrow V_\alpha(x)=$symmetric stable law with characteristic exponent $1\leqq\alpha<2$) it is shown that the sequence $\{S_n\}$ is *equidistributed* in the sense that if each $h_i(x)$ is bounded and summable on $(-\infty, \infty)$ with integral $\bar{h}_1$, $\bar{h}_2\neq 0$, then $\lim_{n\rightarrow\infty} U_n=\bar{h}_1/\bar{h}_2$ in probability. The same result holds if the $h_i(x)$ are Riemann integrable in some finite interval, vanishing elsewhere, even when $F(x)$ is not absolutely continuous, provided it is not a lattice distribution and has mean 0 and finite third absolute moment. For a large class of $F(x)$ belonging to the domain of normal attraction of the symmetric Cauchy law this property is preserved under translation. On the other hand, if the mean of $F(x)$ is $\neq 0$ then $\{S_n\}$ is not equidistributed. As an incidental result the limiting distribution of sums $\sum_1^n h(S_j)$, properly normalized, is found in the case of equidistribution, generalizing results of Chung and Kac (*Remarks on fluctuations of sums of independent random variables*, Memoirs of the American Mathematical Society, no. 6, 1951). (Received January 23, 1953.)

TOPOLOGY

319. R. D. Anderson: *Some monotone (and monotone open) images of R^3.*

Let R^p denote the p-dimensional cube. The author shows that for any positive integer p, there exists a monotone mapping of R^3 onto R^p. With the previously announced result of the author to the effect that if there is a monotone mapping g of a compact n-manifold M (with or without boundary) ($n \geqq 3$), then there is a monotone open mapping of M onto $g(M)$, this establishes the theorem that for any positive integer p, there is a monotone open mapping of R^3 onto R^p. (Received January 13, 1953.)

320. S. I. Goldberg: *On the "Euler characteristic" of a semi-simple Lie algebra.*

Let L be a semi-simple Lie algebra of dimension n over a field of characteristic 0 and denote the qth Betti number of L by R^q. By definition $\sum_{q=0}^{n}(-1)^q R^q$ is the Euler characteristic $\chi(L)$ of L. From transcendental properties of compact Lie groups (Hopf's theorem and Weyl's unitary trick) it has been shown that $\chi(L)=0$ (C. Chevalley and S. Eilenberg, Trans. Amer. Math. Soc. vol. 63 (1948) pp. 85–124). In this paper an elementary algebraic proof of this result is given. (Received January 16, 1953.)

321*t*. V. L. Klee, Jr.: *Some spaces which lack the fixed-point property.*

This note complements some well known fixed-point theorems by showing that certain classes of metric spaces do *not* have the fixed-point property. Specifically, if X is a noncompact metric space which is either (a) connected, locally connected, and locally compact, or (b) a convex subset of a normed linear space, then X contains a topological ray T as a closed subset. But then, since X must admit a retraction onto T, one obtains (1) A locally compact connected ANR X is compact if and only if every null-homotopic map of X into X admits at least one fixed-point; (2) A convex subset of a normed linear space is compact if and only if it has the fixed-point property. (Received January 8, 1953.)

322*t*. E. A. Michael: *On a problem of S. Kaplan.*

An abelian topological group is called a P-group if the natural homomorphism from G into its second character group is a homeomorphism onto. S. Kaplan asked [Duke Math. J. vol. 17 (1950) pp. 419–435] whether every inverse limit of P-groups is a P-group. The answer is "no." Since M. Smith has proved [Ann. of Math. vol. 53 (1952) pp. 248–253] that every Banach space is a P-group, and the author has proved [Memoirs of the American Mathematical Society, no. 11, 1952] that every complete locally convex topological linear space is an inverse limit of Banach spaces, it is sufficient to give an example of a complete locally convex space which is not a P-group. Such an example is provided by the space $C(I)$ of continuous, real-valued functions on the closed unit interval I, in the topology of uniform convergence on compact-and-countable subsets of I. The completeness of $C(I)$ was established in the above paper of the author; that $C(I)$ is not a P-group follows from a criterion in the above paper of M. Smith, and the fact that I, considered as a set of continuous linear functionals on $C(I)$, is compact in the compact-open topology but not equicontinuous. (Received January 16, 1953.)

L. W. COHEN,
Associate Secretary

BOOK REVIEWS

Intégration (Chap. I–IV). By N. Bourbaki. (Actualités Scientifiques et Industrielles, no. 1175.) Paris, Hermann, 1952. 2+237+5 pp. 2500 fr.

The volume under review contains the first four chapters of the sixth book of the first part of the series that the authors started publishing in 1939. One difficulty in the way of a complete critical evaluation of the volume is that at several points the proofs are made to depend on the results of the fifth book (entitled, according to the advertisements, *Espaces vectoriels topologiques*), and that, at the time this is being written, none of the chapters of the fifth book has appeared yet. Another such chronological difficulty is that the volume is only a half (or less) of the entire projected book; even the world's greatest art expert might hestitate to pronounce judgment on Picasso's latest if he were shown only the bottom half of it. Laboring under these two handicaps, I shall do the best I can to give the reader of this review an idea of what he may expect from Bourbaki's treatment of integration.

Mathematics books may be roughly classified as either textbooks or monographs. There is, of course, no clear line between the two classes, but certain tendencies are generally visible. The author of a textbook is mindful of the troubles of beginners; he tries to provide a systematic introduction to his subject; he is prepared to include a liberal supply of exercises; and, faced with an upper bound on the number of pages in his book, he is willing to barter completeness for clarity. The author of a monograph, on the other hand, addresses himself to the expert; he starts where his predecessors left off; he is not bound to mention even the important applications of the theory; and, faced with an upper bound on the number of pages in his book, he will prefer condensed proofs to the charge of bibliographic highhandedness. The famous *Ergebnisse* series, for example, consists of monographs; the volume under review is a textbook.

The expert in modern integration theory will certainly not find anything to startle him here, except possibly the order of the ideas and some of the terminology used to describe them. Hölder's and Minkowski's inequalities, vector lattices, measures on compact and locally compact spaces, vector-valued integrals, product spaces, outer measure, functions defined almost everywhere, the L^p spaces, approximation by step functions, measurable functions and measurable sets—this is a representative selection of the items listed in the table of contents (in that order), and certainly they are all of the sort one expects to find in a book on integration. No previous knowledge of integration or measure theory is assumed. The important and classical instances of the theory (Lebesgue measure in Euclidean space and measures in discrete spaces) receive some special attention. As for the bibliography, it contains exactly three items: an article by Hölder in the *Göttinger Nachrichten* (1889), the second edition of Minkowski's *Geometrie der Zahlen* (1910), and *Inequalities* by Hardy, Littlewood, and Pólya (1934).

In the universal preface (*Mode d'emploi de ce traité*), a copy of which is included with every volume of the series, the authors make it plain that they are writing for beginners and that the series as a whole is, in principle, self-contained. They promise to provide exercises, and they propose to give only those references to the literature that are likely to be of profit to the reader, omitting those that serve only to establish priority.[1]

There is no point in belaboring the issue; we are dealing with a publication that by intention and *de facto* is a textbook. It must be judged as a textbook. Putting ourselves in the place of a student, we must ask: "Is the subject important, is the book clearly written, and is the material well organized?" Putting ourselves in the place of the supervisor of a Ph.D. thesis in one of the applications of integration (e.g., ergodic theory, probability theory, length and area, Boolean algebra, or integral geometry), we must ask: "Is this the point of view that will help a student to understand and to extend his field of interest?" I say that the answer to the student's question is yes and the answer to the professor's question is no.

It is, to be sure, a very special kind of student who can read this book. The prospective reader has to know substantial portions of

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[1] In passing, one might ask whether the listing of Hölder and Minkowski is consistent with this announced policy. There are three other points in the preface that cannot be said to be in perfect agreement with subsequent developments. (1) "Les références seront groupées dans un exposé historique, placé le plus souvent à la fin de chaque chapitre et où l'on trouvera, le cas échéant, des indications sur les problèmes non resolus de la théorie." Regrettably, there is only one historical note in this volume and that one discusses nothing but the inequalities of Hölder and Minkowski; there are no indications of unsolved problems. (2) "On s'est efforcé de ne jamais s'écarter de la terminologie reçue sans de très sérieuses raisons." The authors' term for a *vector lattice* is *Riesz space*. What Hausdorff called a *field* (of sets), and what I more recently called a (Boolean) *ring* (of sets) is called a *clan*. (3) " . . . les nécessités de la démonstration exigent que les chapitres, les livres et les parties se suivent dans un ordre logique rigoureusement fixé." Apparent conclusion: one must know the separation theorems for convex sets in topological vector spaces before one can find out what the Lebesgue measure of the closed unit interval is.

Bourbaki's version of general topology (e.g., filters and uniform structures) and at least a modicum of Bourbaki's version of algebra (e.g., tensor products and ordered groups). Moreover, he has to be at home in topological vector spaces; at the very least, he has to be familiar with pseudo-norms and he must know the Hahn-Banach theorem. If, on top of all that, he has sympathy for the axiomatic and abstract approach to mathematics (this must surely be a consequence of the preceding conditions), then he can proceed to read, to understand, and to enjoy himself.

The basic philosophy of the authors is that the proper context for integration theory is a locally compact Hausdorff space. Three arguments are advanced to support this view. (1) The generality in considering "abstract" measures is illusory, since (via the Stone theory of representations of Boolean algebras) every such measure is isomorphic, in some sense, to a measure on a locally compact space. (2) In the great majority of applications, the spaces that arise come equipped with a locally compact topology. (3) In the few instances where no (locally compact) topology is apparent, it is often useful to introduce one.

I should like to call attention to the fact that there is another side to the ledger. (1) It often happens (e.g., in ergodic theory) that the consideration of the Stone representation space is of no help at all. There is, for instance, a theorem of von Neumann (Annals of Mathematics, 1932), the object of which is to conclude that, under suitable hypotheses, every automorphism of the reduced Boolean algebra of a measure space is induced by a measure-preserving transformation of the space itself. The corresponding assertion for a Stone space is trivial, and, in this context, misses the point. (2) While it is true that discrete spaces, Euclidean spaces, and locally compact groups are locally compact, that fact, as an argument for restricting the study of measure theory to locally compact spaces, is a double-edged sword; it argues just as well for restricting the study of topology itself to locally compact spaces. It argues also against measure theory in, say, general metric spaces; the theory of Hausdorff measure is thereby deprived of a right to existence. (3) One of the most fruitful applications of measure theory is the modern theory of probability. In that theory a locally compact topology virtually never comes up. The topologies that workers in stochastic processes occasionally do meet are likely to be (a) not locally compact and (b) not useful.

The authors themselves are apparently not completely of one mind on the issue. My principal objective evidence for this assertion is that five of the six sections into which Chapter IV is divided include, at the end, a few pages (in small print) devoted to the case of abstract measures. If those pages are meant to pacify the large body of analysts who learned measure theory from, for instance, the classic book of Saks, the desired effect will probably not be achieved. Instead of recognizing an old friend in the pages on abstract measure, the analyst will find that the theory there expounded goes back no farther than four years; in fact, that theory is substantially identical with the theory presented in Stone's four notes on integration (Proc. Nat. Acad. Sci. U. S. A., 1948 and 1949).

The first two (short) chapters of the book are devoted to inequalities and vector lattices, respectively. A very general form of the Hölder and Minkowski inequalities (proof based on the Hahn-Banach theorem) is derived in Chapter I; the subsequent (integral) versions of these inequalities are in turn based on the general ones. Chapter II treats the elementary theory of vector lattices; subspaces, product spaces, and positive linear functionals receive their due share of attention.

The main work begins in Chapter III. A measure (on a locally compact space E) is defined as a linear functional μ on the space $\underset{\sim}{K}(E)$ of continuous functions with compact support, subject to the condition that, for every compact set K, the restriction of μ to the functions whose support is contained in K be continuous with respect to the topology of uniform convergence. An alternative notation for the value of the measure μ at the function f (in addition, that is, to the obvious notation $\mu(f)$) is $\int f(x)d\mu(x)$. The justification for this unusual procedure is, of course, the general Riesz theorem, which asserts that there is a one-to-one correspondence between (regular Borel) measures ν in the classical sense and measures μ in the Bourbaki sense, such that μ and ν correspond to each other if and only if $\mu(f) = \int f(x)d\nu(x)$ for all f. The Riesz theorem, however, is not an essential part of the authors' theory; a somewhat disguised version of it appears in the latter part of Chapter IV (p. 165), in small print.[3] (Some traces of an ambivalent attitude are again visible in connection with the definition of measure. The motivation for studying integration theory, as presented in the introduction, is based on concepts such as length, area, volume, and weight; expressions such as "a unit mass placed at a point" occur throughout the book.) After introducing the basic concepts (e.g., positive measure, bounded measure, the norm and the support of a measure) and some of the relations among them,

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[3] The Riesz theorem for Stieltjes integrals on the line is mentioned in the introduction; no relation between it and the theorem on p. 165 is ever indicated.

Chapter III concludes with a discussion of integrals of vector-valued functions and measures on product spaces.

Chapter IV is the longest one; it contains more than half of the total number of pages. It begins with the definition of the upper integral $\mu^*(f)$ of a positive, lower semi-continuous function f with respect to a positive measure μ (approximation from below by continuous functions). The authors' insistence on avoiding functions of sets whenever possible, and using functions of functions instead, is manifested in the importance of lower semi-continuous functions as a technical device; the point is, of course, that the characteristic function of a set A is lower semi-continuous if and only if A is open, so that lower semicontinuous functions in this theory play the role of open sets. The extension procedure is continued one more step: approximation from above by lower semi-continuous functions yields the upper integral of an arbitrary positive function. The outer measure of a set is defined as the upper integral of its characteristic function. The second section is devoted to "almost everywhere" properties. (A set A is called *μ-negligible* if $\mu^*(A)=0$.) In the third section the L^p spaces are defined, it is proved that they are complete, and the Lebesgue dominated convergence theorem is derived. The fourth section contains various standard results about integrable functions (e.g., the monotone convergence theorem and approximation by set functions). The fifth section studies the concept of measurability and the sixth (and last) section returns to the study of the fundamental inequalities.

Owing, no doubt, to the authors' predilection for using as definiens what for most mathematicians is the definiendum, there are many spots at which the treatment appears artificial. Where others can say immediately that a point x belongs to the support of a measure μ if and only if $\mu(V)\neq 0$ for every neighborhood V of x, the authors are first forced to demand, for every neighborhood V of x, the existence of a continuous function f whose compact support is contained in V and for which $\mu(f)\neq 0$. After defining the space $\mathcal{J}^p$ of all functions f for which the upper integral of $|f|^p$ is finite, the authors define $\mathcal{L}^p$ as the closure in $\mathcal{J}^p$ of the set of continuous functions with compact support—an effective but startling method of enforcing measurability. Some mathematicians might regard as artificial a treatment in which the evaluation of the Lebesgue measure of the unit interval (p. 156) is made to depend on what is essentially the Riesz-Fischer theorem ($\mathcal{J}^p$ is complete, p. 130). The important concept of *clan* (Boolean ring) is defined as follows: a non-empty class Φ of subsets of a set A is a clan if there exists a real algebra $\mathcal{A}$ of numerical functions on A, such that the sets in Φ are exactly the ones whose characteristic function belongs to $\mathcal{A}$. (In order to make the concept usable, it is, of course, necessary that the definition be immediately followed by the theorem: Φ is a clan if and only if it is closed under the formation of unions and relative complements.)

The last example of artificiality that I shall adduce deserves a paragraph to itself. Measurability is the last important concept defined in the volume (p. 180); I quote the definition verbatim. "*Soient E un espace localement compact, μ une mesure positive sur E. On dit qu'une application f de E dans un espace topologique F est mesurable pour la mesure μ* (ou encore *μ-mesurable*) *si, pour toute partie compacte K de E, il existe un ensemble μ-négligeable $N\subset K$ et une partition de $K\cap\complement N$ formée d'une suite* (*finie ou infinie*) (K_n) *d'ensembles compacts, tels que la restriction de f à chacun des K_n soit continue.*" Three comments should be made. First, the history of the definition should be obvious to anyone who has studied measure theory elsewhere; we are looking at Lusin's theorem. Second, the apparent purpose of the definition is to minimize the role of set-algebra in integration theory; although subsequently we must be told of the characterization in terms of inverse images (e.g., the proof that a lower semi-continuous function is measurable depends on that characterization), the original appearance of the concept is doggedly kept on the functional level. Third, the effect of the definition is to help perpetuate the myth that the measurability of a function can only be defined by reference to a measure—a myth quite as unfounded as the (fortunately moribund) myth that the continuity of a function can only be defined by reference to a metric. Both concepts are set-theoretic, and the similarity between them is worth emphasizing.

Some minor technical points should be mentioned. The distinction between $\mathcal{L}^p$ (the function space) and L^p (the quotient space of equivalence classes) is rigorously maintained; the authors effectively demonstrate that that can be done without getting hopelessly mired in notation. The section (pp. 79–89) on weak vector integration is excellent. It is clear and helpful and puts into evidence the essential role played by the condition: "the closed convex hull of a compact set is compact." *In re* terminology: "integrable set" seems to me to be a good phrase (better than "measurable set of finite measure"); on the other hand, "completely additive" instead of "countably additive" is both unfashionable and misleading. There is a slight confusion about the notation for double integrals. On p. 42 it is asserted that it is sometimes desirable to use a symbol such as $\int\int f(x)d\mu(x)$. What is probably meant is that the "double" integral $\int f(x, y)d\nu(x, y)$, where ν is the product measure of, say, λ and μ, is alternatively denoted by $\int\int f(x, y)d\lambda(x)d\mu(y)$; the corresponding iterated integrals are denoted by symbols such as $\int d\lambda(x)\int f(x, y)d\mu(y)$. The threat to use two integral signs with only one differential is never actually carried out. Aside from this triviality, I noticed no errors in the book, trivial or otherwise. I caught only three minor misprints; the only one that might worry a student for a few seconds is on p. 121, line 9: f_n should be $\bar{f}_n$.

The topics centering around the names of Fubini (decomposition of measures), Radon and Nikodym (absolute continuity), and Haar (group invariance), are not discussed in this volume; according to the introduction, they will be treated in the subsequent chapters.[3] The introduction indicates also that the authors are planning eventually to apply their theory (probability) and to generalize it (distributions).

My conclusion on the evidence so far at hand is that the authors have performed a tremendous *tour de force*; I am inclined to doubt whether their point of view will have a lasting influence.

PAUL R. HALMOS

Introduction to modern prime number theory. By T. Estermann. Cambridge University Press, 1952. 10+75 pp. $2.50.

The main purpose of this tract is "to enable those mathematicians who are not specialists in the theory of numbers to learn some of its non-elementary results and methods without too great an effort." Actually, the book is devoted to the limited object of proving the Vinogradov-Goldbach theorem that every sufficiently large odd number is the sum of three primes; in the course of proving this result, the author supplies the necessary results on characters and primes in arithmetic progressions. Only a few elementary number-theoretic results are assumed, these being quoted from Hardy and Wright's book *An introduction to the theory of numbers*; Cauchy's residue theorem is also assumed.

The book is a very carefully thought out exposition which lays bare the whole nature of the proof and unremittingly avoids all things not needed in the final proof. This leads to a work which is somewhat austere although not so formal as Landau's *Vorlesungen über Zahlentheorie*; in common with Landau's book, Estermann's tract gives few references to the literature. Nevertheless, the author admirably succeeds in his aim. The proofs are clear and remarkably concise. Some idea of the conciseness may be gained from the fact that the only comparable work—that of N. G. Čudakov, in Russian, entitled *Introduction to the theory of Dirichlet's L-functions*, 1947—is a little under three times as long; the length of Čudakov's book is partly due to the fact that, unlike Estermann, he makes excursions to round out the background. (Actually, Čudakov's book is less complete than Estermann's in one respect; namely, the former does not contain a proof of a fundamental estimate of Vinogradov which occupies half of the last chapter of Estermann's book.)

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[3] In Chapter III, the sole assertion resembling Fubini's theorem is stated for continuous functions with compact support only.

One of the essential steps in the proof is to obtain a suitable estimate for $\pi(m; k, l)$ which is defined as the number of primes p between 1 and m which are of the form $kn+l$. Chapter I is devoted to proving the following result for $\pi(m)=\pi(m; 1, 1)$, the number of primes between 1 and m:

$$\pi(m) = \operatorname{ls} m + O(me^{-c(\log m)^{1/2}})$$

where $c=1/200$ and

$$\operatorname{ls} m = \sum_{n=2}^{m} 1/\log n = \int_{2}^{m} du/\log u + O(1) = \operatorname{li} m + O(1).$$

This result is de la Vallée Poussin's refinement of the prime number theorem (later sharpened by Čudakov and others). The proof follows the ideas of Landau and only uses the Riemann zeta function $\zeta(s)$ for $\sigma=\Re(s)>0$. Such function-theoretic results as the Borel-Carathéodory theorem, which gives an upper bound for $|f^{(n)}(0)|$ in terms of an upper bound for $\Re f(z)$ on the unit circle, are proved. These results lay the groundwork for the whole method of proof which is used both in this and the next chapter. The author's decision to obtain the above numerical value for c occasionally forces the reader to use a slide rule and leads to proofs in which the underlying reason for their success is somewhat obscured. The author, perhaps, partly recognizes this when he wryly remarks, after obtaining the contradiction $333>1000/3$, that this reminds him "of the story of the Scotsman who looked suspiciously at his change, and when asked if it was not enough, said: 'Yes, but only just'."

Chapter II begins with the development of the theory of characters χ modulo k. This is done with the aid of the basis theorem for finite abelian groups which is proved in the special case of the multiplicative group of reduced residue classes modulo k. The theory of $L(s, \chi) = \sum_{n=1}^{\infty} \chi(n)/n^s$ is now developed for $\chi\neq\chi_0$, the principal character modulo k. That χ_0 can be disregarded is due to the fact that if $\pi(m; \chi) = \sum_{p\leq m} \chi(p)$, then $0\leq\pi(m)-\pi(m; \chi_0)\leq\phi(k)$ and

$$\pi(m;\,k,\,l) = \sum_{\chi \bmod k} \bar{\chi}(l)\pi(m;\,\chi)/\phi(k)$$
$$= \pi(m)/\phi(k) + \sum_{\chi\neq\chi_0} \bar{\chi}(l)\pi(m;\,\chi)/\phi(k) + O(1).$$

Because of the result of Chapter I, it suffices to deal with $\pi(m;\,\chi)$ for $\chi\neq\chi_0$. The usual zero-free regions of $L(s,\,\chi)$, roughly of the form $\sigma>1-C/\log kt$, are obtained. Also, the author gives his proof of Siegel's result that $L(s,\,\chi)\neq 0$ for real $s>1-k^{-\epsilon}$ if $k>A(\epsilon)$. This work is handled adroitly. No mention is made of the possible exceptional real (Siegel-) zero, nor is the idea of primitive character needed or mentioned; for a more perfect understanding of the underlying situation, this may be a loss but it enables the author to drive towards his goal more quickly. The final result, Theorem 55, is that

$$\pi(m;\,k,\,l) = \mathrm{ls}\; m/\phi(k) + O(me^{-c(\log m)^{1/2}})$$

if $(k,\,l)=1$, $k\leqq\log^u m$, $m\geqq 3$, and u is an arbitrary real number on which O depends; the principal term arises from $\pi(m)$ while the error term arises from the functions $\pi(m;\,\chi)$.

Actually, this form of the result is not that needed by the author at the bottom of page 63. What is needed is the slightly more general result that if $(k,\,l)=1$, $k\leqq\log^u M$, $M\geqq 3$, and $m\leqq M$, then

$$\pi(m;\,k,\,l) = \mathrm{ls}\; m/\phi(k) + O(Me^{-c(\log M)^{1/2}}).$$

This result can be obtained, for example, by first developing the corresponding form of Theorem 52; all that is required for this purpose is that b be defined as $\exp(\{\log(M+1/2)\}^{1/2}/160)$ and that the two cases $M>A_1$ and $M\leqq A_1$ be considered. Also, once this is done, the first half of the proof of Theorem 54 may be dispensed with.

The Vinogradov-Goldbach theorem is proved in the last chapter by the usual Farey dissection of the unit circle. The most difficult and least transparent part of this proof (although technically elementary) is the following weakened form of the Vinogradov estimate:

$$\left|\sum_{p\leqq v} e^{2\pi i p h/q}\right| \leqq n/\log^3 n$$

provided $(h,\,q)=1$ and

$$n/\log^3 n < v \leqq n, \qquad \log^{15} n < q \leqq n/\log^{15} n.$$

The proof occupies eight pages and here too the author has effected a considerable simplification.

At the end of the book is an index and a seven page section of theorems and formulae for reference. The book has the usual excellent typography of the Cambridge tracts and is almost free of misprints.

LOWELL SCHOENFELD

Sur les espaces fibrés et les variétés feuilletées. By W. T. Wu and G. Reeb. (Actualités Scientifiques et Industrielles, no. 1183.) Paris, Hermann, 1952. 157 pp.

This monograph consists of the theses of the authors at the University of Strasbourg under the direction of C. Ehresmann, and contains detailed accounts of original contributions of the authors, whose main results have been announced in the Comptes Rendus de l'Académie des Sciences à Paris. The contents of the two papers are not directly related, although both can be said to be concerned with certain aspects of the theory of differentiable manifolds.

The paper of Wu Wen-Tsun has the complete title: *Sur les classes caractéristiques des structures fibrées sphériques.* Its starting-point is the so-called universal bundle theorem. In the cases in which the author is interested, the fiber bundle has as base space a finite polyhedron B and as structural group G one of the three groups: the orthogonal group O_m in m variables, the proper orthogonal group $\hat{O}_m$ in m variables, and the unitary group O'_m in m complex variables. The universal bundle theorem asserts that, B and G being given, there exists a bundle with the base space B_0 and the same structural group G such that the given bundle is induced by a mapping $f:B\rightarrow B_0$ and that this mapping f is defined up to a homotopy. It was perhaps Pontrjagin who first observed that the dual homomorphism $f^*:H(B_0, R)\rightarrow H(B, R)$ of the cohomology ring of B_0 into the cohomology ring of B, relative to a coefficient ring R, is thus completely determined by the bundle. In our three cases we can take as B_0 respectively the following Grassmann manifolds: the manifold $R_{n,m}$ of all m-dimensional linear spaces through a point O in a real Euclidean space of dimension $n+m$, the manifold $\hat{R}_{n,m}$ of such oriented linear spaces, and the manifold $C_{n,m}$ of all linear spaces of dimension m through a point O in a complex Euclidean space of complex dimension $n+m$, with n sufficiently large in all three cases. The author's notation for $C_{n,m}$ is slightly confusing. In many places, such as on pages 7, 8, 48, etc., it would be clearer to write $C_{n',m'}$ for $C_{n,m}$.

The fruitfulness of this approach is based on the fact that these Grassmann manifolds are relatively "rich" in homology properties. The homology groups of these manifolds were determined by Ehresmann, on adopting the notion of Schubert varieties in algebraic geometry. Since f^* is a multiplicative homomorphism, it suffices to know its effect on a multiplicative base of $H(B_0, R)$. Moreover, we can restrict ourselves to the cohomology classes of dimensions $\leqq \dim B$. The cases of interest are the following:

(1) The structural group G is O_m or $\hat{O}_m$ and the coefficient ring R is the finite field of two elements. In this case a multiplicative base of $H(B_0, R)$ is formed by the k-dimensional cohomology classes W_2^k, $1 \leqq k \leqq m$. Their images under f^* are called the Stiefel-Whitney classes.

(2) G is O_m or $\hat{O}_m$ and R is the field of real numbers. There are classes P_0^{4k}, $1 \leqq k \leqq m/4$, which generate the elements of dimension $\leqq m$ of $H(B_0, R)$. Their images under f^* are called the Pontrjagin classes.

(3) G is O'_m and R is the ring of integers. There are classes C^{2k}, $1 \leqq k \leqq m$, of dimension $2k$ which generate the elements of dimension $\leqq 2m$ of $H(B_0, R)$.

These cohomology classes are what the author calls characteristic classes. It is the study of their properties and their geometrical applications which constitutes the main aim of this paper.

The paper is divided into five chapters. Chapter I contains an exposition of the homology properties of Grassmann manifolds. The results are essentially due to Ehresmann, with modifications by Pontrjagin for the case $\widehat{R}_{n,m}$. The section on the relation between two dual Grassmann manifolds is new and should be found useful in geometrical applications.

The author considers Chapter II as containing his main contribution in this paper. For simplicity call a fiber bundle with the structural group G a G-structure. If G' is a subgroup of G, a fundamental problem in fiber bundles is whether a G-structure is equivalent to a G'-structure. The author studies the cases when (G, G') are respectively the pairs $(O_m, \hat{O}_m)$ and $(\hat{O}_{2m}, O'_m)$. For definiteness of description let us restrict ourselves to the pair $(\hat{O}_{2m}, O'_m)$. There is a canonical mapping $\phi' \colon C_{n,m} \to \widehat{R}_{2n,2m}$. If an $\hat{O}_{2m}$-structure is induced by a mapping $f \colon B \to \widehat{R}_{2n,2m}$, it follows from the universal bundle theorem that it is equivalent to an O'_m-structure if and only if there exists a mapping $g \colon B \to C_{n,m}$, such that f and $\phi' \circ g$ are homotopic. The study of the homology properties of the mapping ϕ' is thus carried out. From this one derives the following necessary conditions for a given $\hat{O}_{2m}$-structure to be equivalent to an O'_m-structure

$$(1) \qquad W_2^{2k} = C_2^{2k}, \qquad W_2^{2k+1} = 0$$

$$(2) \qquad (-1)^k P_0^{4k} = \sum_{h=0}^{2k} (-1)^h C_0^{2h} \cup C_0^{4k-2h},$$

where C^{2k} are the characteristic classes (with integer coefficients) of the O'_m-structure and the subscripts 2 and 0 denote respectively their reductions to coefficients mod 2 and to real coefficients. Neither set of these conditions is trivial, even for the tangent bundle of an even-dimensional orientable manifold. In fact, the author gives examples of even-dimensional orientable manifolds whose three-dimensional Stiefel-Whitney class does not vanish. (Whitney proved that the three-dimensional Stiefel-Whitney class of a four-dimensional orientable manifold always vanishes.) Meanwhile, the author derives from (2') the theorem that a sphere of dimension $4k$ has no almost complex structure.

Chapter III is devoted to the study of the homotopy properties of Grassmann manifolds. The main result is a homotopy classification of the mappings of a four-dimensional complex into $C_{n,2}$, $n \geqq 2$. This result is applied in Chapter IV to derive a necessary and sufficient condition for a four-dimensional oriented manifold to have an almost complex structure. The author even determines all possible almost complex structures on it. (In the statement of Theorem 10, p. 74, it would be clearer to mention that the manifold M is of dimension 4.) The same idea is applied to derive a necessary and sufficient condition that a six-dimensional oriented manifold has an almost complex structure.

From an $\hat{O}_m$-structure and an $\hat{O}_n$-structure over the same base space we can define, following an idea of Whitney, an $\hat{O}_{m+n}$-structure. The relations between the characteristic classes of these three structures constitute the duality theorems of the Whitney type. Chapter V gives the statements and proofs of such duality theorems. Whitney himself has not published his proofs, claiming that they are very difficult. The present method of looking at the problem as the study of the homology properties of a certain canonical mapping of Grassmann manifolds seems to be the proper approach to this question.

This thesis was finished in the summer of 1949. In the next two years Wu pushed this study much further. Among other results he has determined the Steenrod squares on a Grassmann manifold and has obtained necessary conditions that a sphere bundle be the tangent bundle of a differentiable manifold. (Cf. Comptes Rendus Paris, v. 230, pp. 508–511, 918–920.)

The second paper of this book, by G. Reeb, is entitled: *Sur certaines propriétés topologiques des variétés feuilletées*. Its purpose is the study of the global properties of the integral manifolds of a completely integrable Pfaffian system on a manifold. The given data consist of a differentiable manifold V_n of dimension n and class 2 and on it a field E_q of elements of contact of dimension q, $1 \leqq q \leqq n-1$, and class one, which is completely integrable. The latter means that every point $x \in V_n$ has a neighborhood Ω_x and a homeomorphism h_x of class one of Ω_x into an open subset of the Euclidean space R_n of n dimensions, such that the elements of contact of E_q go into parallel elements. If we suppose E_q to be spanned by q linearly independent vectors, each of class one in a certain neighborhood, it follows from a well known theorem on differential equations that the condition of complete integrability of the field can be expressed as differential equations in the components of the q vector fields. The unpleasant feature of the situation is that these differential equations are quadratic, and not linear. The question of the existence of a field E_q on V_n, not necessarily completely integrable, is a question on fiber bundles, equivalent to that of the existence of a cross section in a certain associated bundle. The author mentions as a fundamental problem the following: If a field E_q exists on V_n, does there exist one which is completely integrable?

This is undoubtedly an important problem in differential geometry. The author believes that the answer would be negative, a conjecture which the reviewer is willing to share. But no counter-example is known. On the contrary, on the three-sphere and on some other three-dimensional manifolds, the author gives examples of E_2, which are completely integrable (p. 112).

The reviewer wishes to point out that there are analogous situations in which the answer to such a question is negative. For the general problem can be described as that of deciding whether a fiber bundle has a cross section satisfying a system of differential equations, when a cross section does exist without satisfying the differential system. The simplest case is that of a parallelisable manifold in which the structural group of the tangent bundle can be reduced to the identity. For a manifold V_n to be parallelisable it is necessary and sufficient that there exist n linear differential forms ω^i, $1 \leqq i \leqq n$, of class one, which are everywhere linearly independent. Their exterior derivatives can be written

$$d\omega^i = \sum_{j,k=1}^{n} c^i_{jk}\omega^j \wedge \omega^k, \qquad i = 1, \cdots, n, \; c^i_{jk} + c^i_{kj} = 0,$$

where c^i_{jk} are functions in V_n and are completely determined. The question whether a parallelisable V_n has a set of ω^i satisfying the further condition that the c^i_{jk} are constants is a problem of the above type. When V_n satisfies a completeness condition, which it does if, for instance, its universal covering manifold is compact, this amounts to saying that V_n is a Lie group. This is not always true for a parallelisable manifold; for example, the seven-dimensional sphere is parallelisable but is not a Lie group. There are other questions of similar nature, but we shall not bother with them here.

The object of the author is to study the integral manifolds of the completely integrable field of elements E_q. Since the classical work of Poincaré on the integral curves of ordinary differential equations this problem has received much attention. It is well known that even for $q=1$ and even under analyticity assumptions the global behavior of the integral curves can be rather "wild." Complete information is known in the case of integral curves in the plane, owing to the work of Kaplan.

A natural step in such a study is to introduce a new topology in V_n, whose open sets are unions of integral manifolds. The manifold V_n with this topology is called a leaved manifold (variété feuilletée) and a connected component of it a leaf (feuille). Chapter A of this paper is devoted to the study of some of the basic properties of this topology. Its last section contains several examples of leaved manifolds, which should prove useful in any further study of the subject.

Chapter B studies the topological properties of neighboring leaves, the results of which are called theorems of stability. Three types of stability theorems are proved, the first concerned with the behavior of leaves in the neighborhood of a leaf, the second with the effect on the leaves when the leaved structure is slightly modified, and the third with the case in which $q=n-1$, where more precise information can be obtained. As examples of such results we mention the following: If a leaf is compact and has a finite fundamental group, every open neighborhood of it has a saturated open neighborhood such that every leaf which meets it is compact. In particular, if the leaved structure is of class 2 and if all the leaves are compact and simply connected, they are homeomorphic and define a fiber structure in V_n, whose base space is a manifold of dimension $n-q$. The third stability theorem implies as a corollary the following theorem: If $q=n-1$ and if V_n is compact and connected and has a compact leaf with a finite fundamental group, every leaf is compact and has a finite fundamental group.

Chapter C is concerned with the case $q=n-1$ and studies the isolated singularities of a completely integrable Pfaffian form. In a neighborhood with the coordinates x^i, $i=1, \cdots, n$, such a form can be approximated by a form $\omega' = \sum_{i,j=1}^{n} a_{ij}x^j dx^i$, where a_{ij} are constants. It is proved that the matrix (a_{ij}) is either symmetric or of rank 2. In the former case the singularity is called a center, if the form $\sum_{i,j=1}^{n} a_{ij}x^i x^j$ is positive definite. When V_n is compact, with $n \geqq 3$, a completely integrable field E_{n-1} with a nonempty set of centers as singularities must have leaves which are homeomorphic to an $(n-1)$-sphere. Moreover, this happens only when there are exactly two centers and when V_n is homeomorphic to an n-dimensional sphere. The chapter is concluded by a more detailed study of the case when the form is analytic.

There does not seem to be any doubt to the reviewer that both studies contain valuable contributions to the topology and differential geometry of manifolds. We also believe that they only mark a beginning of further fruitful investigations.

SHIING-SHEN CHERN

Grundzüge der theoretischen Logik. By D. Hilbert and W. Ackermann. 3d ed. Berlin, Springer, 1949. 8+155 pp.

Principles of mathematical logic. By D. Hilbert and W. Ackermann. Trans. by G. G. Leckie and F. Steinhardt; ed. and with notes by R. E. Luce. New York, Chelsea, 1950. 12+172 pp.

The first edition of this book appeared in 1928. According to the preface, it was based on Hilbert's lectures of 1917–22. It was reviewed, somewhat unsympathetically, by Langford in this Bulletin, Vol. 36, pp. 22 ff. The second edition appeared in 1938; it was reviewed by Rosser in this Bulletin, Vol. 44, p. 474, and by Quine in Journal of Symbolic Logic, Vol. 3, p. 83. The third edition and the English translation of the second edition, with both of which this review is concerned, appeared almost simultaneously in 1949–50. They have been previously reviewed in the Journal of Symbolic Logic, Vol. 15, p. 59, by Church, and Vol. 16, p. 52, by Zubieta, respectively.

The book was intended as an introductory textbook of mathematical logic in a narrow sense. The Hilbert school never subscribed to the identification of mathematics and logic, and regarded "mathematical logic," "theoretical logic," and "logical calculus" as synonymous designations for a preliminary stage in the subject of "foundations of mathematics," which many Americans prefer to call "mathematical logic" in a broader sense (cf. Quine's book of 1940). Anything depending on an axiom of infinity or similar assumption would belong to the latter subject but not to the former. Nevertheless, the authors regard the narrower subject as an essential step to the broader. Thus, in the first preface, signed by Hilbert, it is stated that the book is intended as a preparation for a further book by him and Bernays, no doubt referring to the great work of which the first volume appeared in 1934.

The plan of the book has remained the same in all the editions. There are four chapters, dealing respectively with propositional algebra, the algebra of classes or monadic predicates, the restricted calculus of predicates, and various extended calculuses of predicates. The first and third chapters begin with an intuitive discussion, introduce the formalization gradually and naturally, and conclude with a formal system and its properties of deducibility (consistency, independence, completeness, decidability, etc.). This plan is modified in the second and fourth chapters, which are briefer and less formal. The second chapter is something of a side issue; it treats the class interpretation of Boolean algebra, to which has been added an operation of universality, so as to form a modal algebra (essentially the Lewis system S5) adequate for one formulation of the traditional syllogism. The fourth chapter is more of a survey; but it gives a good idea of the intolerable complexity of the theory of types.

The changes made in the various editions have, in fact, been relatively minor. For those made in the second edition, see the previous reviews. In the third edition some changes occur in the second, third, and fourth chapters. In the second chapter the second edition contains an intuitive treatment with a reference, for all details, to a paper by Wajsberg; in the third edition the intuitive discussion is somewhat amplified, there is a decision process on an intuitive basis for the case where one modal operation does not include another, and there is no reference whatever to Wajsberg. In the third chapter certain technical errors are corrected (see Church review), and new results on the decision problem are incorporated. In the fourth chapter there is an amplified and more accurate treatment of the theory of types. The second edition contained a brief but useful bibliography; the third contains only footnote references in the text.

As a text the book has become a classic. Such terms as "conjunctive normal form" and "restricted predicate [originally 'functional'] calculus" have become well known largely through its influence. Despite certain shortcomings it is still the best introduction for the student who seriously wants to master the technique. Some of the features which give it this status are as follows:

The first feature is its extraordinary lucidity. A second is the intuitive approach, with the introduction of formalization only after a full discussion of motivation. Again, the argument is rigorous and exact with full attention to the difference between a rule and an axiom. A fourth feature is the emphasis on general extra-formal principles (in more recent terminology epitheorems or, in some cases, metatheorems) such as normal forms, replacement rules, decision procedures, etc. Finally, the work is relatively free from bias. In its origin it was a judicious combination of the algebraic methods of Schröder with the logistic standpoint of Frege and Russell. (Thus, the emphasis on propositional interpretation agrees with the latter, while the conjunctive normal form is essentially Schröder's "Entwicklung." There are also notable analogies with the Post paper of 1921 (probably because both drew from the same sources). It therefore represents a union of the two principal tendencies of the time. Since the different schools of mathematical thought differ not so much in regard to logic (in the narrow sense) as in regard to its relation to the foundations of mathematics, the book represents a rather complete treatment of the heart of its subject, acceptable from any point of view. All together, the book still bears the stamp of the genius of one of the great mathematicians of modern times.

On the other hand, there are signs of obsolescence. Much water has spilled over the dam in the twenty-five years since this book was first published. This book is confined entirely to the classical logic; whereas we now know there are a variety of generalized systems, some with interesting applications, which have much the same relation to classical logic that various generalized geometries do to Euclidean. The new techniques developed by Gentzen are barely mentioned. Of course, it is quite true, as some insist, that one cannot do anything with those techniques that one cannot do with the older ones; just as one can go anywhere with a horse and buggy that one can reach with a Cadillac. The mere mentioning of such techniques is not enough; mathematical logic is now capable of an approach which is as different from that in this book as the latter was from its predecessors. Moreover, the separation between logic proper and mathematics can no longer be maintained. Recent foundational studies (recursive arithmetic, combinatory logic including the theories of lambda conversion, Post's formalized syntax, etc.) show that important theories can be constructed without the aid of any logical calculus, and that these are sufficient for portions of mathematics; so that logic is founded on mathematics, as the intuitionists have long held, rather than the reverse. An elementary text along these lines has not yet appeared; but it is certainly possible, and it is to be hoped that it will not be long delayed. Until it appears the present text will lack the completeness that it had when it was first issued.

So much for the German book. Let us turn now to the English translation. This was made from the second German edition without the consent, or even the knowledge, of the surviving author. (This situation has, incidentally, caused some unfavorable comment which the reviewer heard during his visit to Europe in 1950–51.) On the whole the translation is a good one; the reviewer thinks that the translators have succeeded rather well in their stated aim, "both to give an exact English rendering of the sense and intent of the original text and also, so far as possible in a different language, to reproduce something of its manner and style." There are to be sure a number of what the reviewer considers mistranslations, such as "number" for "Anzahl," (instead of "cardinal number,"—"number" is the translation of "Zahl"), "sentence" for "Aussage," and "expression" for "Satz"—the reviewer's objections to the second of these will be stated below—and (cf. review by Zubieta) "Ersetzung" should be translated "replacement" rather than "substitution" (=Einsetzung). The translators and/or editor have also corrected some technical inaccuracies in the text of Chapter 3, essentially the same as those made in the third edition (see Zubieta's review), have added a few pages of "editor's notes," and have revised the bibliography and index.

It remains to comment on the use of "sentence" as a translation for "Aussage" (and of certain related words correspondingly). The editor and translators are evidently quite conscious that this changes the sense of the original and go to some length to justify it as a "correction" in the editor's preface and notes. It is evidently due to the fact that they believe that in logic the objects of study are linguistic symbols and expressions. The reviewer's view is that in logic it is irrelevant what the objects of study actually are (so long as they satisfy the formal conditions). If one chooses to regard these objects as symbolic, one is at liberty to do so; but in that case one must distinguish symbols used (*U*-symbols) and those mentioned (*O*-symbols). If the *O*-symbols can also be used, then a quite elaborate machinery must be employed to maintain the distinction between use and mention. All this bewildering machinery can be avoided without entailing confusion if one does one of two things: (a) adopt *O*-symbols which are actually meaningless, so that if one occurs in a context it can designate only itself; or (b) avoid specifying that the objects are linguistic, adopting a terminology which does not suggest that eventuality. In either case all symbols are *U*-symbols, and the ordinary conventions of good linguistic usage suffice to avoid confusion. Hilbert preferred (a), and the book under review does also; but the preference is not insisted upon, and the natural connotation of "Aussage" leaves the reader free to adopt (b). (This is part of that freedom from bias which was previously commented on. In either case there is no "recurrent carelessness in maintaining a strict distinction" between use and mention, as charged in the editor's preface.) The English word "proposition" has the same connotation. Moreover, it is not necessary, in using it, to settle the question, mentioned in the editor's Note 1, as to what propositions are. But the word "sentence," particularly when accompanied by such a preface, suggests an insistence on the linguistic viewpoint. This imposes on the work a bias which was not there originally; and it interferes somewhat with the intuitiveness of the approach also. If, as the editor himself comments in his Note 1, it is not necessary to decide the nature of the objects of logical study, why insist on a terminology which commits one to a particular view of it? It is a pity that what is otherwise an excellent translation should be marred by such pedantry.

To sum up, the book of Hilbert and Ackermann is one of the classics of the logical literature. In spite of the fact that it reflects the state of the science twenty-five years ago, with some changes of detail but no thorough-going revision, it is still the best textbook in a Western European language for a student wishing a fairly thorough treatment. The English translation is well done. The reviewer regrets certain features of it, and in particular regrets that it was published without consultation of the surviving author; but that does not alter the fact that it is a real service to English speaking students of the subject. Its content does not differ from that of the latest German edition in any important way.

HASKELL B. CURRY

Introduction to the theory of games. By J. C. C. McKinsey. New York, McGraw-Hill, 1952. 10+371 pp. $6.50.

This book is intended as a textbook for advanced undergraduate and graduate students. It fills, perhaps uniquely at present, a wide existing need which the now classical book of von Neumann and Morgenstern cannot satisfy on this level. In addition to the normal interest of mathematicians in the theory of games there is also the great interest of economists and many applied mathematicians in the theory; much of what is now called operations analysis, military and otherwise, makes copious use of this theory. This textbook, therefore, which presupposes essentially a knowledge of advanced calculus, will be useful to students and workers in several fields. In the scope of about 360 pages it discusses the principal topics which, by general agreement, should fit into an introductory text. Most of the book is devoted to zero-sum two-person games, but there are several chapters on n-person games and non-zero-sum games. Separate chapters are devoted to games in extensive form, continuous games, separable games, and games with convex payoff functions. By way of applications there are chapters on linear programming and statistical inference. The final chapter is devoted to unsolved problems.

The author's graceful style is an attractive feature of the book. He is professor of a subject (philosophy) where style is sometimes more important than content, and he writes on a subject (mathematics) where content is always all-important. The combination of attractive features of both subjects is a very felicitous one. In the process of formulation of new theory the author passes back and forth in a non-confusing way from a discussion of practical problems to the construction of the mathematical models, to the greater clarification of the former and the richer motivation of the latter. In the development of the mathematical theory the need for the introduction of new definitions is justified by reference to the development of the theory, and the choice of a particular definition over several possible alternative definitions is justified by a discussion of their consequences. Even books on abstract mathematical disciplines could emulate this procedure with profit; the result is so much more readable than an unbroken succession of definitions and theorems.

The reviewer's chief criticism is that some of the author's proofs are too formalistically algebraic. The result is to make these proofs unnecessarily long and to conceal from the student the basic facts which underlie the proofs. (This is in such contrast to the author's general procedure.) We shall cite a few examples of this. (1) The proof of the fundamental theorem of game theory (page 34). Actually the discussion of a graphical method of solution on p. 52 et seq. can, with minor additions, be made to include a rigorous and perspicuous proof. Another possibility is the very elegant and short proof of Karlin (*Contributions to the theory of games*, ed. by Kuhn and Tucker, Annals of Mathematics Studies, no. 24, Princeton, N. J., p. 135). (2) The proof of the fundamental theorem for continuous games (p. 186). The proof due to Wald (e.g., *Statistical decision functions*, New York, Wiley 1950, p. 38, Theorem 2.2) is elementary and proves much more in a much smaller compass. (3) The proof that the characteristic function is superadditive (p. 307, Theorem 15.1, (III)). The characteristic function of a set of players is the total amount these players can expect to obtain if they form a coalition, even if their strategy is known to their opponents. Obviously, therefore, the characteristic function of a coalition of two sets of players cannot be less than the sum of the characteristic functions.

It is to be regretted that the chapter on statistical inference, which could have been one of the most interesting in the book, is not on the same high level as the rest of the book. It would have been easy to use simple, interesting, and meaningful examples; many such are available in the literature for the cases where minimax solutions are simple to give. Indeed, the proof itself that a given decision function for a meaningful example is minimax could be very instructive. The author correctly makes the point that one cannot regard Nature as a malevolent opponent of the statistician. The statistician who adopts a minimax strategy is simply following a conservative policy. Yet the student may be forgiven if he is puzzled by such expressions as (1) "... the possible mixed strategies which nature (sic) may use," (2) "*S* (the statistician) does not know exactly what mixed strategy nature uses," or (3) "the probability with which nature played her various strategies." Certainly many leading statisticians would deny that in many realistic problems the notion of an a priori distribution is tenable. The author must be well acquainted with the epistemological problem involved. As a matter of fact, modern statistics began with an emphatic rejection of the notion that statistical inference was possible only by use of a priori distributions. (This denial alone is, of course, not a sufficient condition for progress, and some argue that it is not necessary or even desirable.) On p. 283 the author makes the point that the statistician is not especially interested in whether the two-person game between Nature and the statistician is determined or not. At this point the perplexed reader may rightly ask what the connection is between game theory and statistical inference. The answer (unfortunately not given by the author) is that it is often of great technical help to the statistician to know that the "game" is determined. In many problems one finds a minimax decision function by first obtaining a least favorable a priori distribution (optimal strategy for Nature viewed as an opponent of the statistician). The determinateness of the game is also a powerful tool for proving complete class theorems (e.g., Wald, loc. cit., Section 3.6).

Incidentally, the notion of a "best" strategy (p. 84), as compared with merely "optimal" strategies, has long been known in statistics where Wald introduced the notion of "admissibility." A discussion of this idea and the notion of a complete class (due to Lehmann) would have enriched the chapter on statistical inference. These ideas form an interesting bond between the two theories.

It would also have been interesting and profitable to describe some of the contributions to the theory of games made by statisticians in the course of their researches in statistics. For example, elsewhere the author cites as a noteworthy phenomenon an example of a continuous game where every pure strategy is employed in the optimum strategy. Such examples are commonplace in the statistical literature; several interesting ones (designed for another purpose) are to be found in Ann. Math. Statist. (1950) p. 190. Other citable results are those on the equivalence of behavior strategies and mixed strategies under general conditions (Ann. of Math. (1951) p. 581), and nontrivial results on the elimination of randomization (Ann. Math. Statist. (1951) p. 1 and p. 112).

The above criticisms should be regarded as directed at minor blemishes of a highly meritorious piece of work. The mathematical public is indebted to the author for an excellent and highly readable book, which this reviewer read with pleasure.

J. Wolfowitz

Leçons d'analyse functionelle. By F. Riesz and B. Sz.-Nagy. Budapest, Akadémiai Kiadó. 8+448 pp. About $7.50.

This work is superb.

For the field which it covers, it cannot be approached now nor will be soon by other books. It is not presented as a treatise for specialists, the essential purpose of which is to report advanced and complex results. Nor is it written as a textbook for the young student. Its aims are much higher and much more elegant. And in accomplishing these aims its authors have put together a magnificent advanced treatise and a most excellent though not elementary text. The purpose of the work is to set down, within the spirit and context of the undertaking, a certain coherent and central portion of mathematics in final and definite form. And within the spirit of the undertaking, this version is final and correct. Whether it is the only possible such version is another question, the answer to which is not important at this point. The hallmark of the work is its balance and good taste: in the choice of subjects, in the extent and detail in which they are developed, in the methods used to present them, and in the critical question of style and exposition.

The subjects treated are the modern theory of integration and differentiation, and the theory of linear operators which is based upon these concepts. Thus we find discussion of the space L^2 of square integrable functions, of abstract Hilbert space, of the space C of continuous functions. The latter is connected to integration theory by the fundamental correspondence between linear functionals and measures. This leads to a brief treatment of the spaces L^p, $p\geqq 1$, of reflexive spaces, and finally, of Banach spaces. For these various spaces, the operator theory discussed includes integral operators, completely continuous operators in general, completely continuous symmetric (that is, self-adjoint) operators, bounded symmetric operators, unbounded self-adjoint operators, and spectral theory in general Banach spaces.

The method of presentation is a mixture of the inductive and the axiomatic. Thus integration is first developed for the line, then in n-space, and finally abstract integrals and measures are treated. Similarly, in the realm of normed spaces, there is first a thorough treatment of L^2, then the Hilbert space is axiomatized. After discussion of L^2, L^p, $p \geqq 1$, and C, a general Banach space is introduced. The first operators handled are integral operators. From there one generalizes to the completely continuous ones. For the symmetric case there is the succession of stages of generalization mentioned in the preceding paragraph. This method has a multitude of advantages. The historical development of a subject is always humanistically rich. It so happens that in this field, a historical development written in full knowledge of what we now understand is thoroughly satisfactory. This is not to say that the authors insist uniformly on the time-sequence procedure. For instance, at the very beginning, the order of discussion is: differentiation, integration, and measure,—a precise about-face of the traditional sequence. The inductive development has disadvantages which will sometimes discourage the younger student. Thus after he has labored assiduously studying completely continuous operators in L^2, finally establishing the validity of Fredholm's alternative, he is told in one page how to change all the arguments so far presented to apply to a general Banach space. This amounts to receiving a commission to rewrite ten pages of the book to suit the new exigencies. It is doubtful whether these instructions will convince the reader who meets this situation for the first time.

The book is in two parts. The first one on modern variables covers approximately one third of the work. We are told that this section was written by F. Riesz and that Part II on integral equations and linear operators was essentially set down by B. Sz.-Nagy. The appearance finally of Riesz' treatment of real variables is a long awaited event which we all greet with joy. The reviewer has heard many stories of earlier manuscripts which were all ready twenty and more years ago but which due to an exceptional (and exasperating) desire for perfection never reached the printer. A recent inspection of a note-book that this writer kept of daily conversations with F. R. reveals that the treatment of real variables discussed a generation ago is precisely the one now before us. Here it is in the exceptional style and the warm and particularly personal French of its sponsor. The presentation of the entire theory in 140 pages is a remarkable tour de force both of mathematical profundity and of stylistic power. Part II was written by Nagy although it is very heavily influenced by Riesz. The presentation of the junior member of this partnership is of the finest. A more capable and polished writer could scarcely be found. We find in him the same infinite care for smooth and natural development which characterize his teacher. If a difference is to be pointed out, it would be that Nagy is less likely to "épater" the reader with a mild looking but in truth devastating lemma than his kindly malicious preceptor.

Before discussing the substance of the work, a few more general remarks may be made. The mechanics of subdivision within chapters is reduced to a minimum—theorem 3.1.7.12 is nowhere to be found. Instead we have Lebesgue's theorem or Fubini's or Beppo Levi's, etc. This is very satisfactory to the reader, although as a device, it is frequently employed (by other authors) to torture the facts. There is a most useful bibliography citing the works of about one hundred and fifty authors. The errata are infrequent and will cause no difficulty to the reader. Some of these are listed on the last page. The typography is most agreeable. It is a pleasure to be able to report that it is possible to import the book into this country. Some thirty copies were obtained after a delay of three months for a class in integration now being given by this writer.

Part I treats in that order: differentiation, integration, measure; the space L^2; and the Lebesgue-Stieltjes integral. It begins with a three page proof of Lebesgue's theorem on the differentiability almost everywhere (a.e.) of a monotone function. Then comes Fubini's important theorem on the termwise differentiability of a series whose terms are monotone functions of the same type. This theorem yields numerous applications. Next is a section on functions of intervals, their integrals and derivatives. After two brief fundamental theorems for these, it is easy to establish that: a bounded function $f(x)$ is Riemann integrable if and only if it is continuous a.e.

The integral is next defined. First for the class C_0 of step functions in an obvious way. Then for the limits of increasing sequences of step functions whose integrals are bounded—this gives class C_1. Class C_2 is the set of all functions $f(x)-g(x)$ where $f(x)$, $g(x)\in C_1$. This is a very characteristic device with a large number of possible variations and applications. The point of this development is that it incorporates into a definition one of the most important properties of the integral: The integral of a limit is the limit of the integrals. Next comes the theorem (Beppo Levi) that class C_3 (limits of increasing functions in C_2 whose integrals are bounded) is identical with C_2. Essentially of the same character are the classic theorem of Lebesgue and a lemma of Fatou. This completes the development of the integral (in ten pages). It is now possible to develop the inequalities of Schwarz, Hölder and Minkowski fundamental to the study of L^2 and L^p. Finally measurable functions are defined and then measurable sets. A measurable function is one which is the limit a.e. of a sequence of integrable functions. A measurable set is one such that its characteristic function is measurable. The properties of measurability are dealt with in two pages.

Now comes the study of indefinite integrals and their derivatives. With the help of Fubini's theorem (above) it is trivial to prove that every integrable function is the derivative a.e. of its indefinite integral. Using the concept of absolute continuity, the characterization of all indefinite integrals is then taken up.

There is now a section on the space L^2. The well known method of Riesz is applied to the proof of the Riesz-Fisher theorem. Linear functionals and orthonormal systems are studied. The notions of strong and weak convergence are introduced. In germ form one finds here a multitude of phenomena and theorems which will subsequently be reproduced in Banach spaces (for example: The sequence $\{A_n\}$ of linear functionals in L^2 cannot converge for every $f(x) \in L^2$ without being uniformly bounded).

Functions of several variables may be treated by an obvious generalization or by the following mapping procedure. Suppose that we consider two variables. In the theory of integration for the line it is sufficient to consider step functions which are constant on intervals of the form $m3^{-n} \leqq x \leqq (m+1)3^{-n}$; similarly for squares. Now one may establish a 1-1 correspondence between the 3^{2n} squares each of area 3^{-2n} of the unit square and the 3^{2n} intervals each of length 3^{-2n} of the unit interval. This correspondence may be established for each n in the customary way so as to "preserve inclusion." This process now yields a 1-1 correspondence between step functions in two dimensions defined over the planar grid and those in one dimension defined over the linear grid. This correspondence preserves integrals. It may be extended to limits of increasing sequences of step functions. With the help of this correspondence many problems in n dimensions are directly referable to the appropriate theorem in one dimension. Fubini's theorem on the reduction of a double to an iterated integral presents new phenomena. It is handled by noting the fact that it is trivial for step functions, then by appropriate passage to the limit.

Other definitions of the Lebesgue integral are considered. This leads to the proof of the theorems of Egoroff and Lusin.

The final chapter of Part I considers the Lebesgue-Stieltjes integral. First is proved the theorem of Riesz (so fundamental to all modern thinking on abstract integration) to the effect that every bounded linear functional on the space C of functions continuous on $a \leqq x \leqq b$ is of the form $\int_a^b f(x)d\alpha(x)$ where $\alpha(x)$ is of bounded variation—and conversely. This is done by extending the functionals from C to a larger class by precisely the same method used in extending the Lebesgue integral from the class C_0 of step functions to the classes C_1 and C_2 (see above). Questions of the uniqueness of $\alpha(x)$ and concerned with the convergence of functionals are considered. The Lebesgue-Stieltjes integral is sketched briefly from many points of view. Part I closes with a development of the Daniell integral. This is a linear functional defined by extension from a class C_0 of functions $\phi(x)$ defined on an abstract set. The important properties of C_0 are that it be an ordered linear space which is also a lattice (in other words, using the terminology of Bourbaki in his recent book on measure: C_0 est un espace de Riesz). The treatment of this integral throws penetrating light on the Lebesgue and the class of Lebesgue-Stieltjes integrals.

Part II considers first integral equations. This introduces all the phenomena subsequently needed in the study of general linear operators: boundedness, weak, strong, uniform convergence, and the elementary algebraic structure of the ring of operators including inversion and commutativity. If the kernel $K(x, y)$ belongs to L^2 on the square $a \leqq x, y \leqq b$, and gives rise to the operator K on the space L^2 of all functions $h(x)$, $a \leqq x \leqq b$: $Kh(x) = \int_a^b K(x, y)h(y)dy$, then a study is made of the relation between the element $K(x, y) \in \mathrm{L}^2$ and the operator K over L^2. A kernel of the form $\sum_1^n \phi_i(x)\psi_i(y)$ is said to be of finite rank. Any kernel $K(x, y)$ may obviously be approximated in the norm of L^2 by kernels of finite rank. Also, the mapping from kernels into transformations is an isomorphism (into).

Next we find a very thorough discussion of the Fredholm theory. Three methods are given. The first method considers to begin with kernels of finite rank. For these the complete theory is essentially a branch of elementary algebra. To proceed to the general case, it is shown that an arbitrary kernel $K(x, y)$ may be written in the form $K(x, y) = L(x, y) + [K(x, y) - L(x, y)]$ where $L(x, y)$ is of finite rank and the second term $H(x, y)$ is sufficiently small so that $(I-H)^{-1}$

exists. This method is due to E. Schmidt. The second method is the classic one of Fredholm and is sketched rather briefly.

The third method, due to Riesz, is of a geometric nature and applies to any completely continuous operator K. Let $T=I-K$. Let $\mathfrak{M}_n$ be the closed linear manifold of elements f for which $T^nf=0$. Then $(0)=\mathfrak{M}_0\subseteq\mathfrak{M}_1\subseteq\mathfrak{M}_2\subseteq\cdots$. It is shown that there is a first index ν after which equality holds at each step: $\mathfrak{M}_\nu=\mathfrak{M}_{\nu+1}=\cdots$. Now let $\mathfrak{N}_n$ be the range of T^n. Then it may be shown that $\mathfrak{N}_n$ is closed. Clearly $L^2=\mathfrak{N}_0\supseteq\mathfrak{N}_1\supseteq\mathfrak{N}_2\supseteq\cdots$. It may be shown that there is a first index μ after which equality holds at each step: $\mathfrak{N}_\mu=\mathfrak{N}_{\mu+1}=\cdots$. Finally $\nu=\mu$. In the case $\nu=\mu=0$, T has a bounded inverse. If $\nu>0$, $\mathfrak{M}_\nu+\mathfrak{N}_\nu=L^2$ the sum being disjoint, and the pair $(\mathfrak{M}_\nu, \mathfrak{N}_\nu)$ reduces T and hence K. From here to the proof of the Fredholm alternative is a brief step. This Fredholm theory is applied to the solution of Dirichlet and Neumann problems for smooth curves.

The following chapter (V) is devoted to the introduction of abstract Hilbert and Banach spaces. There is the obvious development on linear functionals, adjoint spaces, reflexive spaces, adjoint operators, bases, biorthogonal systems. There is also an interesting section on the form of completely continuous linear transformations in the space C of continuous functions.

The next four chapters are devoted to an exposition of the facts that are associated with symmetric transformations in Hilbert space. Chapter VI treats the completely continuous symmetric transformation. It is shown that if A is such a transformation and if $\{f_n\}$ is a sequence such that $\|f_n\|=1$ and $\|Af_n\|\to\|A\|$ (the norm of A) then there exist solutions to the equation $A\phi=\mu_1\phi$ where $|\mu_1|=\|A\|$. Once such a ϕ has been found, the process of determining the structure of A is based on the application of the same procedure to the subspace of the Hilbert space $\mathfrak{H}$ orthogonal to ϕ. Thus one obtains the sequence of characteristic values $\mu_1, \mu_2, \cdots$ with $|\mu_1|\geqq|\mu_2|\geqq\cdots$ and the associated characteristic elements $\phi_1, \phi_2, \cdots$ which form an orthonormal set. There is also a discussion of the direct determination of the nth characteristic value of A and of the relation existing between the characteristic values of A_1, A_2 and $A=A_1+A_2$. These results are due to Hilbert, Courant, and Weyl. The Hilbert-Schmidt theory of symmetric kernels is treated. (It should be pointed out that a good part of the discussion on completely continuous operators deals with incomplete Hilbert spaces.) It is then applied to two interesting problems. The first one concerns the vibrating string in a formulation given to it recently by Nagy. The second one gives a proof due to Weyl and Rellich of the fundamental theorem of H. Bohr on almost periodic functions to the effect that the generalized Fourier series associated to an almost periodic function $f(x)$ converges in the mean to $f(x)$.

Next (chapter VIII) comes the study of bounded symmetric, normal, and unitary operators. First is a construction and uniqueness proof for the positive square root of a positive symmetric operator. The spectral resolution of an operator A is led to by the necessary discussion of projections and functions of operators. Suppose $u(\lambda)$ is a function on $m \leqq \lambda \leqq M$ where m and M are the bounds of A, $mI \leqq A \leqq MI$; if $u(\lambda)$ may be approximated by a bounded increasing or decreasing sequence of polynomials then the function $u(A)$ may be defined. This process is closely analogous to that for extending the integral from the class C_0 of step functions to the class C_{I}—see the discussion given earlier. Here we proceed from the class of polynomials to the class of lower or upper semi-continuous functions. This gives the functions E_λ which occur in the resolution of the identity of A and lead to the formula $A = \int_{m-0}^{M} \lambda dE_\lambda$. A second proof of this formula consists in calculating $|A| = \sqrt{A^2}$. The orthogonal manifolds occurring in the projection E_0 are then obviously related to the range and null-manifold of $A^+ = 2^{-1}(A + |A|)$. Unitary and normal transformations are subsequently discussed. There is a section devoted to the Fourier-Plancherel theorem and to Watson transforms.

Now comes a chapter on unbounded transformations. There is first a discussion of the algebra of transformations relative to domains of definition; thus we find a collection of facts concerning $T_1 + T_2$, $T_1 T_2$, T^*; for example $(T_1 T_2)^* \subseteq T_2^* T_1^*$. Clearly this type of ground clearing is necessary but it leaves a strange taste in its wake. One feels that this material is being treated in a form which is not final; the definition of the permutability of T with a bounded B is a case in point: $BT \subseteq TB$. As soon as one turns to the graph of a closed transformation one feels much happier. The collection of results on this subject includes the striking theorem that for a closed transformation T, the operators $B = (I + T^*T)^{-1}$ and $C = T(I + T^*T)^{-1}$ are bounded and B is positive definite self-adjoint. Examples are given (differential operators) of transformations which are symmetric but not self-adjoint. The first demonstration here given of the integral representation of a self-adjoint transformation based on the above transformation B and C with $T = T^*$ is that of Riesz and Lorch. The second is the original one of von Neumann and transforms the given problem to one on unitary transformations by means of the Cayley transform. The Cayley transform is studied in the case of symmetric non self-adjoint operators. This leads to the notion of the deficiency index. There is given a proof due to Friedrichs of the theorem that a semi-bounded symmetric transformation with lower bound c, $cI \leqq S$, may be extended to a self-adjoint transformation A such that $cI \leqq A$. The more general methods of Krein are discussed in some detail. The problem here is: Let S be symmetric and semi-bounded $cI \leqq S$; let $\gamma \leqq c$. To determine all self-adjoint A such that $A \supseteq S$ and $\gamma I \leqq A$.

Chapter IX begins with a thorough discussion of the calculus based on a self-adjoint transformation A. This calculus assigns a transformation $u(A)$ to a function $u(\lambda)$ measurable with respect to $(E_\lambda f, f) = \|E_\lambda f\|^2$ by means of the formula

$$(u(A)f, g) = \int_{-\infty}^{\infty} u(\lambda)d(E_\lambda f, g).$$

Each such function of A has the property that it permutes with every bounded symmetric transformation which permutes with A,—in symbols $u(A) \smile\smile A$. The converse theorem essentially due to von Neumann is established: If T is a closed transformation for which $T \smile\smile A$, then T is a function $u(A)$ of A providing the Hilbert space is separable. Another theorem of the same author is demonstrated: If $\{X_n\}$ is a denumerable set of permutable self-adjoint transformations there exists a self-adjoint A such that each X_n is a function of A: $X_n = x_n(A)$. From this point forward to the end, the character of the book changes somewhat. Whereas previously all proofs had been given in detail, from now on, the reader is referred to the literature for a complete discussion of some of the stated results.

There is no discussion of spectral multiplicity. We greet its absence like that of a very good friend who is quite demanding and hard to take. The chapter ends with a discussion of the perturbation of the spectrum of a self-adjoint operator. Proof is given of Rellich's theorem: If $A_n \rightarrow A$ (strongly) then for the corresponding resolutions of the identity, $E_{n\lambda} \rightarrow E_\lambda$ at every point of continuity of E_λ. This device, used in a particular case, led to the first proof of the spectral resolution of A given by Hilbert. For analytic perturbations of A, the following theorem of Rellich and Nagy is proved: Let $A(\epsilon) = A + \epsilon A^{(1)} + \epsilon^2 A^{(2)} + \cdots$ be a continuous family of self-adjoint operators. Suppose (μ_1, μ_2) is a real interval having only one point $\lambda^{(0)}$ in the spectrum of $A = A(0)$, and that of finite multiplicity m. Then for ϵ sufficiently small there are $2m$ series

$$\begin{aligned} \lambda_i(\epsilon) &= \lambda^{(0)} + \epsilon\lambda_i^{(1)} + \epsilon^2\lambda_i^{(2)} + \cdots, \\ \phi_i(\epsilon) &= \phi_i^{(0)} + \epsilon\phi_i^{(1)} + \epsilon^2\phi_i^{(2)} + \cdots, \end{aligned} \qquad (i = 1, \cdots, m)$$

representing respectively characteristic values and orthogonal characteristic functions of the operator $A(\epsilon)$. That is, the characteristic values and vectors are analytic functions of ϵ. The applications of this theorem to situations in physics are obvious.

The penultimate chapter is devoted to groups and semi-groups of transformations. First comes an elegant proof (due to Nagy) of the theorem of Stone on strongly continuous one-parameter groups of unitary transformations U_t in Hilbert space—the theorem asserting that $U_t = \exp(iAt)$ where A is self-adjoint. A second demonstration based on Bochner's representation of functions of positive type by Stieltjes integrals follows. An immediate application is the mean ergodic theorem of von Neumann. Up to this point, the preceding third of the book has been devoted exclusively to operators in Hilbert space and indeed to normal (including the self-adjoint and unitary) ones: Now the authors branch out to the study of operators over Banach spaces. There are proofs of two theorems of Hille. One asserts the existence on a dense manifold of the infinitesimal generator A of a strongly continuous group T_t of bounded transformations: $A = \lim_{h \to 0} A_h$ where $A_h = h^{-1}(T_h - I)$. Although an exponential representation of T_t is impossible (see unitary case above), the second theorem asserts: $T_t f = \lim_{h \to 0} \exp(tA_h)f$. An intriguing application of this yields the Weierstrass approximation theorem for continuous functions. We turn to ergodic theorems for which the additive semi-group is the set of positive integers. For unitary transformations the elegant proof of Riesz is set forth. The extension of this by Lorch to bounded semi-groups in reflexive spaces: $\|U^k\| \leq C$, $k = 1, 2, \cdots$, follows. We mention lastly a type of argument originating with G. Birkhoff and applied by F. Riesz to uniformly convex spaces.

The principal section of the last chapter (XI) is devoted to the development of spectral analysis in general Banach spaces by an extension of the Cauchy calculus of residues. Briefly, the extension is carried out as follows: If T is an arbitrary bounded linear transformation, the points z in the complex plane for which $R_z = (T - zI)^{-1}$ exists constitute an open set—the resolvent set. The complementary set, the spectrum $\sigma(T)$ of T, is closed and bounded. Suppose C is simple closed rectifiable curve lying in the resolvent set, then $\int_C R_z dz$ has a meaning in the uniform topology. Furthermore, since R_z is an "analytic function" of z—in accordance with a classical equation: $R_z = R_\zeta + (z - \zeta)R_\zeta^2 + (z - \zeta)^2 R_\zeta^3 + \cdots$—the above integral is unchanged if the curve C is altered slightly. Thus the value of the integral depends only on the portion of the spectrum of T which lies interior to C. It is clear that one may also consider an integral

$\int_C z^n R_z dz$ and that this is equal to $T^n \int_C R_z dz = \int_C T^n R_z dz$ since $(T^n - z^n I)R_z$ is a polynomial in T and z whose integral around C is zero. Consideration of these integrals reveals a remarkable collection of facts concerning the structure of T. The existence of this method was recognized early by Riesz. The times seem to have been so unprepared for it that in all he devoted to it only too brief pages (pp. 117–119) in his book: *Les systèmes d'équations linéaires à une infinité d'inconnues* (1913). For various reasons, this material was soon forgotten. In part, this was due to the fact just cited. In part, events conspired to focus all attention on the symmetric case in Hilbert space for which the present theory was not then needed. It is probable that certain annoying misprints in Riesz' book adjoined to the summary exposition which is there found contributed to the complete disappearance of this theory here presented in germ form. The fact is that in the next quarter century no mention of this material is to be found in the literature of its day.

The interest of the present writer in extending reducibility phenomena from Hilbert space first to reflexive spaces and subsequently to the most general Banach space led to his rediscovering Riesz' earlier work in the form which is presented in the present book. The essential facts are these. For a simple curve C lying in the resolvent set of T, the transformation $P = -(2\pi i)^{-1} \int_C R_z dz$ is a projection, $P^2 = P$, which reduces T, $PT = TP$. Thus the manifolds $\mathfrak{M}$ and $\mathfrak{N}$ determined by P reduce T. The spectrum of T in $\mathfrak{M}$ (where $P(\mathfrak{M}) = \mathfrak{M}$) consists of that part of the spectrum of T lying interior to C. $P = 0$ if and only if the interior of C contains no points of the spectrum. $P = I$ if and only if the entire spectrum of T lies interior to C.

An important and interesting relation exists between the norm $\|T\|$ of T and the spectrum T. Let $|z| \leqq r_T$ be the smallest disc which contains the spectrum $\sigma(T)$ of T. The number r_T is called the spectral radius of T. Then it is easy to see that $\|T\| \geqq r_T$. Since, as follows from an elementary calculation, $\sigma(T^n) = [\sigma(T)]^n$, $\|T^n\| \geqq r_T^n$. Now, in fact, the following theorem holds: $\lim_{n \to \infty} \|T^n\|^{1/n} = r_T$. This theorem was first stated in the present general form by Gelfand in his paper on normed rings (1941). A special case concerned with ring of absolutely convergent trigonometric series had been elaborated by Beurling three years earlier.

This so-called *spectral radius theorem* is a simple consequence of a fundamental identity which appears in the work of this reviewer that deals with the calculus of residues and has already been mentioned above. It is there proved that if T is a linear transformation having no singularities on the unit circle C: $|z| = 1$, then for the projection P

given by the contour integral $P = -(2\pi i)^{-1}\int_C R_z dz$, we have (in the uniform topology)

$$P = \lim_{n\to\infty} (I - T^n)^{-1}.$$

The facts which concern us here may be set down in the form of a

THEOREM. *The following hypotheses concerning* T *are equivalent*:
(a) *The spectrum of* T *lies interior to the circle* $|z| = 1$, *that is* $r_T < 1$.
(b) $T^n \to 0$ (*uniformly*).
(c) *The* $\lim_{n\to\infty} \|T^n\|^{1/n}$ *exists, equals* r_T, *and is less than* 1.

PROOF. Assume (a). If $r_T < 1$, then $P = I$ (see a remark made three paragraphs above). Thus $I = \lim_{n\to\infty} (I - T^n)^{-1}$; hence

$$I = \lim_{n\to\infty} (I - T^n) \text{ or } \lim_{n\to\infty} T^n = 0.$$

This gives (b).

Assume (b). If $T^n \to 0$, then $\|T^n\| \to 0$ and thus lim sup $\|T^n\|^{1/n} \leqq 1$. Now, clearly, the spectral radius $r_T^{-1}T$ is 1 and for $\epsilon > 0$, that of $S = (r_T + \epsilon)^{-1}T$ is less than 1. Thus applying the results just obtained for T to the transformation S, we have lim sup $\|S^n\|^{1/n} \leqq 1$. This implies since ϵ is arbitrary that lim sup $\|T^n\|^{1/n} \leqq r_T$. Since, as mentioned earlier, $\|T^n\|^{1/n} \geqq r_T$, we have lim inf $\|T^n\|^{1/n} \geqq r_T$. Combining these two results yields lim $\|T^n\|^{1/n} = r_T$. This is (c).

Now assume (c). We have immediately, $\|T^n\| \to 0$. Since for the spectra of T and T^n, $[\sigma(T)]^n = \sigma(T^n)$, it follows that $[\sigma(T)]^n \to 0$. Hence the spectrum of T lies in the interior of the circle $|z| = 1$. This is (a). This completes the proof of our theorem.

We return to an examination of the present book. The work of Beurling on absolutely convergent series is developed in accordance with his methods. As a corollary one obtains the theorem of Wiener: If a periodic function $f(x)$ has an absolutely convergent Fourier series and does not vanish, then $1/f(x)$ has the same property. There is a section devoted to an operational calculus following the ideas of Riesz, Gelfand, and Dunford. If $u(z)$ is holomorphic on a domain which contains the spectrum of T, and is bounded by a curve C, then the integral $(2\pi i)^{-1}\int_C u(z)R_z dz$ defines the function $u(T)$.

The book terminates with a discussion of the latest work of von Neumann on spectral sets. This theory applies to bounded transformations in Hilbert space. The principal theorem follows: Let C_1 be the disc $|z| \leqq 1$. Suppose $u(z)$ is holomorphic in C_1 and such that $|u(z)| \leqq 1$ in C_1. Then for any T with $\|T\| \leqq 1$, $\|u(T)\| \leqq 1$. A spectral set Z of a transformation T is now defined to be a closed set of the extended complex plane such that for all rational functions $u(z)$ satisfying $|u(z)| \leq 1$ on Z, the transformation $u(T)$ exists and $\|u(T)\| \leq 1$. It is proved that unitary and symmetric transformations are characterized by the fact that they have respectively the unit circle and the real axis as spectral sets.

E. R. LORCH

Über die Klassenzahl abelscher Zahlkörper. By Helmut Hasse. Berlin, Akademie, 1952.

This is a highly technical book, whose object is the derivation of a formula for the class-number h of an arbitrary absolute abelian field K and the study of this formula. Such a formula had been proved by Kummer for cyclotomic fields (i.e. fields generated by a root of unity) and in the general case by several authors (Fuchs, Beeger, Gut). The source of these formulae is of course the fact that the class number appears in the expression of the residue at $s=1$ of the zeta function $\zeta_K(s)$ of K. Using the product decomposition of ζ_K into L-series, one is reduced to the computation of the values $L(1; \chi)$ at 1 of the L-series corresponding to those characters $\chi \neq 1$ which are associated to K by class field theory. The numbers $L(1; \chi)$ appear as infinite series; the main problem is to express them in closed form, which is done by making use of Gaussian sums.

The resulting formula appears in the form $h=h_0h^*$, where h_0 is the class-number of the maximal real subfield K_0 of K, while h^*, the "second factor" of h, turns out to be an integer >0. The fact that these two factors h_0 and h^* are actually integers is not obvious from the expressions for these numbers which appear in the formula itself. One of the aims of the author is to transform these expressions in such a way as to render their arithmetic nature more apparent. This in itself would not appear so very fascinating a task: when we express the number of zeros of an analytic function in a region by a contour integral, we do not take pains to establish independently that the value of this integral is an integer. However, in the process of so doing, new properties of h_0 and h^* appear which lead to a certain number of new results on class numbers of fields.

The second chapter of the book is concerned with the transformation of the expression for h_0. Here the striving to obtain for h_0 an expression which exhibits it as an integer is not entirely successful. Two different lines of attack are followed which yield results for two different kinds of fields K. The end results of the two methods are in the following form: the product of h by some integer c is expressed as the index in the group of all units of a certain sub-group generated by explicitly given units. The number c is 1 for certain categories of fields. As a consequence of the formulas he obtains, the author obtains generalizations of Weber's theorem to the effect that the largest real subfield of the field of 2^νth roots of unity has an odd class number; for instance, h_0 is odd if K_0 is cyclic of degree 2^r and if its conductor is a power of a prime.

The third chapter is concerned with the second factor h^*, which occurs of course only in the case where K is imaginary. It is first proved by class-field theory that h^* must be an integer. This being done, the author concentrates on the study of the expression of h^*. One of the factors which occur in it is the index in the group of units of K of the subgroup generated by the units of K_0 and by the roots of unity of K. This index Q is always 1 or 2, but whether it is 1 or 2 cannot be ascertained by the méthods of class field theory alone: this is a question which involves some non-abelian features. The author gives various criteria to compute Q. Next, he succeeds in the case of h^* in giving an expression of this number which exhibits it as an integer. As a result, the author obtains the following generalizations of Weber's theorem: if the conductor of K is a power of 2, then h is odd—if the conductor of K is a power of 3, then h^* is not divisible by 3. It is known since Kummer that, for the field of pth roots of unity, p a prime >2, then h can only be divisible by p or by 2 when h^* itself is. It is proved that the part of this statement concerning the divisibility by 2 extends to the case of arbitrary cyclic imaginary fields K; moreover, necessary and sufficient conditions for h to be odd are given in that case.

The book concludes with a table of values of h^* for all fields of conductors <100, and of some other quantities relative to these fields.

The amateur of special concrete arithmetical facts will be highly gratified by the reading of this book, which abounds in various examples. However, it must be admitted that the extraordinary ingenuity displayed by the author in the derivation of these results can only confirm the impression that class numbers behave in a rather chaotic fashion or, at any rate, that if they are governed by general laws, then we have as yet no inkling of what these laws may be like.

C. CHEVALLEY

NOTES

A fund has been established to endow a scholarship at Newcomb College in memory of Marie J. Weiss.

The first two volumes of the Collected Mathematical Works of Harald Bohr have been published by the Danish Mathematical Society with the support of the Rask-Orsted Foundation and the Carlsberg Foundation; the third volume will appear shortly. The three volumes contain about 1000 pages each, and are edited by Erling Følner and Børge Jessen. They may be obtained, bound in buckram, at a total price of 110 Danish crowns ($15.90), post free, from the Danish Mathematical Society, Institute of Mathematics, University of Copenhagen, Blegdamsvej 15, Copenhagen.

A conference on functions of a complex variable, with emphasis on topological methods, will be held from June 17 to June 30, 1953, in Ann Arbor under the auspices of the Department of Mathematics of the University of Michigan. About a dozen mathematicians from the United States and abroad will deliver invited addresses. There will also be opportunities for short contributed papers. The 1953 Summer Session at the University of Michigan will give special attention to complex variable theory in the form of courses and seminars. Information can be obtained by writing to Professor W. Kaplan, 274 West Engineering Building, Ann Arbor, Michigan.

The Fourth Symposium on Plasticity will be held at Brown University, Providence, Rhode Island, September 1–3, 1953. Persons interested in receiving further information should write to Professor H. J. Weiss, Graduate Division of Applied Mathematics, Brown University, Providence 12, Rhode Island.

Professor Otto Neugebauer of Brown University has been awarded the Dannie Heineman Prize of $5000 for his book "The Exact Sciences in Antiquity."

Professor Lamberto Cesari of Purdue University has been awarded a National Science Foundation Grant.

Professor Claude Chevalley of Columbia University has been elected to membership in the National Academy of Sciences.

Professor Einar Hille of Yale University is on leave in France and has been awarded a Guggenheim Fellowship and a Fulbright Award.

Professor Witold Hurewitz of the Massachusetts Institute of Technology has been awarded a Fulbright Grant and is on leave of absence at the University of Paris.

Professor C. O. Oakley of Haverford College is on leave of absence and has received a Ford Foundation Grant.

Professor Moses Richardson of Brooklyn College has been awarded a Ford Foundation Fellowship and is on leave of absence at Princeton University.

Assistant Professor E. H. Spanier of the University of Chicago has been awarded a Guggenheim Fellowship.

Professor André Weil of the University of Chicago has been awarded a Guggenheim Fellowship.

Professor G. T. Whyburn of the University of Virginia has been awarded a Fund for the Advancement of Education Grant and is on leave of absence at Stanford University.

Dr. Benjamin Bernholtz of the University of Connecticut has accepted a position as mathematician with the Hydro-Electric Power Commission of Ontario, Toronto, Ontario, Canada.

Dr. A. A. Brown of the Operations Evaluation Group, Massachusetts Institute of Technology, has accepted a position as staff member with Arthur D. Little, Inc., Cambridge, Massachusetts.

Mr. B. M. Brown has accepted a position as applied science representative with the International Business Machines Corporation, New York, New York.

Professor R. H. Cameron of the University of Minnesota will be on leave of absence during the academic year 1953–1954 at the Institute for Advanced Study.

Dr. R. E. Carr of the North American Aviation Company has accepted a position as head of the Computer Operations Group, Jet Propulsion Laboratory, California Institute of Technology.

Visiting Associate Professor K. L. Chung of Cornell University has been appointed to an associate professorship at Syracuse University.

Professor J. A. Dieudonné of the University of Nancy is on leave of absence and has been appointed to a visiting professorship at Northwestern University.

Dr. D. L. Fuller of the General Aniline and Film Corporation has accepted a position as technical director with the American Cyanamid Company, New Orleans, Louisiana.

Mr. J. H. Hoelzer of Williams College has been appointed to an associate professorship at the Tennessee Polytechnic Institute.

Mr. O. H. Hoke of the Naval Air Test Center, Patuxent River, Maryland, has accepted a position as mathematician at the White Sands Proving Ground, New Mexico.

Mr. B. F. Howard of Johns Hopkins University has accepted a position as assistant research engineer with the Bendix Radio Corporation, Towson, Maryland.

Mr. C. Y. Lee of the University of Washington has accepted a position as member of the technical staff with the Bell Telephone Laboratories. Inc., New York, New York.

Mr. W. E. Moore of Johns Hopkins University has accepted a position as mathematician with the Redstone Arsenal, Huntsville, Alabama.

Assistant Professor E. D. Nering of the University of Minnesota will be on leave of absence during the academic year 1953–1954 and has accepted a position at the Goodyear Aircraft Corporation, Litchfield Park, Arizona.

Associate Professor Ruth E. O'Donnell of Duquesne University has accepted a position as senior scientist with the Westinghouse Electric Corporation, Pittsburgh, Pennsylvania.

Mr. B. E. Owens of Yale University has accepted a position as assistant project engineer with the Sperry Gyroscope Company, Great Neck, New York.

Dr. L. G. Peck of the Los Alamos Scientific Laboratory is on leave of absence at the Institute for Advanced Study and New York University.

Associate Professor Murray Rosenblatt of the University of Chicago is on leave of absence at the Institute of Mathematical Statistics, Stockholm, Sweden.

Dr. W. J. Schart of the Consolidated Vultee Aircraft Corporation has accepted a position as design specialist with the Electronics Laboratory, Convair, San Diego, California.

Professor W. H. Sellers of Aldeson-Broaddus College has accepted a position as flutter and vibration technician with Fairfield Aircraft, Hagerstown, Maryland.

Mr. David Singer of Corbett-Tinghir and Company, New York, has accepted a position as structural research engineer with the Armour Research Foundation, Illinois Institute of Technology.

Mr. Judson C. Smith of the University of Washington has accepted a position as mathematician at the Naval Ordnance Test Station, China Lake, California.

Assistant Professor C. T. Taam of the University of Missouri has been appointed to an assistant professorship at the Catholic University of America.

The following promotions are announced:

A. F. Bausch, University of Chicago, to an assistant professorship.
Raoul Bott, University of Michigan, to an assistant professorship.
Harold Grad, New York University, to an associate professorship.
Philip Hartman, Johns Hopkins University, to a professorship.
Banesh Hoffmann, Queens College, to a professorship.

Eugene Isaacson, New York University, to an associate professorship.

Peter Lax, New York University, to an associate professorship.

W. W. Leutert, Ballistic Research Laboratories, to chief of the Computing Laboratory.

R. D. Murphy, Equitable Life Assurance Society of the United States, to President.

David Nelson, George Washington University, to an associate professorship.

I. E. Segal, University of Chicago, to an associate professorship.

H. N. Shapiro, New York University, to an associate professorship.

H. L. Turrittin, University of Minnesota, to a professorship.

The following appointments to instructorships are announced: University of Maryland: Mr. Sol Schwartzman; University of Nebraska: Mr. R. A. Moore, Dr. M. F. Ruchte; University of Oklahoma: Mr. D. W. DuBois; Pennsylvania State College Extension Center, Altoona, Pennsylvania: Mr. F. H. Pott; Tulane University of Louisiana: Mr. R. E. Allan, Dr. A. L. Shields; University of Wisconsin Extension Center, Wausau, Wisconsin: Mr. P. W. Gillett; Yale University: Dr. J. T. Schwartz.

Professor Emeritus H. C. Van Buskirk of the California Institute of Technology died on November 21, 1952 at the age of eighty years. He had been a member of the Society for thirty-one years.

The note concerning E. W. Barankin in the March issue of the Bulletin is in error, and should read as follows:

Associate Professor E. W. Barankin of the University of California, Berkeley, is on leave of absence and has accepted a position as Research Mathematician at the Institute for Numerical Analysis, National Bureau of Standards, Los Angeles, for the current academic year.

The following two hundred and seven doctorates, with mathematics, mathematical physics, or statistics as a major subject, were conferred during 1952 in universities in the United States and Canada. The university, month in which degree was conferred, minor subjects (other than mathematics), and title of the dissertation are given in each case if available.

I. T. A. C. Adamson, Princeton, June, *The cohomology structure of formations.*

José Adem Chahin, Princeton, June, *Iteration of the squaring operations in algebraic topology.*

A. C. Allen, Princeton, June, *Some theorems on harmonic functions and related topics.*

A. G. Anderson, Michigan, February, *The prediction of quantitative characteristics in polygenic systems.*

J. W. Armstrong, Purdue, June, minor in physics, *Lagrange and Hermite interpolation in several variables.*

E. L. Arnoff, California Tech., June, minor in physics, *Partial differential and difference equations.*

B. J. Ball, Texas, June, *Concerning continuous and equicontinuous collections of arcs.*

J. B. Bartoo, Iowa, June, minor in astronomy, *Certain theorems on order statistics.*

W. F. Bauer, Michigan, February, *Modified Sturm-Liouville problems and associated integral transforms.*

R. L. Beinert, Cornell, September, *Abstract parametrization of an elliptic curve.*

C. A. Bennett, Michigan, June, *Asymptotic properties of ideal linear estimators.*

Benjamin Bernholtz, California Tech., June, minor in physics, *Complex function theory for functions with arguments and values in locally convex linear topological spaces.*

R. L. Blair, Iowa, August, minor in astronomy, *Ideal lattices and the structure of rings.*

I. E. Block, Harvard, March, *Singular integrals and the decomposition of functions of summable square.*

E. K. Blum, Columbia, June, *Theory of analytic functions in Banach algebras.*

V. W. Bolie, Iowa State, March, minor in physics, *Periodic orbits in the neighborhood of libration points in certain rotating systems.*

W. W. Boone, Princeton, June, *Several simple, unsolvable problems of group theory related to the word problem.*

J. J. Brandstatter, California, Los Angeles, April, *The algorithm of Schwarz in the mathematical theory of elasticity of an anisotropic solid for doubly-connected domains.*

J. P. Brewster, Duke, June, minor in education, *Initial conditions for differential systems.*

J. A. Bronzo, Brown, May, *On the propagation of elastic waves into a semi-infinite solid.*

A. R. Brown, Jr., Yale, June, *Non-uniqueness of the initial value problem for a class of generalized heat equations.*

H. A. Brown, Chicago, August, *Two classes of non-normal operators.*

R. K. Brown, Rutgers, June, *Some mapping properties of the mean sums of power series and a problem in typically-real functions of order p.*

R. J. Buehler, Wisconsin, February, minor in physics, *On functions* *satisfying certain differential inequalities.*

C. W. Burrill, Iowa, June, minor in astronomy, *Characterising systems of continuous functions.*

D. W. Bushaw, Princeton, June, *Differential equations with a discontinuous forcing term.*

R. K. Butner, Iowa, June, minor in astronomy, *Polynomials associated with certain differential equations.*

R. K. Butz, Georgia, June, minor in physics, *The minimum function of a matrix B relative to a matrix A.*

Jean M. Calloway, Pennsylvania, June, *On the discriminant of arbitrary algebraic number fields.*

W. H. W. Chen, Washington, June, minors in electrical engineering and physics, *On the singular points of the homogeneous ordinary differential equation.*

Fred Chong, Iowa State, March, minor in physics, *Solutions by dual integral equations of mixed boundary value problems in elasticity.*

Joshua Chover, Michigan, June, *On homogeneous measure and operator decompositions of Hilbert space.*

I. J. Christopher, Oregon, June, *The asymptotic density of integers in more than one dimension.*

W. H. Clatworthy, North Carolina, August, *Partially balanced incomplete block designs with $r<k$.*

Haskell Cohen, Tulane, August, *Codimension.*

H. R. Coish, Toronto, June, *Theory of internal conversion—exact solution.*

H. S. Collins, Tulane, August, *Completeness and compactness in linear topological spaces.*

Paolo Comba, California Tech., June, minor in aeronautics, *Geometry of finite dimensional moment spaces and applications to orthogonal polynomials.*

R. M. Conkling, Florida, June, minor in education, *On metric algebras.*

K. L. Cooke, Stanford, January, *The asymptotic behavior of the solutions of linear and non-linear differential difference equations.*

B. V. Dean, Illinois, June, minor in electrical engineering, *Near rings and their isotopes.*

R. Y. Dean, California Tech., June, minor in physics, *Laplace transform in commutative Banach algebras.*

G. P. Dineen, Wisconsin, June, *The exponential function of a variable matrix.*

Jim Douglas, Jr., Rice, June, I. *Two equivalence methods for sequences of random variables*, II. *The expansion of measures in series of orthogonal measures.*

Lloyd Dulmage, Toronto, June, minor in relativity, *The relation between continuity and angle sum in hyperbolic geometry.*

L. K. Durst, California Tech., June, minor in physics, *Apparition and periodicity properties of equianharmonic divisibility sequences.*

Meyer Dwass, North Carolina, August, *On the large sample power of certain rank order tests.*

S. E. Dyer, Texas, June, *Certain conditions under which the sum of the elements of a continuous collection of continua is an arc.*

W. G. Dyer, Iowa State, December, minors in theoretical and applied mechanics, *Approximate solutions of boundary value problems by* (1) *minimization of least square error, by* (2) *use of finite difference.*

Victor Elconin, California Tech., June, minor in physics, *Transformation of linear spaces and linear operators by inverse reversion.*

J. W. Ellis, Tulane, August, *Complementary spaces of Cartesian products of groups and linear spaces.*

M. P. Emerson, Illinois, June, minor in education, *Dualities of modular lattices.*

J. C. English, Florida, August, minor in physics, *Stress concentrations in infinite orthotropic plates.*

E. R. Fadell, Ohio State, March, *Unessential identifications in singular homology theory.*

A. M. Feyerherm, Iowa State, July, *Distribution of Kronecker products.*

G. B. Findley, Florida, June, minor in physics, *The effect of certain types of small holes on the stresses in a uniformly loaded isotropic plate.*

W. T. Fishback, Harvard, March, *On the theorem of Bertini on the variable singular points of a linear system.*

R. C. Fisher, Kansas, October, minor in physics, *A study of line congruences by means of a modified tensor calculus.*

I. B. Fleischer, Chicago, August, *Locally symmetric spaces.*

Bernard Fleishman, New York, June, *On the periodic solutions to an initial-value problem for a Duffing-type non-linear wave equation.*

W. C. Foreman, Kansas, June, minor in physics, *Generalized vectors and the matrix equation* $X^{(k)}=B$.

J. F. Foster, Jr., Iowa, August, minor in astronomy, *A study of errors in visual methods of measurement of the velocities of meteors.*

K. A. Fowler, Michigan, February, *Investigations on finite groups of even order.*

N. S. Free, California, Berkeley, September, *On the embedding of Lipschitzian manifolds.*

J. E. Freund, Pittsburgh, August, *Some methods of estimating prior probabilities from heterogeneous populations.*

Morris Friedman, Chicago, December, *Problems in the calculus of* *variations for which the second variation vanishes.*

K. D. Fryer, Toronto, June, *A class of permutation groups of prime degree.*

F. B. Fuller, Princeton, January, *The homotopy theory of coincidences.*

L. D. Gates, Jr., Iowa State, July, minor in physics, *Differential equations in the distribution of Schwartz.*

S. G. Ghurye, North Carolina, August, *Some problems in the theory of stochastic difference equations.*

F. J. Ginivin, Notre Dame, June, *Natural topologies in a group with applications in the Galois theory of infinite extensions.*

W. L. Gordon, Tulane, August, *On the coefficient group in cohomology.*

R. L. Graves, Harvard, March, *The Fredholm theory in Banach spaces.*

Franklin Graybill, Iowa State, June, *On quadratic estimates of variance components.*

L. W. Green, Yale, June, *Manifolds without conjugate points.*

Simon Gruenzweig, Pittsburgh, August, *Affine and non-affine projective transformations in four-space which leave invariant the family of surfaces representing regular functions of a complex variable.*

E. L. Griffin, Jr., Chicago, June, *On semifinite rings.*

W. C. Guenther, Washington, December, *On testing whether or not a given percentile of one distribution is less than or equal to a given percentile of another distribution.*

V. H. Haag, Pittsburgh, August, *On the summability of Fourier series by regular circle methods.*

Felix Haas, Massachusetts Tech., June, minor in physics, *Theorem about characteristics of differential equations on closed manifolds.*

G. J. Habetler, Carnegie, June, *"$N+4$" dimensional relativity.*

B. F. Hadnot, Georgia, August, minor in physics, *Summability of series of the form* $\sum c_n g(s+n)$.

Mary-Elizabeth Hamstrom, Texas, June, *Concerning webs in the plane.*

C. B. Hanneken, Illinois, October, minor in education, *Irreducible quintic congruences.*

C. D. Harris, Carnegie, June, *On the occurrence of unconstrained plastic flow in a Prandtl-Reuss and Mises bodies.*

R. M. Hayes, California, Los Angeles, June, *Iterative methods of solving linear problems on Hilbert space.*

Emilie V. Haynsworth, North Carolina, June, *Bounds for determinants with dominant main diagonal.*

D. G. Higman, Illinois, June, minor in philosophy, *Focal series in finite groups.*

A. T. Hind, Jr., Georgia, August, minors in physics and education, *Concerning the solution of certain partial difference equation boundary value problems and the convergence of approximate solutions of the vibrating membrane problem to its exact solution.*

W. M. Hirsch, New York, February, *On the maximum cumulative sum of independent random variables.*

J. H. Hornback, Illinois, February, minor in education, *Integral equations related to the representation of functions by potentials.*

H. M. Horwitz, Ohio State, December, *On the homotopy axiom in general cohomology theory.*

J. R. Jackson, California, Los Angeles, May, *Abstract function spaces and their homotopy theory.*

A. T. James, Princeton, October, *Group methods in normal multivariate distribution theory.*

T. A. Jeeves, California, Berkeley, June, *Identifiability and almost-sure estimability of linear structures in n-dimensions.*

W. E. Jenner, Michigan, June, *Block ideals and arithmetics of algebras.*

Hyman Kamel, Pennsylvania, February, *Relational algebra.*

S. T. Kao, Catholic, June, minors in physics and chemistry, *A further generalization of the theorem of Droz-Farny.*

A. F. Kay, Harvard, March, *Methods in the location of zeros of families of polynomials of unbounded degree in circles, sectors, and other regions.*

J. C. Kiefer, Columbia, May, *Contributions to the theory of games and statistical decision functions.*

L. A. Kokoris, Chicago, August, *New results in power-associative algebras.*

Adnah Kostenbauder, Syracuse, September, *Multiplicative semigroups of continuous functions.*

Jack Kotik, Massachusetts Tech., February, minor in language, *Existence and uniqueness theorems for water waves.*

M. D. Kruskal, New York, February, *The bridge theorem for minimal surfaces.*

R. H. Kyle, Princeton, June, *Branched covering spaces and the quadratic forms of knots.*

Pedro Laborde-Montaner, New York, February, *On equivalent formulations of a generalized prime number theorem for arithmetic progressions.*

M. D. Lamoree, Purdue, June, minor in physics, *On the Bosanquet-*

Linfoot summability of the successively derived and conjugate Fourier series.

J. R. Larkin, Kansas, June, minor in physics, *Normal determinants and expansions in modified sequences.*

Solomon Leader, Princeton, June, *Set function spaces and orthogonal expansions of finitely additive set functions.*

P. J. Leah, Toronto, November, minors in quantum and statistical mechanics, *Multiplicators for moment sequences.*

L. M. LeCam, California, Berkeley, June, *On some asymptotic properties of maximum likelihood estimates and related Bayes' estimates.*

S. L. Levy, Brown, May, *The influence of magnetization on ultrasonic attenuation in some ferromagnetic single crystals.*

C. H. Lindahl, Iowa State, March, *Overlapping Pfaffians with application to utility theory.*

G. R. Livesay, Illinois, June, minor in philosophy, *An extension of the notion of unicoherence with an application to mappings of spheres.*

A. E. Livingston, Oregon, June, *Some Hausdorff means which exhibit the Gibbs' phenomenon.*

J. M. A. Maranda, McGill, October, *On p-adic integral representations of finite groups.*

E. W. Marchand, Rochester, June, *Spherical trigonometry in n-dimensional space.*

L. F. Markus, Harvard, March, *A topological analysis of differential equations in the large.*

E. W. Martin, Jr., Ohio State, August, *Representations of lattices.*

S. H. Maslen, Brown, May, *Supersonic conical flow.*

R. V. Mendenhall, Ohio State, June, *On Lebesgue measure and integration in an abstract space.*

M. R. Mickey, Jr., Iowa State, July, *An application of sequential tests to a problem of quality control.*

H. D. Mills, Iowa State, July, *Equivalent functions of strategies in the theory of games.*

J. C. Moore, Brown, May, *Some applications of homology theory to homotopy problems.*

J. T. Moore, Chicago, June, *Primary central division algebras.*

J. C. Morelock, Florida, August, minor in education, *Invariants with respect to special projective transformations.*

E. R. Mullins, Jr., Illinois, October, minor in philosophy, *A straight line plane with preassigned circles.*

W. M. Myers, Jr., Ohio State, March, *A functional associated with a continuous transformation.*

D. M. Nead, Indiana, June, minor in physics, *A theorem on the* *limit passage, viscosity $\mu \rightarrow 0$ heat conduction $K \rightarrow 0$, in the solutions of the equations of fluid flow.*

Morris Newman, Pennsylvania, June, *A structure theorem for certain modular subgroups with applications to the construction of modular identities.*

T. A. Newton, Georgia, June, minor in philosophy, *A study of De La Vallée Poussin's theorem and polynomials associated with the Hölder mean.*

F. I. N. Niordson, Brown, May, *Transmission of shock waves in thinwalled cylindrical tubes.*

R. S. Novosad, Chicago, August, *Relations between homotopy and homology groups.*

E. E. Osborne, California, Los Angeles, June, *Systems of commuting matrices.*

R. H. Owens, California Tech., June, minor in physics, *An asymptotic study of a class of ordinary linear differential equations of the second order.*

W. L. Parker, California, Berkeley, June, *On uniquely solvable Boolean equations.*

F. P. Pedersen, Southern California, June, minor in physics, *On spaces with negative curvature.*

L. L. Pennisi, Chicago, June, *An indirect sufficiency proof for the problem of Lagrange with differential inequalities as added side conditions.*

C. M. Petty, Southern California, August, minor in philosophy, *Minkowski geometries.*

P. E. Pfeiffer, Rice, June, *Equivalence of measures on infinite product spaces.*

R. S. Pierce, California Tech., June, minor in physics, *Homomorphisms of function lattices.*

Pasquale Porcelli, Texas, May, *Concerning uniform completeness of sets of reciprocals of linear functions.*

L. E. Pursell, Purdue, August, minor in physics, *Some algebraic structures associated with smooth manifolds.*

Gustave Rabson, Michigan, February, *Fourier series on compact groups.*

Howard Raiffa, Michigan, February, *Arbitration schemes for generalized two-person games.*

F. B. Reis, Massachusetts Tech., February, minor in aeronautical engineering, *Studies of correlation and spectra in homogeneous turbulence.*

H. J. Reiter, Rice, June, *Investigations in harmonic analysis.*

H. G. Rice, Syracuse, January, *Classes of recursively enumerable sets and their decision problems.*

J. D. Riley, Kansas, June, minor in physics, *Contributions to the theory of functions of a bicomplex variable.*

Joan E. Robinson, Bryn Mawr, June, *Continuity of transformation groups in topological spaces.*

Hartley Rogers, Jr., Princeton, October, *Some results on definability and decidability in elementary theories.*

J. C. Rogers, Cornell, September, *Cross sections in fibre bundles.*

P. G. Rooney, California Tech., June, minor in physics, *Inversion and representation theorems for the Laplace transformation.*

W. L. Root, Massachusetts Tech., February, minor in electrical engineering, *H-systems and rings of operators.*

G. F. Rose, Wisconsin, February, *Jaskowski truth tables and their realizability.*

David Rosen, Pennsylvania, June, *A class of continued fractions associated with certain properly discontinuous groups.*

W. C. Royster, Kentucky, June, *Convexity and starlikeness of analytic functions.*

J. A. Schatz, Brown, May, *Representation of normed algebras with an involution.*

Samuel Schecter, Syracuse, June, *Boundary value problems for the minimal surface equation.*

D. R. Scholz, Washington University, June, *Some minimum problems in the theory of functions.*

Augusta L. Schurrer, Wisconsin, August, *On the locations of the zeros of the derivative of rational functions of distance polynomials.*

J. T. Schwartz, Yale, June, *Linear elliptic differential operators.*

W. R. Seugling, California, Los Angeles, January, *On the non-linear theory of elasticity.*

H. S. Shapiro, Massachusetts Tech., September, minor in language, *Extremal problems for polynomials and power series.*

V. L. Shapiro, Chicago, June, I. *Square summation and localization of double trigonometric integrals,* II. *Summability of double trigonometric integrals,* III. *Circular summability C of double trigonometric series.*

Henry Sharp, Jr., Duke, June, minor in physics, *Strongly homeomorphic imbedding of F-sigma sets in a subset of* E_N.

R. H. Shaw, George Washington, February, *On the proof of a theorem of Frobenius without the use of group characters.*

A. L. Shields, Massachusetts Tech., September, minor in language, *On additive properties of real numbers.*

R. J. Silverman, Illinois, June, minor in meteorology, *Invariant extensions of linear operators.*

R. O. Skinner, Carnegie, September, *On the quantization of interacting gravitational, electromagnetic and electron fields.*

Stephanie Sloyan, Catholic, June, minors in physics and education, *On the Kantor lines associated with a poristic system of simple cyclic quadrangles.*

K. W. Smillie, Toronto, June, minors in quantum and statistical mechanics, *A mathematical treatment of certain movements of fish—an application of the theory of Markov processes.*

N. E. Smith, McGill, May, *A statistical problem in the geometry of numbers.*

W. N. Smith, Wisconsin, February, *The expansion problem for second order differential systems of the mildly irregular type.*

W. L. Stamey, Jr., Missouri, June, *New metric properties of spherical and elliptic spaces.*

C. W. Steeg, Jr., Massachusetts Tech., February, minor in physics, *Three-dimensional aerodynamic trajectories.*

R. G. Stoneham, California, Berkeley, January, *Stresses in thick-walled cylinders.*

W. L. Strother, Tulane, May, *Continuity for multi-valued functions and some applications to topology.*

R. F. Tate, California, Berkeley, June, *Contributions to the theory of random numbers of random variables.*

J. S. Thale, Northwestern, June, *Univalence of continued fractions and Stieltjes transforms.*

W. R. Thickstun, Jr., Maryland, June, minor in physics, *A nozzle flow for a compressible fluid.*

G. H. M. Thomas, Wisconsin, June, *Simultaneous partitioning of two sets.*

D. J. Thompson, Iowa State, August, *A theory of sampling finite universes with arbitrary probabilities.*

F. B. Thompson, California, Berkeley, January, *Some contributions to abstract algebra and metamathematics.*

Lee D. Thompson, Indiana, September, minor in physics, I. *Converse of a well-known theorem on integral means,* II. *Conjugate classes of subharmonic functions.*

W. J. Thomsen, Iowa, June, minor in astronomy, *Calculations on the fall of large meteorites.*

W. V. Tilston, Toronto, November, minor in quantum mechanics, *Contributions to the theory of antennas.*

A. J. Tingley, Minnesota, December, minor in physics, *A gen-*

eralization of the Poisson formula for the solution of the heat flow equation.

H. E. Tinnappel, Ohio State, June, *On the topological index.*

Marvin Tomber, Pennsylvania, June, *Lie algebras of type* F.

C. K. Tsao, Oregon, June, *A general class of discrete distributions and mixtures of distributions.*

M. J. Walsh, Illinois, June, minor in philosophy, *The paracompactness of the CW-complex and gradient mappings in locally convex spaces.*

K. C. Walters, Florida, June, minor in education, *Effect of uniform displacement on the stress distribution of a wood plate.*

M. T. Wechsler, Michigan, June, *A characterization of certain topological spaces by means of their groups of homeomorphisms.*

Arnold Wendt, Wisconsin, June, *A generalization of a theorem in schlicht function theory.*

R. E. Wheeler, Kentucky, June, *A variable probability distribution function.*

Phillips Whidden, Carnegie, June, *On a criterion for measuring closeness of probability distributions.*

H. H. Wicke, Iowa, August, minor in astronomy, *The geometry of Hilbert space.*

R. A. Willoughby, California, Berkeley, January, *p-algebras.*

N. Z. Wolfsohn, Harvard, June, *On differentiable maps of Euclidean n-space into Euclidean m-space.*

K. G. Wolfson, Illinois, June, minor in physics, *An ideal theoretic characterization of the ring of all linear transformations.*

G. N. Wollan, Georgia, June, *On Euler methods of summability for double series.*

J. W. Wray, Illinois, February, minor in physics, *Non-analytic functions of a complex variable representable by Lebesgue-Stieltjes integrals with a Cauchy kernel.*

Morris Yachter, New York, February, *On the existence of periodic solutions of the differential equation:* $x+f(x)=\epsilon f(x, \dot{x}, T, \epsilon)$.

C. T. Yang, Tulane, May, *Equivalence of the Alexander-Kolmogoroff and Čech cohomology theories.*

K. T. Yen, Brown, May, *On the construction of high speed flows.*

J. W. Young, Florida, August, minor in education, *Non-classical orthogonal polynomials.*

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SOLVING LINEAR ALGEBRAIC EQUATIONS CAN BE INTERESTING

GEORGE E. FORSYTHE[1]

1. Introduction. The subject of this talk is mathematically a lowly one. Consider a system of n linear algebraic equations in n unknowns, written

$$Ax = b. \tag{1}$$

Here A is a square matrix of order n, whose elements are given real numbers a_{ij} with a determinant $d(A) \neq 0$; x and b denote column vectors, and the components of b are given real numbers. (Complex numbers would offer no essential difficulty.) It is desired to calculate the components of the unique solution $x = A^{-1}b$; here A^{-1} is the inverse of A.

Such problems arise in the most diverse branches of science and technology, either directly (e.g., the normal equations of the least-squares adjustment of observations) or in an approximation to another problem (e.g., the difference-equation approximation to a self-adjoint boundary-value problem for a partial differential equation). These two are very frequent sources of numerical systems; note that $A > 0$ (i.e., A is symmetric and positive definite) in both examples. The order n is considered to range from perhaps 6 or 8 up to as large a number as can be handled. Stearn [111], for instance, mentions the solution of a system of order 2300 by the U.S. Coast and Geodetic Survey. The accuracy demanded of an approximate solution ξ varies widely; even the function which is to measure the accuracy of ξ varies or is unknown. Some "customers" want to make the length $|b - A\xi|$ small; some, $|\xi - A^{-1}b|$; others have apparently thought only in terms of getting $A^{-1}b$ exactly.

We all know that each component of the solution $A^{-1}b$ can be expressed as a quotient of determinants by Cramer's rule. We have all evaluated determinants of orders 3, 4, and possibly 5, with a_{ij} integers; it is quite easy and rather boring. I therefore suspect that the average mathematician damns the practical solution of (1) as being both trivial and dull.

An address delivered (under the title *Solving linear equations is not trivial*) before the Eugene meeting of the Society, June 21, 1952, by invitation of the Committee to Select Hour Speakers for Far Western Sectional Meetings; received by the editors December 6, 1952.

[1] Sponsored in part by the Office of Naval Research, USN.

One defense of the numerical analyst (preface of [81]) is to show that in many practical cases (say for decimal fractions a_{ij} and $n \geq 10$) getting $A^{-1}b$ efficiently (or at all) is actually not trivial, but demands both know-how and planning. This point is especially well taken against those whose program for solving (1) would be to evaluate $n+1$ determinants from their definition, employing $n!$ multiplications per determinant. For $n=26$, for example, $(n+1)!$ is approximately 10^{28}, a number of multiplications which would take the SWAC[3] some 10^{17} years. Actually, only about $(1/3)n^3$ multiplications are needed to solve (1); see Bodewig [12]. For $n=26$, $(1/3)n^3$ is approximately 6000, and the multiplications would take the SWAC about 3 seconds.

It is my hope, on the other hand, to arouse the mathematician's interest by showing (§2) the diversity of approaches to the solution of (1), and by mentioning (§§3 to 6) some problems associated with selected iterative methods. The newest process on the roster, the method of conjugate gradients, is outlined in §7. §8 touches on the difficult general subject of errors and "condition," while a few words are hazarded in §9 about the effect of machines on methods.

Whether or not the subject proves interesting, the bibliography is intended to make the field look at least respectable! It is a representative condensation of the more than 500 titles in the author's file, most of which are in [27]. There are bibliographies on related subjects in Collatz [16], Dwyer [20], Engineering Research Associates [22], Frame [36], Frankel [37], Franklin and Hankam [38a], Harvard Computation Laboratory [47], Higgins [55], Kuroš, Markuševič, and Raševskiĭ [71], Motzkin [82], Schwerdtfeger [106], and Taussky [116].

It is remarkable how little is really understood about most of the methods for solving (1). They are being used nevertheless, and yield answers. This disparity between theory and practice appears to be typical of the gulf between the *science* and the *art* of numerical analysis.

The following notations are occasionally used: (i) "$p_k \simeq q_k$ (as $k \to \infty$)" means that (for the vectors or numbers p_k, q_k) $|p_k - q_k| = o(|p_k|)$; (ii) "$p_k \sim q_k$ (as $k \to \infty$)" means that (for the numbers p_k, q_k) $p_k = O(q_k)$ and $q_k = O(p_k)$.

2. **Survey of methods.** It usually surprises the uninitiated to learn

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[3] National Bureau of Standards Western Automatic Computer, an electronic machine which can perform 2600 multiplications per second. See [61] for a description, now out of date.

the variety of methods actually used to solve systems (1). Surveys of methods are given by Bodewig [12], Dwyer [20], Faddeeva [23], Forsythe [26], Fox [33; 34], Frazer, Duncan and Collar [39], Hotelling [58], Householder [59; 60], Zurmühl [131], and others, but the spectrum of known methods is considerably broader than any of these surveys suggests. A classification of elimination methods is given by Jensen [65]. A tentative classification of all known methods, with a bibliography of about 450 titles, is in [27].

As mentioned in §1, one can solve a system (1) explicitly by use of determinants, while explicit solutions in other forms are sometimes available. Friedrich and Jenne [40], for example, describe the use of continued fractions.

The best known methods are based on systematic elimination of unknowns from equations in the system (1) in the fashion of high-school algebra, as described by Gauss [42]. The elimination amounts to triangularizing the matrix A by premultiplying it by a triangular matrix, as Banachiewicz [5] and Turing [122] point out. The process can be rendered very efficient numerically by consolidating operations; see, for example, Benoit [7], Dwyer [20], and Turing [122]. When A is positive definite, the method is equivalent to the successive orthogonalization of the unit vectors in the A metric by the Gram-Schmidt process [99]. With other metrics the orthogonalization process yields different methods for solving (1). All these elimination methods can also be performed on any matrix of submatrix blocks formed by partitioning A, an idea suggested by Boltz [13] and based on relations given by Schur [105]. The various elimination methods are *direct*, a term defined at the end of §2.

There is a group of direct methods related to the characteristic polynomial ϕ of some matrix. For example, if one can learn that $\phi(A) \equiv c_n A^n + \cdots + c_1 A + I = 0$, then, as Bingham [8] notes, one can compute $A^{-1}b = -c_n A^{n-1}b - \cdots - c_2 Ab - c_1 b$. Similar remarks apply when $\phi(H)$ is known for an operator H associated with the solution of (1); see §4. There are related methods involving the successive orthogonalization of the vectors $Ax_0, A^2x_0, \cdots, A^n x_0$ in the I, A, A^{-1}, or other metrics; one of these is given in §7.

There is an unpublished direct method of H. Lewy using the theory of congruences in n dimensions, applicable mainly when the components of $A^{-1}b$ are integers. It is based on the use of stencils for solving the system $A^*x = b^*$, where $a_{ij}^* = 0$ (if a_{ij} is even) and $a_{ij}^* = 1$ (if a_{ij} is odd), and where b_i^* is similarly defined.

Of a quite different nature is a group of iterative processes devised by Jacobi [63], Nekrasov [88], Richardson [98], and others, and subsumed by Cesari [15], Wittmeyer [127], and Geiringer [44] in a general class of linear iterative processes to be discussed in §3. In these methods one starts with an arbitrary first vector x_0 approximating $A^{-1}b$. For $k=0, 1, 2, \cdots$, the components of the kth approximant vector x_k are systematically corrected, often in the cyclic order $1, 2, \cdots, n$, but sometimes in blocks or otherwise. After one cycle of the corrections has been completed, the components of the resulting vector x_{k+1} will have been obtained by solving a matrix equation $Bx_{k+1}+Cx_k=b$ for x_{k+1}, where $B+C=A$. The different linear processes are distinguished by the choice of B, which must be an easily invertible matrix. For example, B can be the lower triangle of A [88], the diagonal of A [63], a scalar matrix [98], diagonal blocks of A (von Mises and Pollaczek-Geiringer [123], Hertwig [49]), etc. (When B is the lower triangle of A, the iterative method is commonly called the "Seidel process," or the "Gauss-Seidel process." But, as Ostrowski [91] points out, Seidel [107] mentions the process but advocates *not* using it. Gauss nowhere mentions it. Nekrasov [88] studies its properties and says it is Seidel's process. It will henceforth be called the "cyclic single-step process.") Under appropriate conditions mentioned in §3, x_k approaches $A^{-1}b$, but ordinarily no x_k equals $A^{-1}b$.

Another group of iterative processes includes those of Gauss [41], Seidel [107], Southwell [109; 110], Motzkin [2], and others. These are called "relaxation methods" in [109, 110]. They are discussed by Black and Southwell [10], Fox [32], Temple [118], and others, and have proved especially useful in engineering work. They are difficult to define precisely, since the computer is essentially advised to use all the art and artifice he can muster to find x such that $r=b-Ax$ is near 0. Their predominant feature, however, is that the components of x are corrected, not in a predetermined order, but in an order of "worst first." If this feature is adopted as defining relaxation, the iteration function (defined in §3) depends on x in a *piecewise linear* fashion, and the analytical character of the processes is completely different from that of the related linear processes. Relaxation has been studied recently in connection with solving systems of linear equalities; see Agmon [2], and Motzkin and Schoenberg [83].

Other nonlinear iterative processes include the least-squares iterative methods discussed in §§5, 6, and 7. They start with Cauchy [14], and are synthesized by Temple [118], Rosser [unpublished], Hestenes and Stein [53], and others. Special cases include certain linear processes which essentially deal with a positive definite matrix. For example, Kaczmarz [67] and Tompkins [120] interpret the system

(1) as restricting $A^{-1}b$ simultaneously to n hyperplanes. A first guess x_0 is projected successively on each hyperplane in cyclic order. Then the distance $|x_k - A^{-1}b|$ decreases monotonically to 0. De la Garza [19] proposes a related process. When $A > 0$, the cyclic single-step method and all the relaxation methods are also least-squares methods. This fact is the basis of many studies of the relaxation methods; see [118] and Ostrowski [93].

Least-squares processes which are not linear include the gradient methods of [14], developed in [118] and by Kantorovič [68] and Birman [9]; see §5. Their culmination appears to be the conjugate gradient method of Hestenes [50; 54], Lanczos [72], and Stiefel [114; 54]; this is a finite iteration described in §7. A gradient method in a more general metric is mentioned in §6.

Seemingly very different are the *Monte Carlo* methods for solving (1), employing random sampling of a certain chance variable whose expectation is $A^{-1}b$. One such method, devised by von Neumann and Ulam [unpublished], is described by Forsythe and Leibler [28]; Wasow [125] devises another Monte Carlo method. Both are based on properties of discrete random walks in a space of n points. When the system (1) represents Laplace's or related difference equations in one or more dimensions, the Monte Carlo methods have a longer history; see the exposition and bibliography in Curtiss [18]. The methods are theoretically fascinating, but there is little evidence yet of their practical utility for solving linear systems.

The iterative processes for solving (1) are likely to converge slowly, and a number of tricks have been devised to speed them up, called *acceleration* processes. Accelerations of linear processes may themselves be either linear or nonlinear; some are described in §4. A number of processes for accelerating the nonlinear gradient methods are mentioned at the end of §5, and the conjugate gradient method of §7 may be considered also as an acceleration.

The distinction between direct and iterative methods is ordinarily stressed in numerical analysis; see [23], for example. Applied to systems (1), a *direct* method is one which yields $A^{-1}b$ exactly in a finite number of arithmetical operations, if the latter are performed without round-off error. An iterative process can ordinarily yield $A^{-1}b$ only as the limit of a sequence of exact arithmetical operations. However, it must be remembered that as soon as calculations are rounded off (as ordinarily occurs in machine calculation), direct methods disappear except for quite simple problems, and all methods become iterative; see the end of §8. The practical distinction between methods for solving (1) then appears to depend on: (a) the speed of the convergence to $A^{-1}b$, and (b) the simplicity of the computation at each stage of the iteration. A two-way classification of methods might well be based on their behavior with respect to properties (a) and (b), and that of [27] was roughly of this type. Two difficulties make such a classification imprecise. First, the theory of round-off error is too poorly developed to yield rates of convergence to $A^{-1}b$. Second, the practical criteria of simplicity in machine computation vary too greatly among different machines.

One may also distinguish whether the matrix A is altered in the course of solution, as in elimination, or whether it is retained in its original form, as in the conjugate gradient process. This is probably a crucial distinction in practice, for, when the original A is retained, the round-off errors seem to accumulate more slowly; see §9.

3. Linear iterative processes.

An iterative process for solving (1) (or other equation) is determined by the functions F_k wherewith the $(k+1)$th approximant to $A^{-1}b$, x_{k+1}, is derived from the earlier approximants $x_0, x_1, \cdots, x_k$. If the only argument of F_k is x_k, the iterative process is said (Schröder [103]) to be of *first degree*: $x_{k+1}=F_k(x_k)$. If the function F_k is independent of k, the process is called *stationary*. A bibliography on iteration as such has been prepared by Schwerdtfeger [106].

As elsewhere in mathematics, the most studied functions are the linear ones. We introduce the (most general) *linear iterative process of the first degree* by the definition

$$x_{k+1} = F_k(x_k) = H_k x_k + v_k, \tag{2}$$

where the H_k are square matrices and the v_k are vectors. If the iterative process described by (2) is to solve (1), it seems essential that the solution $A^{-1}b$ be a *fixed point* of F_k:

$$A^{-1}b = F_k(A^{-1}b) = H_k A^{-1}b + v_k. \tag{3}$$

We demand that (3) hold. It follows that

$$v_k = (I - H_k)A^{-1}b = M_k b. \tag{4}$$

If the H_k and M_k are independent of b, then (6) follows from (4). Thus:

The most general stationary linear iterative process of the first degree for solving (1) *which is independent of b and which satisfies* (3) *is defined by the functions*

$$x_{k+1} = F_k(x_k) = H_k x_k + M_k b, \tag{5}$$

where the square matrices H_k and M_k depend only on A and satisfy the *relations*

$$H_k + M_k A = I. \tag{6}$$

If the process is stationary, then

$$x_{k+1} = Hx_k + Mb, \tag{7}$$

where the square matrices H and M depend only on A and satisfy the relation

$$H + MA = I. \tag{8}$$

Stationary processes of the type (7), (8) have been studied by Cesari [15], Wittmeyer [127], Geiringer [44], and others. They include the cyclic single-step iteration of Nekrasov [88] and Liebmann [74], and those of Jacobi [63], Richardson [98], Frankel [38], Young [129], and many others. In the cyclic single-step process, for example, one writes A as the sum of two matrices $B = (b_{ij})$ and $C = (c_{ij})$, where

$$b_{ij} = \begin{cases} a_{ij} & (i \geqq j) \\ 0 & (i < j) \end{cases} \quad \text{and} \quad c_{ij} = \begin{cases} 0 & (i \geqq j) \\ a_{ij} & (i < j). \end{cases}$$

Then $Bx_{k+1} + Cx_k = b$, or $x_{k+1} = -B^{-1}Cx_k + B^{-1}b$; it is assumed that no $a_{ii} = 0$. Thus $H = -B^{-1}C$, while $M = B^{-1}$.

Let us consider the convergence of the linear processes, under the assumption that all arithmetic operations are carried out with perfect accuracy. From (2) and (3) it follows that

$$x_{k+1} - A^{-1}b = H_k(x_k - A^{-1}b), \tag{9}$$

whence

$$x_k - A^{-1}b = K_k(x_0 - A^{-1}b), \tag{10}$$

where $K_k = H_{k-1}H_{k-2} \cdots H_1H_0$. In order that $x_k - A^{-1}b \to 0$ for arbitrary x_0 it is therefore necessary and sufficient that

$$\lim_{k\to\infty} K_k z = 0, \qquad \text{for every vector } z. \tag{11}$$

In practice condition (11) is usually known to hold only in certain special cases, such as when:

(a) all H_k are polynomials in some one matrix—for example, A;
(b) all $H_k = H$ (stationary process);
(c) for some norm function, all $\|H_k\| \leqq 1 - \epsilon < 1$.

Henceforward we consider only case (b): stationary linear processes. Then $K_k = H^k$, and it is known (see [94], for example) that (11) holds if and only if each eigenvalue $\lambda_i(H)$ of H is less than one in absolute value. Thus, we have derived the well known result that *in* *the stationary linear process* $x_k \to A^{-1}b$ *for all* x_0 *if and only if all* $|\lambda_i(H)| < 1$.

This result, while exceedingly important, hardly begins to solve the practical computer's problem. He raises the following questions: (i) How can one tell whether all $|\lambda_i(H)| < 1$? (ii) If all $|\lambda_i(H)| < 1$, how fast will the convergence be? (iii) If the convergence is intolerably slow (as the computer realistically expects), how may it be speeded up? To a considerable extent these questions are answered by a knowledge of the asymptotic behavior of the error vector $x_k - A^{-1}b$ as $k \to \infty$. Question (iii) will also be dealt with in §4.

The general theory of the Jordan canonical form [78; 94] of the matrix H can be applied to discuss the asymptotic behavior of $x_k - A^{-1}b$. Suppose, as is usually the case, that H has the diagonal canonical form

$$H = \sum_{i=1}^{n} \lambda_i c_i r_i^T, \tag{12}$$

where c_i, r_i^T are respectively the column and row eigenvectors of H belonging to λ_i. Suppose that $x_0 - A^{-1}b = \sum_1^n \gamma_i c_i$. From (10) one then finds that

$$x_k - A^{-1}b = H^k(x_0 - A^{-1}b) = \sum_{i=1}^{n} \gamma_i \lambda_i^k c_i. \tag{13}$$

If λ_n is a unique eigenvalue of maximum modulus, and if $\gamma_n \neq 0$, then

$$x_k - A^{-1}b \simeq \gamma_n \lambda_n^k c_n \qquad (\text{as } k \to \infty). \tag{14}$$

This indicates that asymptotically $x_k \to A^{-1}b$ along a certain line, a fact which is often useful in accelerating the convergence of $\{x_k\}$.

When λ_n is unique but $\gamma_n = 0$, the relation (14) does not hold for exact operations. However, a calculation involving round-off errors will soon have the practical effect of making $\gamma_n \neq 0$. If several eigenvalues dominate in absolute value, formula (14) must usually be modified to include several terms. In any case, when (12) holds, formula (13) shows that x_k approaches $A^{-1}b$ *linearly*—i.e., that

$$| x_k - A^{-1}b | \sim \left(\max_i |\lambda_i|\right)^k \qquad (\text{as } k \to \infty). \tag{15}$$

When (12) fails (i.e., when H does not have a diagonal Jordan canonical form), formulas (13) and (15) have to be altered. The important fact, however, is that the basic mathematical theory is already in existence for dealing with linear iterative processes—at least as long as round-off errors are not considered in detail. (It is commonly supposed that a nondiagonal canonical form would never arise in practical analysis. However, if one uses the Liebmann [74] process to solve the usual difference equation corresponding to the Dirichlet problem in one or more dimensions, the matrix H turns out to have a nondiagonal canonical form; see [129].)

Recent developments in stationary linear processes have included a suggestion of Aitken [4] and others for making $H>0$ (i.e., a positive definite symmetric matrix), so that all $\lambda_i(H)$ will be real and positive, insuring that (13) will in fact be dominated by one term. In the cyclic single-step process this can be achieved by solving for the components in the "to-and-fro" order $1, 2, \cdots, n, n-1, \cdots, 2$ and repeat.

According to Ostrowski [91], for the cyclic single-step process it was first proved by Pizzetti [95] that, if $A>0$, all $|\lambda_i(H)|<1$. A converse recently proved by Reich [97], and more simply by Ostrowski [92], asserts that *if A is symmetric and each $a_{ii}>0$, then all $|\lambda_i(H)|<1$ if and only if $A>0$.* Collatz [17], Stein and Rosenberg [113], and others have studied the relations between the $\lambda_i(H)$ for the cyclic single-step process and those for the related total-step process of Jacobi [63]. Other recent developments [38; 129] include the alteration of H by simple devices so that $\max_i |\lambda_i|$ is reduced as much as possible. In the Liebmann process for the Dirichlet problem, a certain systematic "over-relaxation" has just this effect.

4. **Acceleration of stationary linear processes.** Since ordinarily the stationary linear process (7) seems to converge slowly, or even diverge, it is of the greatest practical importance to devise methods to improve the convergence. Any procedure which replaces x_k by a vector closer in some sense to $A^{-1}b$ is loosely called an *acceleration* of the iterative process (7). An acceleration may be considered as a *summability process* applied to the sequence $\{x_k\}$ defined by (13). The best understood accelerations are, like the iterations themselves, *linear processes*, in which the improved vector is a linear combination of $x_0, x_1, \cdots, x_k$.

The simplest linear acceleration is useful when H has a unique dominant eigenvalue λ_n. By (14)

$$x_{k+1} - A^{-1}b \simeq \lambda_n(x_k - A^{-1}b) \qquad (\text{as } k \to \infty).$$

Hence the vector

$$\frac{x_{k+1} - \lambda_n x_k}{1 - \lambda_n} \tag{16}$$

may be expected to give a better approximation to $A^{-1}b$ than x_k gives; (16) may be useful even when λ_n is known only approximately. This idea has been applied by Flanders and Shortley [24], Lyusternik [77], and others.

The same idea can be extended to take care of two or more dominant λ_i. Let E be the operator increasing by one the subscript k of x: $x_{k+1}=Ex_k$. One can write (16) in the form

$$P_1(E)x_k, \tag{17}$$

where $P_1(E)=(1-\lambda_n)^{-1}(E-\lambda_n)$ is a polynomial of degree one in E. Let m be an integer. Consider the expression

$$P_m(E)x_k, \tag{18}$$

where P_m is some polynomial of degree m. How shall we determine P_m so that (18) is a good approximation to $A^{-1}b$? If we can answer this effectively, then (18) furnishes a useful linear acceleration of the process (7).

In order that the operation (18) not worsen an already good approximation x_k, we must have

$$P_m(1) = 1. \tag{19}$$

For if $x_k=A^{-1}b$, then, by (3), $x_{k+1}=x_{k+2}= \cdots =A^{-1}b$, so that $P_m(E)x_k=P_m(1)x_k=P_m(1)A^{-1}b$.

If $m=n=$ the order of H, and if $\phi(\lambda)$ is the *characteristic polynomial* of H, then the choice $P_m(E)=\phi(E)/\phi(1)$ is perfect, in that $P_m(E)x_k$ is exactly equal to $A^{-1}b$, for all x_k. But ordinarily ϕ is unknown, and to obtain it would involve precise knowledge of all the eigenvalues λ_i of H.

Suppose that the eigenvalues λ_i, although not known precisely, are known all to be in a certain closed set R of the complex plane (for example, a real interval). By (8), since $d(A)\neq 0$, the number 1 cannot be an eigenvalue of H. Hence we may assume that R is bounded away from the point 1. Now the eigenvalues of $P_m(H)$ are $\{P_m(\lambda_i)\}$. Since, by (13),

$$P_m(E)x_k - A^{-1}b = P_m(H)(x_k - A^{-1}b) = \sum_{i=1}^{n} P_m(\lambda_i)\gamma_i c_i,$$

it is essential for the success of the acceleration that all $|P_m(\lambda_i)|$ be small. This leads to the approximation problem of determining the polynomial P_m of degree m such that (19) holds and such that

$$\max_{\lambda\in R} |P_m(\lambda)| \text{ is a minimum.} \tag{20}$$

Such polynomials, named generically after Čebyšev (= Chebyshev = Tschebyscheff = $\cdots$), arise frequently in numerical analysis.

If R is a real interval not containing 1, the minimizing polynomial $P_m(\lambda)$ is proportional to the ordinary Čebyšev polynomial $T_m(\lambda)$ for the interval R; see W. Markoff [79]. For just this reason the polynomials T_m have been used several times in matrix problems to suppress the effects of unwanted eigenvalues; see Abramov [1], Flanders and Shortley [24], Gavurin [43], and Lanczos [73]. (A recent treatment of Čebyšev approximation problems in the complex plane is in [80].)

The real difficulty in numerical analysis is that R is not known. How can the information gained from computing $x_0, x_1, \cdots, x_k, \cdots, x_{k+m}$ be used effectively to localize R, so that the Čebyšev problem can be formulated? After R is localized, how can the corresponding $P_m(E)$ be determined approximately? These are important questions which are in need of treatment. The use of symmetric matrices H helps a great deal, by restricting R to the real axis.

Another acceleration device is due to Schulz [104]. Suppose $A = I - H$, and that (21) converges. A linear process analogous to (7) obtains A^{-1} as the sum of a Neumann series

$$(I - H)^{-1} = I + H + H^2 + \cdots, \tag{21}$$

often slow to converge. By [104] one can obtain the partial sum $X_n = I + H + \cdots + H^{2^n - 1}$ of (21) by n iterations (22),

$$X_{k+1} = X_k(2I - AX_k), \qquad X_0 = I, \tag{22}$$

a total of $2n$ matrix multiplications. The use of (22) to get A^{-1} is sometimes called Newton's process. Can this idea be adapted to solving the system (1) without getting A^{-1} first?

For accelerating the convergence of any linear process one also has the δ^2-process of Aitken [3; 4] and its extensions by Shanks [108] and Samuelson [101]. Lubkin [76] has studied it as a nonlinear sequence-to-sequence summability process. It requires no knowledge of the $\lambda_i(H)$. Let y_k represent an arbitrary, but fixed, component of x_k. Then the functional character of y_k in a linear process is given by

$$y_k = y_\infty + \sum_{i=1}^{n} s_i \lambda_i^k, \tag{23}$$

where y_∞ is the desired component of $A^{-1}b$, and the λ_i, s_i are numbers. To determine y_∞ from (23) it is theoretically sufficient to have exactly $2n+1$ successive values of y_k—for instance, $y_0, y_1, \cdots, y_{2n}$. In practice the elimination of the s_i and λ_i would be too tedious, but frequently one λ_i, say λ_n, predominates in (23). In the δ^2-process one then ignores s_i, λ_i for $i=1, \cdots, n-1$, and from three successive y_k obtains the following estimate for y_∞, useful for moderately large k:

$$\text{estimate of } y_\infty = \frac{y_k y_{k+2} - y_{k+1}^2}{y_k - 2y_{k+1} + y_{k+2}} = y_k - \frac{(\Delta y_k)^2}{\Delta^2 y_k}.$$

In [108] Shanks assumes that m eigenvalues dominate (23), ignores s_i, λ_i for $i=1, \cdots, n-m$, and by an elimination obtains an estimate for y_∞ in terms of $2m+1$ successive y_k. For $m=2$, for example, one can show that the

$$\text{estimate of } y_\infty = \begin{vmatrix} y_k & \Delta y_k & \Delta^2 y_k \\ \Delta y_k & \Delta^2 y_k & \Delta^3 y_k \\ \Delta^2 y_k & \Delta^3 y_k & \Delta^4 y_k \end{vmatrix} \Bigg/ \begin{vmatrix} \Delta^2 y_k & \Delta^3 y_k \\ \Delta^3 y_k & \Delta^4 y_k \end{vmatrix}.$$

5. **Least-squares methods.** A variational approach to solving (1) seems very fruitful. In a general treatment over the complex field (the present treatment is confined to the real field), Hestenes and Stein [53] take a matrix $R>0$ and let $(z^T R z)^{1/2} = |z|_R$ be the *R-length* of a vector z. (The T stands for transposition.) If $B = A^T R A$ and $c = A^T R b$, then $B>0$ also. Let the deviation of x from $A^{-1}b$ be measured by

$$(24) \quad f(x) = |Ax - b|_R^2 = |x - A^{-1}b|_B^2 = x^T B x - 2x^T c + |b|_R^2.$$

Starting from an initial vector x_0, one can move x in various directions with the object of minimizing $f(x)$. Clearly the minimum is attained just when $x = A^{-1}b$.

To simplify the exposition we now assume $A>0$ and take $R = A^{-1}$. Then

$$(25) \qquad f(x) = x^T A x - 2x^T b + b^T A^{-1} b.$$

Although $f(x)$ is not computable unless $A^{-1}b$ is known, it is sufficient in practice to minimize the computable function $f(x) - b^T A^{-1} b$. Fix x, and let $r = b - Ax$ be the *residual* of the system (1) at x. Let $d \neq 0$ determine a direction. Since $f(x+\alpha d) = \alpha^2 d^T A d - 2\alpha d^T r + f(x)$, the value of the real parameter α for which $f(x+\alpha d)$ is a minimum is

$$(26) \qquad \alpha^* = d^T r / d^T A d;$$

this α^* is called the *optimum* α (corresponding to x and d).

For any $\alpha = \beta\alpha^*$ one can compute $f(x+\alpha d)$ from the relation $f(x) - f(x+\alpha d) = (2\alpha\alpha^* - \alpha^2) d^T A d$, i.e., $f(x) - f(x+\beta\alpha^* d)$

$=\beta(2-\beta)\alpha^{*2}d^TAd$. Thus (when $d^Tr\neq 0$), $f(x+\beta\alpha^*d)<f(x)$ for $0<\beta<2$. The greatest reduction in $f(x)$ comes when $\beta=1$.

We now describe a *general least-squares iterative process* for solving (1). There must be prescribed: (i) a start x_0; (ii) a sequence of directions $\{d_k\}$; (iii) a sequence of ratios $\{\beta_k\}$. (In [53], but not here, R must also be prescribed.) For each $k=0, 1, 2, \cdots$, one determines $\alpha=\alpha_k^*$ by (26) so that $f(x_k+\alpha d_k)$ is minimized. Then one lets

$$x_{k+1} = x_k + \beta_k\alpha_k^* d_k, \tag{27}$$

where $r_k=b-Ax_k$ and $\alpha_k^*=d_k^Tr_k/d_k^TAd_k$. (It is not specified that the β_k and the d_k be determined a priori; they may depend on x_k.) If the sequences $\{\beta_k\}$ and $\{d_k\}$ satisfy (28) and (29), it is shown in [53] that, independently of x_0, $f(x_k)\to f(A^{-1}b)$, as $k\to\infty$, so that $x_k\to A^{-1}b$. The conditions are:

$$0 < \delta_1 \leqq \beta_k \leqq 2 - \delta_1 < 2 \qquad \text{(all } k\text{)}; \tag{28}$$

$$0 < \delta_2 = |d_k^Tr_k|/(|d_k|\cdot|r_k|) \qquad \text{(all } k\text{)}. \tag{29}$$

In [53] an alternate to (29) states that the d_k recurrently span the space in a certain uniform manner.

Among others, the following two least-squares processes are also linear iterative processes: (a) the d_k and β_k are independent of the x_k; (b) $d_k=r_k$ but $\alpha_k=\beta_k\alpha_k^*=\alpha$ is a constant. When in (a) all $\beta_k=1$ and the d_k are the coordinate unit vectors in cyclic order, one has the cyclic single-step process of Nekrasov [88]. The use of $f(x)$ is very useful in studying any single-step process. In (b) one can write the linear iteration function of §3 in the form $F(x)=x+\alpha(b-Ax)=(I-\alpha A)x+b$; the process is due to Richardson [98] and to von Mises and Pollaczek-Geiringer [123], and converges whenever $0<\alpha<2/\max_i\lambda_i(A)$.

In the general case, however, the least-squares process is nonlinear. When $d_k=r_k=-2^{-1}\operatorname{grad} f(x_k)$, it is called a *gradient* method (or method of *steepest descent*). When $\beta_k=1$, one has the *optimum* gradient method (since $f(x_k+\alpha d_k)$ is minimized as a function of α), proposed by Cauchy [14] and studied by Temple [118], Kantorovič [68], Birman [9], and by Hestenes and Karush [52] for the eigenvalue problem. Some variations of the method are treated by Krasnosel'skiĭ and Kreĭn [70a].

By (25) the surfaces $f(x)=$constant are similar and similarly situated ellipsoids whose common center $A^{-1}b$ we seek. Any approximant x_k lies on a certain ellipsoid S_k of the family. The gradient, $-2r_k$, lies in the normal $\pi_1(x_k)$ to S_k at x_k. Now x_{k+1} is the unique point of

$\pi_1(x_k)$ for which $f(x)$ is minimized. Since $f(x)$ is a quadratic function of distance along the normal, x_{k+1} is located halfway between the two intersections of $\pi_1(x_k)$ with S_k. Moreover, x_{k+1} is the point where $\pi_1(x_k)$ is tangent to an ellipsoid of the family.

Let $0<\lambda_1 \leqq \lambda_2 \leqq \cdots \leqq \lambda_n$ be the eigenvalues of A. Kantorovič [68] shows that

$$\left[\frac{f(x_{k+1})}{f(x_k)}\right]^{1/2} \leqq \frac{\lambda_n - \lambda_1}{\lambda_n + \lambda_1} = \mu_1 < 1 \qquad (\text{all } x_k). \tag{30}$$

From this it follows that $f(x_k) \downarrow 0$, so that $x_k \to A^{-1}b$ and the process converges. As we mentioned in §3, the practical computer's problem merely begins with such knowledge. Some experience convinces him that (a) μ_1 is frequently very close to 1, and (b) after a few steps $[f(x_{k+1})/f(x_k)]^{1/2}$ becomes and remains very close to μ_1; see [25]. Observation (a) is borne out by the remark [6, p. 59] that, for certain matrices A of type C^TC, it is likely that, as $n \to \infty$, $\lambda_n/\lambda_1 \sim n^2$, whence $1-\mu_1 \sim n^{-2}$. For finite-difference approximations to the Laplace operator over rectangular regions Frankel [38] shows that $1-\mu_1 \sim n^{-1}$, where n is the number of lattice points. As to (b), for any A the value μ_1 can always be attained by $[f(x_{k+1})/f(x_k)]^{1/2}$ when $x_k - A^{-1}b$ assumes certain directions in the plane spanned by u_1 and u_n (defined below). From (a) and (b) we conclude that the optimum gradient method is frequently too slow for practical use.

As a guide to the possible acceleration of the method it would be valuable to know the asymptotic behavior of $x_k - A^{-1}b$, if arithmetical operations are done exactly. But, because x_{k+1} is obtained from x_k by a rational cubic transformation, theorems are hard to prove! It is a conjecture of Forsythe and Motzkin [30], proved only for $n=3$ in [29], that in the optimum gradient method the error vector $x_k - A^{-1}b$ is asymptotically a linear combination of the eigenvectors u_n, u_1 of A belonging to the largest (λ_n) and least (λ_1) eigenvalue of A. (If there are eigenvectors of A orthogonal to $x_0 - A^{-1}b$, one disregards the corresponding eigenvalues in determining λ_1 and λ_n.) A proof of the conjecture for $n \geqq 4$ would be very desirable because, when the conjectured asymptotic relationship holds, for all sufficiently large k the points x_k, x_{k+1}, x_{k+2}, and $A^{-1}b$ are asymptotically coplanar. Thus one could accelerate the convergence of the optimum gradient method by occasionally minimizing $f(x)$ in the plane through the end points of x_k, x_{k+1}, and x_{k+2}. The method has been used successfully in experiments [25] for $n=6$ on an IBM Card-Programmed Calculator, where the average value of $f(x_{k+1})/f(x_k)$ over a series of about 100 steps was reduced from .9733 (optimum gradient method) to .6245 (optimum gradient method with acceleration every ninth step).

A second idea to speed up the optimum gradient method, the *β-optimum* method proposed by Hartree [46] and by Hestenes and Stein [53], is to give β_k some constant value β in the range $0<\beta<1$. In the test problem of [25], Stein [112] finds $\beta=0.9$ to be approximately best, and with it the average value of $f(x_{k+1})/f(x_k)$ is found [25] to be .8204; in [112], .8065 is found for a shorter run. Although the convergence of the β-optimum method is slower (in this test) than that of the accelerated optimum gradient method, the former has the considerable advantage of being shorter to code for automatic machinery. The success of the β-optimum method is perhaps due to an inherent instability of the transformations. In 2 dimensions the transformation of $x_k-A^{-1}b$ to $x_{k+1}-A^{-1}b$ has no stable fixed direction when β is slightly less than 1 [Motzkin and the author, unpublished].

6. **General descent methods.** The function f of (24) has the following significant properties: $f(A^{-1}b)=0$; $f(x)>0$ if $x\neq A^{-1}b$; f is a convex function of x. Any f with these properties might serve as the basis for what Professor Motzkin calls a *descent* method. In such a method one has a first guess x_0, and sequences $\{d_k\}$ and $\{\beta_k\}$ as in §5. As before, one finds $\alpha=\alpha_k^*$ minimizing $f(x_k+\alpha d_k)$, and selects $x_{k+1}=x_k+\beta_k\alpha_k^*d_k$.

Other suitable f would be

$$f(x)=\sum_{i=1}^{n}|r_i|, \tag{31}$$

$$f(x)=\max_i |r_i|. \tag{32}$$

Methods employing the latter norm functions f, with $d_k=-\operatorname{grad} f(x_k)$, are somewhat related to—but apparently do not include—the piecewise linear iterative processes. Agmon [2] discusses a relaxation method for linear inequalities from somewhat this point of view. Zuhovickiĭ [130] gives a gradient method in the metric (32) for minimizing $f(x)$ for incompatible systems $Ax=b$.

7. **Method of conjugate gradients.** It is easy to show that the acceleration step of [25], discussed in §5, is equivalent to finding the unique point x' for which $f(x')=f(x_k+\alpha_0r_k+\alpha_1Ar_k)$ is minimized as a function of the two parameters α_0 and α_1. This suggests a generalization to p parameters discussed by Kantorovič [68] and Birman [9]. Let x_0 be any point, and let $r_0=b-Ax_0$. Define $f(x)$ by (25). Let $\pi_p(x_0)$ be the p-space of points $x_0+\alpha_0r_0+\alpha_1Ar_0+\cdots+\alpha_{p-1}A^{p-1}r_0$,

where $\alpha_0, \cdots, \alpha_{p-1}$ are real parameters. Let x_p be the unique point in $\pi_p(x_0)$ for which $f(x)$ is minimized:

$$(33) \qquad f(x_0 + \alpha_0 r_0 + \cdots + \alpha_{p-1}A^{p-1}r_0) = \min.$$

To determine x_p one has to solve a linear system of p equations in the p unknowns $\alpha_0, \cdots, \alpha_{p-1}$.

Define $\lambda_1, \cdots, \lambda_n$ as in §5. Let the interval $\lambda_1 \leqq \lambda \leqq \lambda_n$ be transformed into the interval $-1 \leqq t \leqq 1$ by the affine transformation $t = \tau(\lambda)$, carrying λ_n into -1, and λ_1 into 1. Then $\tau(0) = \delta = (\lambda_n + \lambda_1)(\lambda_n - \lambda_1)^{-1}$. Let $T_n(t)$ be the ordinary Čebyšev polynomial of degree n, normalized so that $T_n(1) = \max_{-1 \leqq t \leqq 1} |T_n(t)| = 1$. In these notations Birman [9] has proved that, for each A,

$$(34) \qquad \left[\frac{f(x_p)}{f(x_0)}\right]^{1/2} \leqq \frac{1}{T_p(\delta)} = \mu_p < 1 \qquad (\text{all } x_0).$$

Thus (34) is the extension to $p > 1$ of (30).

Let $t_i = \tau(\lambda_i)$, so that $t_1 = 1$, $t_n = -1$. For a given A, the value μ_p cannot necessarily be attained by $[f(x_p)/f(x_0)]^{1/2}$. It will be attained, however, for any A with eigenvalues for which the corresponding t_i include all the $p+1$ values t with $|T_p(t)| = 1$. For such A, the x_0 for which $[f(x_p)/f(x_0)]^{1/2} = \mu_p$ are in the subspace spanned by just those eigenvectors u_i belonging to the λ_i for which $|T_p[\tau(\lambda_i)]| = 1$. The case $p = 1$ is exceptional, in that the maximum μ_1 is always attained in (34) = (30), just because the two values $t = \pm 1$ where $|T_1(t)| = 1$ are necessarily τ-images of eigenvalues.

For any fixed integer $p \geqq 1$ the above p-step process can be iterated to yield a convergent procedure for finding $A^{-1}b$. Namely, for $k = 0, 1, \cdots$, one obtains $x_{k+1}^{(p)}$ as the unique point x in $\pi_p(x_k^{(p)})$ for which $f(x)$ is minimized. The optimum gradient method of §5 is the case $p = 1$. If $p \ll n$ we may expect that there are eigenvalues λ_i of A close to the $p+1$ points where $|T_p[\tau(\lambda_i)]| = 1$, and hence that the value μ_p of (34) is almost attained for certain x_0. It is then to be expected that, for most x_0, $[f(x_{k+1}^{(p)})/f(x_k^{(p)})]^{1/2}$ will be approximately μ_p for all large k. Moreover, if $\lambda_1 \ll \lambda_n$, then δ is near 1 and μ_p is approximately $1 - 2p^2(\lambda_1/\lambda_n)$. Thus the minimization in p dimensions may be expected asymptotically to proceed p^2 times as fast as the optimum gradient method ($p = 1$), when $p \ll n$. If $\lambda_n/\lambda_1 = n^2$ (see §5), the iterated p-dimensional minimization in n-space may be expected to converge like the optimum gradient method in n/p dimensions.

The true asymptotic behavior of $x_k^{(p)} - A^{-1}b$ is unknown. Does the vector $x_k^{(p)} - A^{-1}b$ asymptotically lie in a certain $(p+1)$-dimensional subspace, as is conjectured for $p = 1$?

Because the above iterative process requires at each step solving a linear system of pth order, up to very recently the method was considered practical only for $p \ll n$. For $p = n$, in particular, it appeared that determining the minimizing α_i in (33) would involve solving a linear system quite as difficult as $Ax = b$. Then, about 1951, Stiefel [114], Lanczos [72], and Hestenes [50], working independently at first, all discovered that (33) can be minimized by p repetitions of a beautifully simple algorithm. By taking $p = n$ one actually finds $A^{-1}b$ in n steps, except for round-off errors. The resulting *conjugate gradient* method is thus a typical *finite iteration*. An extended exposition is in [54], while brief expositions are given by Rosser [100], Householder [59], and Taussky and Todd [117].

The conjugate gradient method is a nonlinear stationary iterative process. The first approximant, x_0, is arbitrary; one takes $p_0 = r_0 = b - Ax_0$. For any $k \geqq 0$, assume the vectors x_k, r_k, p_k have been determined. Then x_{k+1}, r_{k+1}, p_{k+1} are determined in order as follows:

$$
\begin{aligned}
\alpha_k &= r_k^T p_k / p_k^T A p_k; \\
x_{k+1} &= x_k + \alpha_k p_k; \\
r_{k+1} &= r_k - \alpha_k A p_k; \qquad (35) \\
\beta_k &= - r_{k+1}^T A p_k / p_k^T A p_k; \\
p_{k+1} &= r_{k+1} + \beta_k p_k.
\end{aligned}
$$

Here $r_k = b - Ax_k$, and the significance of p_k, α_k, β_k will appear below; β_k has a different meaning than in §5. In the absence of round-off errors, $x_n = A^{-1}b$; if round-off errors make $x_n \neq A^{-1}b$, one has merely to carry the algorithm on for $k = n+1, n+2, \cdots$, until sufficient accuracy is attained.

The kernel of a number of methods of solving $Ax = b$ for $A > 0$ is the determination of a set of n directions $\{p_k\}$ $(k = 0, \cdots, n-1)$ which are *conjugate* (*A-orthogonal*) in the sense that $p_i^T A p_j = 0$ for $i \neq j$. If the $\{p_i\}$ are known, then

$$
A^{-1} b = \sum_{k=0}^{n-1} (p_k^T b / p_k^T A p_k) p_k. \qquad (36)
$$

A convenient method to get the p_k is to apply the Gram-Schmidt process [99] of successively A-orthogonalizing some set of n linearly independent vectors $\{v_k\}$. In Gaussian elimination (pivotal condensation) the v_k are taken to be the n coordinate unit vectors, as Fox, Huskey, and Wilkinson [35] discovered, and the coefficients defining the orthogonalization build up a triangular matrix.

In the method of conjugate gradients the v_k are the vectors $r_k = b - Ax_k$ $(k=0, \cdots, r-1)$. The beauty of this choice is that r_{k+1} turns out to be automatically conjugate to $p_0, p_1, \cdots, p_{k-1}$. In picking p_{k+1} it is therefore necessary only to alter r_{k+1} in the direction p_k; the calculation of β_k and p_{k+1} in (35) has this object. The calculation of α_k and x_k is an iterative procedure for building up the sum (36).

Recalling that x_p actually minimizes (33), we see that in practice the residual r_p may become so small for some $p<n$ that it is unnecessary even to complete the n steps theoretically required to compute $A^{-1}b$. This occurred for $p=90$ in a successful calculation with $n=105$ of a difficult stress problem reported by Hochstrasser [56] and Stiefel [114]. Such a saving could hardly occur with Gaussian elimination, since the unit vectors have no such intimate connection with the system (1) as do the vectors r_k.

The conjugate gradient method has a geometrical interpretation. As in §5, one seeks the common center $A^{-1}b$ of the ellipsoids $f(x) =$ constant. Let x_0 be prescribed. One gets x_1 by performing one optimum gradient step (§5) in the unrestricted n-dimensional space R_n. Recall the definition of $\pi_1(x_0)$. There is an $(n-1)$-dimensional affine subspace R_{n-1} passing through x_1 and conjugate to $\pi_1(x_0)$. The solution $A^{-1}b$ lies in R_{n-1}. One gets x_2 by taking an optimum gradient step *within* R_{n-1}. The gradient $-2p_1$ of $f(x)$ at x_1 within R_{n-1} is the projection into R_{n-1} of the gradient $-2r_1$ of $f(x)$ at x_1 within R_n. The optimal point in the direction p_1 from x_1 is $x_2 = x_1 + \alpha_1 p_1$. Similarly, one gets x_{k+1} by taking one optimum gradient step from x_k within the $(n-k)$-dimensional affine subspace R_{n-k} through x_k conjugate to $\pi_k(x_0)$. Finally, R_0 is the solution point $A^{-1}b$.

This is a bare description of the method. In [54] Hestenes and Stiefel give an amazing number of its properties, discuss its application to unsymmetrical systems (1), and so on. A few machine experiments [51; 56; 57; 114] with the method suggest good stability with respect to round-off errors, but a theoretical study of the stability would be desirable.

The conjugate gradient method can theoretically be applied to solving a system $Ax=b$, where A is a bounded positive-definite self-adjoint operator on a Hilbert space. One defines $f(x) = (x, Ax) - 2(x, b) + (b, A^{-1}b)$; the inverse operator A^{-1} certainly exists. The method will ordinarily no longer converge in n steps, but the asymptotic behavior of $f(x_p)$ can be discussed. Karush [70] shows that if $A = \omega I + K$, where K is completely continuous and $\omega \neq 0$, then $f(x_p)$ goes to 0 faster than the pth term of any geometrical progression. Hayes [48] treats a general A with lower, upper bounds m, M

$(0<m<M<\infty)$, and proves that $f(x_p) \leqq [1-(m/M)]^p f(x_0)$. More can be proved:

Let $\delta = (M+m)(M-m)^{-1}$. The Birman inequality (34) shows that $[f(x_p)/f(x_0)]^{1/2} \leqq 1/T_p(\delta) < 1$. Hence, since $2T_p(\delta) = [\delta + (\delta^2-1)^{1/2}]^p + [\delta - (\delta^2-1)^{1/2}]^p$, one gets the estimate

$$(37) \qquad \left[\frac{f(x_p)}{f(x_0)}\right]^{1/2p} \leqq \frac{2^{1/p}}{\delta + (\delta^2 - 1)^{1/2}} \to \frac{1}{\delta + (\delta^2 - 1)^{1/2}}, \qquad \text{as } p \to \infty.$$

In another paper the asymptotic nature of $[f(x_p)]^{1/2p}$ will be described more precisely for a class of operators A with a continuous spectrum.

8. Errors and "condition." One must say something about the important but little-understood subject of errors. We may distinguish between: I. errors committed in the course of solving the system by a specific algorithm; II. errors inherent in the system $Ax=b$.

Within I one is concerned with the *truncation* errors and the *round-off* errors of an algorithm, a distinction explained in [124]. The truncation error exists for infinite iterations, and may be identified with $x_k - A^{-1}b$; its behavior has been examined in the above survey, under the assumption that there was no round-off error. The study of round-off error itself is far more difficult, and there seems to have been a complete discussion in connection with only one method, elimination; see von Neumann and Goldstine [124; 45], and also Mulholland [85]. For other studies see Bargmann, Montgomery, and von Neumann [6], Dwyer [20], Satterthwaite [102], Tuckerman [121], and Turing [122].

Any approximate solution ξ of $Ax=b$ can be checked a posteriori by forming the residual $\rho = b - A\xi$. The magnitude of $A^{-1}b - \xi$ can then be estimated by using some tool for examining errors under II. Hence to bound the errors in a calculated ξ it is unnecessary to have a priori knowledge of the accumulation of round-off error. Such knowledge may be important, however, for planning purposes—for example, in deciding in advance how many digital places to carry in order that ρ be reasonably small.

The errors under II have attracted more study. The practical analyst, realizing that the elements of A and b are subject to uncertainty, wishes to know the corresponding uncertainty in $A^{-1}b$; the latter is the *inherent error* of Milne [81]. The usual approach is the approximate one of bounding the principal part of the error, $\delta(A^{-1}b) = A^{-1}(\delta A)A^{-1}b + A^{-1}\delta b$; see Blumenthal [11], Milne [81], Moulton [84], Ostrowski [90], Wittmeyer [126], and Zurmühl [131].

But others (e.g., Janet [64], Ostrowski [89], Lonseth [77]) bound the entire error, finding a region S to which $A^{-1}b$ is rigorously confined, given $\rho = b - A\xi$ and other reasonably computable quantities associated with the system. See also Woodbury [128].

Various persons (e.g., Jürgens [66], Todd [119], and Turing [122]) have attempted to ascribe a *condition* to the system $Ax = b$. In part, the condition should measure the influence on $A^{-1}b$ of small changes in A and b; the larger the change in $A^{-1}b$ for given changes in A and b, the "worse" the condition. Although the condition depends on both A and b, the measures hitherto proposed depend only on A. The belief is widespread that the condition of a system (1) has a decisive influence on the convergence of an iterative solution and on the accuracy of a direct solution; this cannot always be true. Even when it is true for an iterative process, it may be possible actually to take advantage of the poor condition of (1) in converting the slow process into an accelerated method which converges rapidly. There is great need for clarification of the group of ideas associated with "condition."

With the concept of "ill-conditioned" systems $Ax = b$ goes the idea of "preconditioning" them. Gauss [41] and Jacobi [63] made early contributions to this subject. That of Gauss is analyzed and extended in [31].

A convenient means of preconditioning is to premultiply the system with a matrix B, so that one has to solve

$$BAx = Bb. \tag{38}$$

The perfect choice of B would be A^{-1}. A frequent choice is A^T, so that (38) gets a symmetric matrix A^TA, very convenient for many processes, though "worse conditioned" in some senses (Taussky [115]). Is there in any sense a "best" choice of B which is quickly obtainable?

The Gauss elimination process may be written in the form (38), where in the absence of round-off errors BA is a triangular matrix. In some calculations the true solution $A^{-1}b$ comes from iteration of the back solution—i.e., of getting x_{k+1} by an approximate solution of the triangular system $BA(x - x_k) = B(b - Ax_k)$. Where this occurs, we may interpret the forward solution or triangularization as merely a preconditioning of the system (1) into the form (38).

9. **Influence of computing equipment.** The usefulness of a process for solving (1) depends intimately on the properties of the machine on which the calculation takes place, as well as on the special character of A and b. The past decade has seen revolutionary develop-

ments in computing equipment: analogue machinery, desk computers, IBM equipment, and automatically sequenced high-speed digital computers. As a result, computing methods are in no way settled down, and bibliographies are out of date before publication.

Analogue machinery can be very useful, but is not discussed here; for references see Engineering Research Associates [22], Frame [36], and Murray [86].

While direct methods for solving $Ax=b$ have been little mentioned here, they have been very successful since the time of Gauss or earlier. Dwyer [20], Bodewig [12] and others conclude that a compact arrangement of Gaussian elimination is commonly the best method for the computing team of a desk machine, a data sheet, and a trained human being—principally because the number of operations is minimized. Elimination is very successful with IBM machines also, but its superiority over other methods is less pronounced, because it is seldom expedient with IBM equipment to use the compact arrangements which save so many operations. A bibliography on computing methods for IBM machinery is given in [38a].

Let us now consider automatically programmed digital computers like the SWAC. These are much faster than previous computers, but the speed of arithmetic operations and of access to a small store of data (high-speed memory) has been accelerated a great deal more than the operations of input, output, and access to a large store of data. The resulting change in the relative costs of different operations has a profound effect on the choice of computing methods. One soon learns that a variety of processes have been tried and advocated for solving (1); certainly the optimal method depends on the problem, the machine, the operator, and the coder. Moreover, small changes in these factors may radically alter the optimal choice of method.

The following tentative ideas are based on a limited experience with the SWAC, and practically no experience with other machines. The analysis is dominated by the relative shortage of memory cells in the SWAC; it is therefore less pertinent for machines with more storage space, and for the SWAC after the expected addition of a magnetic drum. Assume n to be fairly large, say $\geqq 15$. For simplicity we again confine our attention to matrices $A>0$. As indicated after (38), the forward solution in elimination amounts to building up a new matrix BA, which must be stored somewhere. If BA is kept in the high-speed memory, it occupies critically needed space. If it is output and input as needed (say by rows, as Huskey [61] describes), the programming is complicated and the solution is considerably slowed. If A is a matrix which can be generated internally as needed (for instance, the matrix of the Laplace difference operator), it requires little space, and BA becomes the principal item to be stored. Where A cannot be generated internally, the storage problem gets still worse, because the round-off errors can only be reduced by using A to compute the residual $r=b-Ax$ from time to time, so that both BA and A must be stored.

These considerations suggest that a solution process should preferably work with only one matrix, A itself, and should require relatively little other storage. Since the instructions have to be stored, this suggests a process of simple structure, repeated as often as necessary. A process seeming to require a minimum of storage is the cyclic single-step procedure mentioned in §3; besides A (if it must be stored), one need store only one vector, x, at a time. This method was picked by Reich [96] as best for a digital computer, and is undoubtedly ideal when it converges fast enough. But we may expect that the convergence is frequently too slow. If an acceleration of the types discussed in §4 is needed, the complication in programming may make another procedure preferable. Another method of speeding up the cyclic single-step method is by appropriately overcorrecting at each step, as discussed by Frankel [38] and Young [129] for special systems (1). It seems likely that a careful determination of the optimal overcorrection will sometimes provide adequate convergence, but that it will often fail.

The ordinary relaxation (i.e., piecewise linear) processes require about the same storage as the cyclic single-step methods; it is not clear whether they are essentially faster or not. A suggestion of Motzkin and Schoenberg [83] for extreme overrelaxation is promising but untried.

If the above methods fail, one can switch to the optimum gradient method of §5. This also works with A, which must be stored or generated, and further requires the storage of the two vectors x_k and r_k. (The storage of x_k can be avoided if $x_{k+1}-x_k$ is output at each step, and cumulated later.) Again the method is probably commonly too slow. It can be speeded up either by the β-optimum device of §5, for $\beta<1$, or by Richardson's idea [98] of widely varying $\alpha_k=\beta\alpha_k^*$ over the range of eigenvalues λ_i^{-1} of A^{-1}.

If these tricks fail or require too complex a program, the gradient methods of §7 are available. Besides A, they require the storage of three vectors x_k, r_k, p_k. (As above, the outputting of $x_{k+1}-x_k$ saves storing x_k). Of these methods, there seems to be no reason for not adopting the conjugate gradient method, since for the same storage its convergence is much the best. Programming is simple, as only one routine is needed; all necessary variations in the α_k are provided automatically. A drawback is that, since the α_k and other numbers vary so much in the calculation, it is difficult to provide scaling factors in advance. Consequently one uses "floating binary point" operations, requiring considerable memory space to hold the instructions and multiplying the solution time by a factor which varies on the SWAC from the order of 50 (when A is generated internally) to about one (when A is input at each iterative step). But the method has proved able to cope with some "badly conditioned" matrices, as reported by Hestenes, Hochstrasser, and Wilson [51], and Hochstrasser [57]. It probably approaches the ideal of a machine method which can be relied on to work automatically without special analysis of the particular system (1).

With any method the partitioning of A may greatly increase the speed by enabling subsidiary matrix inversions to take place entirely within the high-speed memory; see [21]. One usually thinks of Gaussian elimination on the submatrix blocks. Would other methods on the blocks be preferable?

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NATIONAL BUREAU OF STANDARDS, LOS ANGELES

THE APRIL MEETING IN NEW YORK

The four hundred ninetieth meeting of the American Mathematical Society was held at New York University in New York City on Thursday through Saturday, April 23–25, 1953. Over 600 persons attended the meeting, including the following 444 members of the Society:

C. R. Adams, L. V. Ahlfors, M. I. Aissen, A. A. Albert, R. G. Albert, F. L. Alt, D. B. Ames, S. A. Amitsur, A. G. Anderson, R. D. Anderson, R. G. Archibald, D. N. Arden, P. N. Armstrong, H. A. Arnold, Emil Artin, M. C. Ayer, George Bachman, W. G. Bade, F. E. Baker, K. Y. Bal, R. P. Bambah, W. E. Barnes, J. H. Barrett, L. K. Barrett, E. H. Batho, J. D. Baum, L. W. Baumhoff, E. G. Begle, F. D. Bennett, Jerome Berkowitz, C. H. Bernstein, D. L. Bernstein, Lipman Bers, Nicholas Bilotta, M. A. Biot, Garrett Birkhoff, D. W. Blackett, Jerome Blackman, W. E. Bleick, I. E. Block, E. K. Blum, H. W. Bode, Paul Boschan, S. G. Bourne, J. W. Bower, C. B. Boyer, A. D. Bradley, C. C. Bramble, Richard Brauer, Eleazer Bromberg, F. E. Browder, A. B. Brown, J. L. Brown, Jr., Lorenzo Calabi, F. P. Callahan, Jr., G. F. Carrier, W. C. Carter, C. R. Cassity, M. N. Chase, Y. W. Chen, Peter Chiarulli, Joshua Chover, Sarvadaman Chowla, S. I. Ciolkowski, F. M. Clarke, E. J. Cogan, Haskell Cohen, L. W. Cohen, R. M. Cohn, H. R. Cooley, G. A. Coon, T. F. Cope, Richard Courant, V. F. Cowling, M. J. Cox, H. S. M. Coxeter, J. B. Crabtree, R. H. Cramer, E. H. Crisler, A. B. Cunningham, P. M. Curran, H. B. Curry, M. L. Curtis, P. C. Curtis, Jr., J. H. Curtiss, M. D. Darkow, D. A. Darling, J. A. Daum, D. R. Davis, Philip Davis, R. B. Davis, R. M. Davis, Karel DeLeeuw, C. R. DePrima, A. H. Diamond, J. B. Diaz, C. E. Diesen, Alexander Dinghas, Avron Douglis, F. G. Dressel, R. J. Duffin, Nelson Dunford, W. L. Duren, Jr., J. S. Dwork, W. F. Eberlein, Albert Edrei, Samuel Eilenberg, C. C. Elgot, Robert Ellis, D. I. Epstein, M. P. Epstein, M. E. Estill, H. J. Ettlinger, Trevor Evans, R. M. Exner, W. H. Fagerstrom, Herbert Federer, W. E. Ferguson, F. A. Ficken, Irwin Fischer, A. D. Fleshler, E. E. Floyd, William Forman, Harold Forstat, R. M. Foster, Phyllis Fox, W. C. G. Fraser, Gerald Freilich, Bernard Friedman, M. B. Friedman, K. O. Friedrichs, Orrin Frink, Jr., R. E. Fullerton, J. W. Gaddum, M. P. Gaffney, A. S. Galbraith, David Gale, H. P. Galliher, Jr., Mariano Garcia, Jr., Boris Garfinkel, G. N. Garrison, F. W. Gehring, Hilda Geiringer, Leonard Geller, B. H. Gere, J. J. Gergen, Murray Gerstenhaber, H. A. Giddings, J. H. Giese, David Gilbarg, B. P. Gill, Wallace Givens, R. D. Glauz, Sidney Glusman, Herbert Goertzel, Karl Goldberg, Samuel Goldberg, S. I. Goldberg, J. K. Goldhaber, Saul Gorn, H. M. Griffin, H. C. Griffith, Emil Grosswald, Laura Guggenbuhl, Felix Haas, V. B. Haas, R. W. Hamming, M. E. Hamstrom, O. G. Harrold, Jr., B. I. Hart, Alvin Hausner, G. A. Hedlund, A. E. Heins, Alex Heller, I. R. Hershner, Jr., J. H. Hett, T. H. Hildebrandt, I. I. Hirschman, Jr., F. E. P. Hirzebruch, S. P. Hoffman, D. L. Holl, Carl Holtom, H. G. Hopkins, Alfred Horn, J. G. Horne, L. A. Hostinsky, A. S. Householder, W. A. Howard, C. C. Hsiung, L. C. Hutchinson, D. H. Hyers, Eugene Isaacson, H. G. Jacob, Jr., Nathan Jacobson, T. J. Jaramillo, W. S. Jardetzky, Fritz John, R. E. Johnson, F. B. Jones, Bjarni Jónsson, M. L. Juncosa, R. V. Kadison, Shizuo Kakutani, Aida Kalish, Wilfred Kaplan, S. N. Karp, W. H. Keen, D. G. Kendall, J. F. Kiefer, A. R. Kirby, M. D. Kirby, J. W. Kitchens, M. S. Klamkin, George Klein, J. R. Kline, Morris Kline, I. I. Kolodner, B. O. Koopman, Saul Kravetz, M. D. Kruskal, R. R. Kuebler, Jr., Serge Lang, R. E. Langer, P. D. Lax, C. Y. Lee, J. R. Lee, Solomon Lefschetz, Joseph Lehner, Marguerite Lehr, Marie Lesnick, M. E. Levenson, D. J. Lewis, J. A. Lewis, J. V. Lewis, B. W. Lindgren, M. A. Lipschutz, E. R. Lorch, Lee Lorch, Mark Lotkin, S. C. Lowell, C. I. Lubin, G. S. S. Ludford, Eugene Lukacs, P. B. McKowen, E. J. McShane, L. A. MacColl, R. W. MacDowell, G. W. Mackey, D. B. MacMillan, H. M. MacNeille, W. G. Madow, Wilhelm Magnus, W. R. Mann, Morris Marden, A. J. Maria, M. H. Maria, M. H. Martin, W. S. Massey, A. P. Mattuck, L. F. Meyers, Joseph Milkman, K. S. Miller, W. H. Mills, R. v. Mises, Don Mittleman, E. E. Moise, Deane Montgomery, Halina Montvila, C. N. Moore, T. W. Moore, W. E. Moore, C. S. Morawetz, Morris Morduchow, G. W. Morgan, William Moser, Wolfe Mostow, B. H. Murdoch, W. L. Murdock, C. H. Murphy, Jr., Bernard Mushinsky, John von Neumann, L. W. Neustadt, D. J. Newman, H. K. Nickerson, Katsumi Nomizu, A. B. Novikoff, F. G. O'Brien, R. E. O'Donnell, E. M. Olson, E. T. Onat, Alex Orden, Robert Osserman, Morris Ostrofsky, R. H. Owens, J. C. Oxtoby, S. V. Parter, Emanuel Parzen, L. E. Payne, L. G. Peck, A. M. Peiser, Anna Pell-Wheeler, A. J. Penico, A. S. Peters, I. D. Peters, F. P. Peterson, B. J. Pettis, C. M. Petty, R. S. Pierce, Edmund Pinney, Everett Pitcher, Morris Plotkin, H. O. Pollak, Hillel Poritsky, Aditya Prakash, C. M. Price, W. W. Proctor, M. H. Protter, Hans Rademacher, Arthur Radin, R. A. Raimi, H. E. Rauch, G. E. Raynor, M. S. Rees, Russell Remage, Moses Richardson, P. R. Rider, P. D. Ritger, Herbert Robbins, J. H. Roberts, R. A. Roberts, M. S. Robertson, G. de B. Robinson, L. V. Robinson, Mark Robinson, Robin Robinson, Selby Robinson, F. V. Rohde, G. F. Rose, I. H. Rose, M. E. Rose, David Rosen, A. S. Rosenthal, Edward Rosenthall, M. F. Rosskopf, J. P. Roth, S. G. Roth, Walter Rudin, J. P. Russell, C. W. Saalfrank, Charles Saltzer, H. E. Salzer, Hans Samelson, J. E. Sammet, Jacob Samoloff, James Sanders, W. C. Sangren, Arthur Sard, W. K. Saunders, A. B. Schacknow, A. T. Schafer, R. D. Schafer, J. A. Schatz, Samuel Schecter, Albert Schild, Abraham Schwartz, G. B. Seligman, D. B. Shaffer, Edna Sheinhart, R. W. Shephard, R. T. Shield, J. R. Shoenfield, S. S. Shu, K. M. Siegel, Robert Simon, David Singer, James Singer, Jerome C. Smith, P. A. Smith, J. J. Sopka, Clifford Spector, S. K. B. Stein, Marvin Stern, R. L. Sternberg, F. M. Stewart, J. J. Stoker, R. R. Stoll, R. L. Swain, Olga Taussky, R. L. Taylor, William Clare Taylor, Feodor Theilheimer, G. L. Thompson, D. L. Thomsen, Jr., R. M. Thrall, D. E. van Tijn, E. W. Titt, John Todd, M. L. Tomber, P. M. Treuenfels, A. W. Tucker, Bryant Tuckerman, R. J. Turyn, S. I. Vrooman, H. E. Wahlert, H. V. Waldinger, J. L. Walsh, C. Y. Wang, Jack Warga, W. R. Wasow, G. C. Webber, M. T. Wechsler, J. V. Wehausen, H. F. Weinberger, J. H. Weiner, Alexander Weinstein, Louis Weisner, Paul Weiss, Bernard Weitzer, David Wellinger, W. J. Wells, Franc Wertheimer, F. J. Weyl, G. N. White, Jr., J. H. White, Jr., A. L. Whiteman, P. M. Whitman, Hassler Whitney, G. T. Whyburn, W. M. Whyburn, H. H. Wicke, Albert Wilansky, Herbert Wilf, J. E. Wilkins, Jr., A. B. Willcox, František Wolf, N. Z. Wolfsohn, Arthur Wouk, Hidehiko Yamabe, Michael Yanowitch, D. M. Young, Jr., Arthur Zeichner, J. A. Zilber, R. E. Zindler, Leo Zippin.

A Symposium in Applied Mathematics, sponsored jointly by the Office of Ordnance Research and the Society, took place on Thursday and Friday.

On Friday, Professor W. S. Massey of Brown University delivered an invited address on *Some new algebraic methods in topology* at a general session, Professor Hans Samelson presiding. On Saturday, Professor Emil Artin of Princeton University delivered an invited address on *Cohomology and class field theory* at a general session, Professor Richard Brauer presiding.

Sessions for contributed papers were held Friday afternoon and Saturday morning and afternoon. Presiding at these sessions were Professors Y. W. Chen, F. A. Ficken, R. E. Langer, Dr. L. A. MacColl, Professors M. H. Martin, C. N. Moore, R. D. Schafer, P. R. Rider, J. H. Roberts, G. de B. Robinson.

The members of the Institute for Mathematics and Mechanics of New York University entertained at tea on Friday at 5:00 P.M.

The Council met on April 24 at 5:00 P.M. in Waverly Building, reconvening after dinner.

The Secretary announced the election of the following fifty-five persons to ordinary membership in the Society:

Mr. James Robert Boyd, San Marcos High School, San Marcos, Texas;
Mr. George Ulrich Brauer, University of Michigan;
Mr. John Leslie Chamberlin, Marquardt Aircraft Company, Van Nuys, California;
Mr. Ward Cheney, University of Kansas;
Mr. Ralph Theodore Dames, Willow Run Research Center, Ypsilanti, Michigan;
Mr. William Frank Darsow, DePaul University;
Dr. William E. Davis, Hercules Powder Company, Wilmington, Delaware;
Professor Meyer Dwass, Northwestern University;
Mr. Ben Elwood Dyer, Ben Dyer, Consultants, New York, New York;
Mr. Daniel O'Connell Etter, Tulane University;
Mr. Arthur Goeffrey Eyles, Arthur G. McKee and Company, Cleveland, Ohio;
Mr. Royal Nathaniel Fitchett, Jr., Rome Air Development Center, Griffiss Air Force Base, Rome, New York;
Dr. Bernard Abraham Fleishman, Johns Hopkins University;
Mrs. Ruth M. Frisch, Syracuse University;
Mr. Arthur Oris Garder, United Gas Corporation, Shreveport, Louisiana;
Rev. Francis Joseph Ginivan, Saint Francis Xavier University, Antigonish, Nova Scotia;
Mr. Ben Theodor Goldbeck, Jr., University of Oklahoma;
Mr. Edward P. Graney, Willow Run Research Center, Ypsilanti, Michigan;
Mr. Glenn William Graves, Willow Run Research Center, Ypsilanti, Michigan;
Mr. Norman Greenspan, Polytechnic Institute of Brooklyn;
Mr. James Stonely Hall, McGill University;
Mr. Alan Howard Halpin, Willow Run Research Center, Ypsilanti, Michigan;
Dr. Charles Dewitt Harris, Carter Oil Company, Tulsa, Oklahoma;
Mr. John Henry Holland, International Business Machines Corporation, Poughkeepsie, New York;
Mr. Robert Edwin Kalaba, The Rand Corporation, Santa Monica, California;
Mr. Martin Krakowski, Carnegie Institute of Technology;
Mr. L. Clark Lay, John Muir College, Pasadena, California;
Mr. Alfred S. Lee, Magnolia Petroleum Company, Dallas, Texas;
Mr. Roger Joseph Lemelin, Willow Run Research Center, Ypsilanti, Michigan;
Mr. Harold George Loomis, Haller, Raymond and Brown, Inc., State College, Pennsylvania;

Dr. Theodore Arceola Love, Tennessee Agricultural and Industrial State College, Nashville, Tennessee;
Mr. Paul Joseph McCarthy, University of Notre Dame;
Mr. Matthew A. Medick, International Business Machines Corporation, New York, New York;
Mr. Tito Abella Mijares, University of Santo Tomas and University of the East, Manila, Philippine Islands;
Mr. Jack Culbertson Miller, Pomona College;
Mr. J. Sayer Minas, University of Illinois;
Mr. Norton Leonard Moise, University of California, Los Angeles, California;
Mr. Raphael Morena-Castro, Western Reserve University;
Mr. Robert Marvin Natkin, Chicago, Illinois;
Miss Donna May Neeb, University of Michigan;
Mr. Monroe Lawrence Norden, Johns Hopkins University;
Mr. Edward Joseph Pellicciaro, University of North Carolina;
Mr. John William Petro, University of Chicago;
Professor Edwin James George Pitman, University of Tasmania;
Mr. Ivan Paul Polonsky, University College, New York, New York;
Mr. Alfred Reichenthal, Arma Corporation, Brooklyn, New York;
Mr. Robert Edward Ross, Avco Manufacturing Company, Bethel, Connecticut;
Mr. Henry H. Ryffel, The Industrial Press, New York, New York;
Mr. Seymour Sherman, Bell Telephone Laboratories, Whippany, New Jersey;
Mr. Robert La Vern Slater, Jr., Chicago Midway Laboratories, Chicago, Illinois;
Mr. Victor Richard Staknis, Boston University;
Mr. Gene Thomas Thompson, Oregon State College;
Professor V. P. Venkatachari, Osmania University, Deccan, India;
Mr. Julius Widrewitz, Griffiss Air Force Base, Rome, New York;
Mr. Herbert Wilf, Columbia University.

It was reported that the following one hundred ninety-three persons had been elected to membership on nomination of institutional members as indicated:

University of Alabama: Miss Susie Lee Ward.

University of British Columbia: Mr. Wilfred Eaton Barnes, Mr. Charles Andrew Swanson, and Mr. Donald Alastair Trumpler.

Brown University: Mr. Harry Geoffrey Hopkins.

California Institute of Technology: Mr. William Daniel Dean, Mr. Wesley R. Guebert, Mr. Juris Hartmanis, Mr. John Beverley Johnston, and Mr. Rodrigo Alvaro Restrepo.

University of California, Berkeley: Mr. Chen-Chung Chang, Mr. Arthur Shapiro, and Mr. George Powell Steck.

University of California, Los Angeles: Mr. Ali Reza Amir-Moez, Miss Olive Jean Dunn, Mr. Harold Parkins Edmundson, Mr. Richard Carl Gilbert, Mr. Sheldon Green, Mr. Charles John August Halberg, Jr., Mr. Eugene Levin, Mr. Mervin F. Muller, and Mr. Chien Wenjen.

Carnegie Institute of Technology: Mr. Carlton Edward Lemke.

University of Chicago: Mr. Jacob Feldman, Mr. Malcolm Goldman, Mr. Samuel Shaheen Holland, Jr., Mr. Brindell Horelick, Mr. Royal Bruce Kellogg, Mr. Richard Leroy McKinney, Mr. Michael Darwin Morley, Mr. Edward Nelson, Mr. Robert Harvey Oehmke, Mr. Eugene Carroll Paige, Jr., Mr. Morris Schreiber, Mr. Jerome Spanier, and Mr. Elias M. Stein.

University of Cincinnati: Mr. James Massey Shaheen.

City College of New York: Mr. Harvey Philip Greenspan and Mr. Samson Mordecai Rosenzweig.

Columbia University: Mr. S. David Berkowitz, Mr. Douglas Grassel Dickson, Dr. Alexander Dinghas, Mr. Max Beerbohm Friedman, Mr. Donald Bashford MacMillan, Mr. Mark Robinson, and Miss Miriam Schapiro.

Cornell University: Mr. Raquel Rosa Heller, Mr. Martin Baer Herskovitz, Mr. David Hertzig, Mr. Leonard F. Kilian, Mr. Elliott Mendelson, Mr. Jerome Sacks, and Mr. David Edward Schroer.

Duke University: Mr. William Jeffrey Coles and Mr. William Rodger Smythe, Jr.

Haverford College: Mr. Robert Trull Ives.

University of Illinois: Mr. Samuel Eli Benesch.

Indiana University: Mr. Theodore Henry Miller Crampton, Mr. James A. Nickel, Mr. Joseph Francis Schell, Miss Jane Anna Uhrhan, and Mr. Robert Weiller.

Institute for Advanced Study: Dr. Shimshon Avraham Amitsur, Dr. Ram Prakash Bambah, Dr. Armand Borel, Dr. Morikuni Goto, Dr. Friedrich Ernst Peter Hirzebruch, Dr. Richard Lee Ingraham, Dr. Gopinath B. Kallianpur, Dr. Yukiyosi Kawada, Dr. Shigeo Sasaki, and Mr. Hidehiko Yamabe.

Iowa State College: Mr. Buchanan Cargal.

Johns Hopkins University: Mr. Robert W. Bass, Mr. Juri Vello Nou, Mr. Raymond Joseph Pipino, Mr. Mark E. Stern, Mr. Shlomo Zvi Sternberg, and Mr. August Martin Wildberger.

University of Kansas: Mr. Kenneth Robert Lucas, Mr. Isaac Namioka, and Dr. Shambhu Dayal Sinvhal.

Kenyon College: Mr. Robert George Busacker.

University of Maryland: Mr. Karl Heinz Dieter and Mr. Benjamin Yee-Chieh Koo.

Massachusetts Institute of Technology: Miss Evelyn Mary Bender, Mr. Harry Floyd Davis, II, Miss Phyllis Fox, Mr. William Brunner Kehl, Mr. Ernest Bronson Leach, and Mr. Jacob Joseph Levin.

Michigan State College: Mr. John Lucian Bagg, Mr. Alton Thomas Butson, Mr. William Gerald Franzen, Mr. Hugo Alexander Myers, Mr. Orrin Edison Taulbee, and Mrs. Patricia James Wells.

University of Michigan: Mr. William Price Brown, Mr. Walter Feit, Mr. Ronald Kay Getoor, Mr. James Raymond Munkres, and Mr. James Maxwell Osborn.

University of Minnesota: Mr. Glen Earl Baxter, Mr. Charles McMurray Braden, Mr. Leon Brown, Mr. Robert Ernest Fagen, Mr. Ernest Raymond Johnston, Mr. Jesse Marshall Shapiro, and Mr. David Zeitlin.

New York University: Mr. Halina Montvila, Mr. Paul David Ritger, and Mr. Peter Ungar.

Northwestern University: Mr. Robert Rolf Christensen.

Ohio State University: Mr. Albert George Fadell, Mr. Robert Morton Haber, Mr. Billy O. Hoyle, Miss Verna Frances Lair, Mr. Shen Lin, Mr. Christoph Johannes Neugebauer, Miss Phyllis Rubin, and Mr. Robert James Thompson.

Oklahoma Agricultural and Mechanical College: Professor Ladislaus J. Fila, Professor Franklin Arno Graybill, and Mr. Aboulghassem Zirakzadeh.

University of Oregon: Mr. Hubert Edwin Chrestenson.

University of Pennsylvania: Mr. Willard Ellis Baxter, Mr. Herbert Morton Gurk, Mr. Burrowes Hunt, Mr. Justin Jesse Price, Mr. Jerome Raymond Ravetz, and Mr. Richard Edmund Williamson.

Princeton University: Dr. Darrel Don Aufenkamp, Mr. Walter Lewis Baily, Mr. Albert Andrew Folop, Mr. John Blackman Fraleigh, Mr. Ralph Edward Gomory, Dr. Stefano Guazzone, Dr. Victor K. A. M. Gugenheim, Mr. Robert Clifford Gunning, Mr. Douglas Weir Hall, Dr. Harlan Duncan Mills, Mr. Sigurdur Helgason, Mr. Brian Hughs Murdoch, Dr. Albert Nijenhuis, Mr. Carl Reading Ohman, Mr. Charles David Parker, Mr. Franklin Paul Peterson, Dr. Clinton Myers Petty, Mr. Barth Pollak, Mr. Bruce Lloyd Reinhart, Mr. Kiron Chandra Seal, and Mr. Samuel James Taylor.

Purdue University: Mr. Robert Arnold Gambill.

Queens College: Mr. Maurice Leonard Richter and Miss Florence Spatz.

Rice Institute: Mr. Thomas Muir Gallie, Jr., and Mr. James Alexander Hummel.

Rutgers University: Mr. John Bender.

College of Saint Thomas: Mr. Paul John Nikolai.

Syracuse University: Mr. Sullivan Graham Campbell, Mr. Edwin Foote Gillette, Mr. Charles Stanley Ogilvy, and Miss Jacqueline Lax Zemel.

University of Tennessee: Miss Gertrude Ehrlich.

University of Texas: Mr. Clair Eugene Abraham, Mr. Steve Armentrout, Mr. James Warren Evans, Mr. John Theodore Mohat, Mr. Charles Stuart Stone, and Mr. James Newton Younglove.

University of Toronto: Mr. Irwin Guttman, Mr. Katsumi Okashimo, Mr. Seymour Schuster, Mr. David Arthur Sprott, Mr. Jakob Abraham Steketee, and Mr. Murray Scott Watkins.

University of Washington: Mr. James Harold McKay and Mr. Douglas Vern Newton.

Wellesley College: Miss Isabella Waldie.

Williams College: Mr. William Hollis Peirce.

University of Wisconsin: Mr. John West Addison, Jr., Mr. Edward H. Batho, Miss Anna Chandapillai, Mr. Nicholas D. Kazarinoff, Mr. Guydo Rene Lehner, Mr. Bruce Leon Lercher, Mr. Robert William McKelvey, Mr. Martin Herbert Pearl, Mr. Carl Joseph Vanderlin, and Mr. Roger Norman Van Norton.

Yale University: Mr. Philip Chadsey Curtis, Mr. Paul Ernest Klebe, Jr., and Mr. Thomas Allen Paley.

The Secretary announced that the following had been admitted to the Society in accordance with reciprocity agreements with various mathematical organizations: Deutsche Mathematikervereinigung: Dr. Arno Jaeger, University of Illinois, and Professor Wilhelm Süss, University of Freiburg; Société Mathématique de France: Dr. Louis Camille Maurice Nollet, Université de Liège, and Dr. Jamil Ahmad Siddiqi, Université de Paris; Svenska Matematikersamfundet: Mr. Göran Björck, University of Stockholm, Professor Lars Göran Borg, Royal Institute of Technology, Professor Otto Albin Frostman, University of Stockholm, and Mr. Arne Elis Pleijel, Kommunala Realskolan, Hagfors, Sweden.

The following appointments by the President were reported: as a committee to nominate officers and members of the Council for 1954: Professors Marston Morse (Chairman), J. C. Oxtoby, A. C. Schaeffer, A. E. Taylor, and H. S. Wall; as a member of the Committee on Applied Mathematics: Garrett Birkhoff (Committee now consists of M. H. Martin (Chairman), R. V. Churchill, F. J. Murray, Eric Reissner, Shizuo Kakutani, and Garrett Birkhoff); as a member of the committee to recommend to the Council the award of the Bôcher Memorial Prize: B. J. Pettis (Committee now consists of L. V. Ahlfors (Chairman), Philip Franklin, and B. J. Pettis); as a committee to select Gibbs lecturers for 1954 and 1955: A. A. Albert (Chairman), J. L. Doob, and C. B. Morrey; as a committee on corporate memberships: C. C. Hurd (Chairman), M. M. Flood, and B. P. Gill; as a committee on arrangements for the summer meeting of 1953: Norman Miller (Chairman), L. W. Cohen, H. A. Elliott, H. M. Gehman, Israel Halperin, R. L. Jeffery, Percy Lowe, and F. M. Wood; as a committee on arrangements for the meeting to be held at Wofford College on November 27–28, 1953: T. L. Jordan (Chairman), John Hill, G. May, E. H. Shuler, and W. M. Whyburn.

The following appointments to represent the Society were reported: Professor C. R. Wylie, Jr. at the inauguration of J. Richard Palmer as Seventh President of Westminster College on January 9, 1953; Professor Walter Strodt at the inauguration of Buell Gordon Gallagher as Seventh President of the City College of New York on February 19, 1953; Rev. J. J. Lynch at the Centenary Celebration of Manhattan College on April 25, 1953; and Professor R. L. Moore at the dedication of Benedict Hall on the campus of the University of Texas on April 13–14, 1953.

The following items were reported for the information of the Council: selection of J. L. Doob as Managing Editor of the Transactions and Memoirs Editorial Committee; Leo Zippin as Chairman of the Mathematical Surveys Editorial Committee; G. B. Price as Managing Editor of the Bulletin Editorial Committee; G. A. Hedlund as Chairman of the Proceedings Editorial Committee; William Feller as Chairman of the Mathematical Reviews Editorial Committee (R. P. Boas has been approved as a substitute for Professor Feller during the period February–August, 1953); A. A. Albert as Chairman of the Colloquium Editorial Committee; C. J. Rees as Chairman of the Committee on Printing and Publishing; J. W. Green to substitute for Professor Feller as Chairman of the Committee to Study Problems of Nominations by Petition; reciprocity agreements have been concluded with the Indian Mathematical Society, the Finnish Mathematical Society, and the Icelandic Mathematical Society; the Committee on Applied Mathematics has voted in favor of postponing the Seventh Symposium in Applied Mathematics to 1955; the McGraw-Hill Book Company has agreed to publish the Proceedings of the

Sixth Symposium in Applied Mathematics; Professor Wassily Leontief has accepted an invitation to deliver the Josiah Gibbs Lecture at the Annual Meeting in 1953; committees to select hour speakers have invited Professor Ernst Snapper to deliver an address at the Stanford, California meeting on May 2, 1953; Professor A. T. Lonseth, Missoula, Montana meeting on June 20, 1953; Professor Salomon Bochner at the Summer Meeting at Kingston, Ontario.

The following actions taken by mail vote of the Council were reported: election of Professors E. E. Moise and J. C. Oxtoby to serve as members of the Executive Committee of the Council for a period of two years beginning January 1, 1953; acceptance by the Council of an invitation from Wofford College, Spartanburg, South Carolina, to hold a meeting there on November 27–28, 1953; and acceptance by the Council of an invitation from the Johns Hopkins University to hold the Annual Meeting in 1953 in Baltimore.

The Council voted to approve the following dates of meetings of the Society: October 24, 1953 at Columbia University; February 27, 1954 in New York City; April 23–24, 1954 at Columbia University; and November 26–27, 1954 at the State University of Iowa.

The Executive Director reported that the National Science Foundation has made a grant to the Society of $6300 to defray expenses of an experimental project to explore the possibilities of publication and distribution of the results of research in the field of mathematics. The Council voted to authorize the President to appoint a committee to advise with the Executive Director concerning this project.

The Secretary reported that the Policy Committee for Mathematics would be asked to nominate a member of the committee to evaluate the present functions and operations of the Bureau of Standards in relation to the present national needs.

The Council voted to approve holding the Sixth Symposium in Applied Mathematics at the City College in Santa Monica, California.

The Council voted to co-sponsor with the Committee on Training and Research in Applied Mathematics of the National Research Council a Conference on Training in Applied Mathematics at the time of the October 1953 meeting in New York and a Conference on Topics in Applied Mathematics at the time of the November 1953 meeting in Evanston, Illinois.

The Council voted to cooperate with the National Science Foundation on a roster of mathematicians and voted to authorize the President to appoint a committee to make recommendations to the Council concerning the size and scope of this roster.

The Council voted to remove its restriction on the exhibition of books at Summer and Annual Meetings that these books be at the graduate level or higher.

The Symposium consisted of the following papers, invited by the Office of Ordnance Research:

(1) M. J. Lighthill: *Mathematical methods in compressible flow theory.*

(2) G. F. Carrier: *Boundary layer problems in applied mathematics.*

(3) Garrett Birkhoff: *Fourier synthesis of homogeneous turbulence.*

(4) G. S. S. Ludford and M. H. Martin: *Non-isentropic flows in a shock tube.* (Presented by Professor Ludford.)

(5) J. H. Giese: *Approximate methods for computing flow fields.*

(6) Lipman Bers: *Results and conjectures in the mathematical theory of subsonic and transonic flows.*

(7) Alexander Weinstein: *The singular solutions and the Cauchy problem in the hodograph method.*

(8) P. Germain: *Remarks on the theory of partial differential equations of mixed type and applications to the study of transonic flows.*

(9) R. v. Mises: *Discussion on transonic flow problems.*

(10) John von Neuman: *Role of computing machines.*

(11) Mark Lotkin: *Some problems on computing machines.*

(12) P. D. Lax: *Generalized solutions of non-linear equations and their numerical calculation.*

(13) L. H. Thomas: *Computation of one dimensional flows including shocks.*

(14) G. E. Hudson: *Deformation of a metallic shell of non-uniform thickness by a detonating wave.*

(15) Hubert Schardin: *Measurements of spherical shock waves.*

Presiding at the sessions of the Symposium were Dr. C. C. Bramble, Dr. C. W. Lampson, Mr. C. L. Poor, Dr. T. E. Stern.

Abstracts of the papers presented are listed below. Where a paper has more than one author, that author whose name is followed by "(p)" presented it. Those papers with "*t*" following their numbers were read by title. Professor Fraïssé was introduced by Professor Alfred Tarski, Dr. Fleishman by Professor F. A. Ficken, Mr. Berg by Professor Lipman Bers, and Mr. Siry by Professor D. W. Hall.

ALGEBRA AND THEORY OF NUMBERS

323*t*. I. T. A. C. Adamson: *Cohomology groups of coset spaces.*

Let G be a group, H a subgroup, not necessarily normal. The left cosets of G modulo H are denoted by $A_\nu = \bar{\sigma}_\nu H$. A "coset space complex" K is constructed, the q-chain group of which, $C_q(K)$, is freely generated by the $(q+1)$-tuples of cosets

$[A_0, \cdots, A_q]$; these groups are made into G-modules by setting $\sigma[A_0, \cdots, A_q] = [\sigma A_0, \cdots, \sigma A_q]$ and extending the action of G to $C_q(K)$ by linearity. The boundary homomorphisms are defined by setting $\partial[A_0, \cdots, A_q] = \sum_{p=0}^{q} (-1)^p [A_0, \cdots, A_{p-1}, A_{p+1}, \cdots, A_q]$, and extending by linearity. The complex is augmentable and acyclic, but not free. If A is a G-module, the q-cochains of K are the G-homomorphisms of $C_q(K)$ into A; cohomology groups are defined in the usual way. The following results are obtained: (1) If $0 \to A' \to A \to A'' \to 0$ is an exact sequence of G-modules such that $H^1(U, A') = 0$ for all subgroups U of H, then there is an exact sequence $\to H^r(K, A) \to H^r(K, A'') \to H^{r+1}(K, A') \to H^{r+1}(K, A) \to$ for all r; (2) If A is a G-module such that $H^r(U, A) = 0$ for $r = 1, 2, \cdots, n-1$, and all subgroups U of H, then the sequence $0 \to H^n(K, A) \to H^n(G, A) \to H^n(H, A)$ is exact, where the homomorphisms are those induced by injection, inflation, and restriction respectively. (Received March 9, 1953.)

324*t*. I. T. A. C. Adamson: *Cohomology groups for non-normal fields.*

Let k be a field for which local or global class field theory holds. Let E be a finite extension of k, not necessarily normal. Let $A(E)$ be a group invariantly attached to E—e.g. the multiplicative group of E or the group of idèles or idèle classes of E. Let F be a normal extension of k containing E, G its Galois group, H the subgroup of G which cuts out E, K the coset space complex of G modulo H. Then the cohomology groups $H^r(A(E))$ are defined to be $H^r(K, A(F))$; they do not depend upon the normal extension F; when E is normal with Galois group G', $H^r(A(E)) = H^r(G', A(E))$. Let $A(E)$ be the multiplicative group of E (local case) or the group of idèle classes of E (global case); then (1) $H^1(A(E)) = 0$; (2) $H^2(A(E))$ is cyclic of order equal to the degree of E over k, generated by c^m, where c is the canonical generator of $H^2(G, A(F))$ and m is the index of H in G; (3) If I is the ideal of the integral group ring of G consisting of elements with sum of coefficients zero, then for $r > 0$, $H^r(A(E))$ is isomorphic to $H^{r-1}(K, I)$. (Received March 9, 1953.)

325*t*. Leonard Carlitz: *q-Bernoulli and Eulerian numbers.*

Frobenius (Sitzungsberichte der Preussischer Akademie der Wissenschaften, 1910, pp. 809–847) showed that many properties of the Bernoulli and related numbers can be derived from the Eulerian function $H_m(x)$ defined by $(H+1)^m = xH^m$, $m \geqq 1$. In the present paper we show that much the same can be done for the q-Bernoulli numbers (Duke Math. J. vol. 15 (1948) pp. 987–1000) by means of a generalized function $H_m(x, q)$ defined by $(qH+1)^m = xH^m$, $m \geqq 1$. In particular if we place q equal to a rational number, then we obtained analogues of the Staudt-Clausen theorem and of Kummer's congruences. (Received February 25, 1953.)

326*t*. Leonard Carlitz: *Some congruences of Vandiver.*

Results of the following kind are proved. Let $\{a_{i,m}\}$, $i = 1, \cdots, k$, denote sequences of integers such that $a_i^m(a_i^{p-1} - 1)^r \equiv 0 \pmod{p^m, p^r}$, where after expansion a_i^m is replaced by $a_{i,m}$ and p is a prime; also let $\lambda_1 + \cdots + \lambda_k \equiv 0 \pmod{p}$. Then $a_1^{m_1} \cdots a_k^{m_k}(\lambda_1 a_1^{p-1} + \cdots + \lambda_k a_k^{p-1}) \equiv 0 \pmod{p^{m_1}, \cdots, p^{m_k}, p^r}$. These results generalized certain congruences proved by Vandiver (Bull. Amer. Math. Soc. vol. 43 (1937) pp. 418–423). (Received February 25, 1953.)

327*t*. Leonard Carlitz: *Some hypergeometric congruences.*

By specializing the parameters in known hypergeometric identities, a large number of congruences are obtained which may be thought of as congruential analogues of various formulas of Ramanujan. For example we cite $(1/p)\sum_0^m (8r+1)$

$\cdot(1\cdot 5\cdots(4r-3)/4\cdot 8\cdots 4r)^4\equiv(1\cdot 3\cdot 5\cdots(4m-1)/(m!)^2 \pmod{p=4m+1}$, $\sum_0^m(-1)^r(4r+1)(1\cdot 3\cdots(2r-1)/2\cdot 4\cdots 2r)^3\equiv -2^{-1}(1\cdot 3\cdots(2m-1)/2\cdot 4\cdots 2m) \pmod{p=4m+3}$, where the moduli are primes. (Received February 25, 1953.)

328*t*. Leonard Carlitz: *The class number of an imaginary quadratic field.*

Let $h(d)$ denote the class number of the imaginary quadratic field $R(d^{1/2})$ of discriminant d. Let p be an odd prime divisor of d and $n\geqq 1$. In this paper we find the residue of $h(d) \pmod{p^n}$ in terms of Bernoulli polynomials. (For $n=1$ see A. Hurwitz, *Mathematische Werke*, vol. 2, Basel, 1933, pp. 208–235). (Received February 25, 1953.)

329*t*. Leonard Carlitz: *The coefficients of singular elliptic functions.*

Let $\mathrm{sn}\, x=\mathrm{sn}\,(x, k^2)$ be an elliptic function that admits of complex multiplication and let the period quotient belong to the imaginary quadratic field of discriminant d. Put $\mathrm{sn}\, x=\sum_0^\infty \alpha_{m+1}x^{2m+1}/(2m+1)!$, $x/\mathrm{sn}\, x=\sum_0^\infty \beta_m x^{2m}/(2m)!$. Then if p is an odd prime such that $(d/p)=-1$, we show that $\alpha_m\equiv 0 \pmod{p^r}$ for $m\geqq pr$; $\beta_m\equiv 0 \pmod{p^{r+e}}$ for $p^e\mid m$, $m>pr$, $p^2-1\nmid m$; $\beta_m\equiv 0 \pmod{p^{r-e-1}}$ for $p^e\mid m$, $p^{e+1}\nmid m$, $m>pr$, $p^2-1\mid m$. (Received February 25, 1953.)

330*t*. Leonard Carlitz: *The Schur derivative of a polynomial.*

Let $f(x)=f(x_1,\cdots,x_k)$ denote a polynomial in k indeterminates with integral coefficients. Define $\Delta f^{p^m}(x)=f'_m(x)=(f^{p^{m+1}}(x)-f^{p^m}(x^p))/p^{m+1}$, $\Delta^{r+1}f^{p^m}(x)=f^{r+1}_m(x)=(f^{(r)}_{m+1}(x)f^{p^{m+1}}(x)-f^{(r)}_m(x)f^{p^{m+1}r}(x^p))/p^{m+1}$. Generalizing the results of Schur it is shown that $\Delta^r f^{p^m}(x)$ has integral coefficients for $1\leqq r\leqq p-1$. Moreover the residue of $\Delta^r f^{p^m}(x) \pmod{p^m}$ is determined. (Received February 25, 1953.)

331*t*. Leonard Carlitz: *Weighted quadratic partitions.*

Let p be a prime >2, and put $S=\sum \exp\{2\pi i(2\lambda_1x_1+\cdots+2\lambda_tx_t)/p^r\}$ summed over all integers $x_1,\cdots,x_t \pmod{p^r}$ such that $a_1x_1^2+\cdots+a_tx_t^2\equiv c \pmod{p^r}$. Assuming that the λ's are integers and that the a_i are prime to p it is shown that S can be expressed in terms of Gauss and Kloosterman sums. If $(\lambda_1,\cdots,\lambda_t,p)=1$, the results are particularly simple. (Received March 6, 1953.)

332*t*. K. T. Chen, R. H. Fox, and R. C. Lyndon: *On the quotient groups of the lower central series.*

A workable algorithm is developed for computing the lower central quotients $Q_n(G)=G_n/G_{n+1}$ of a group $G=G_1$. This rests upon finding, for a free group F, a basis for Hom $(Q_n(F), Z)$, where Z = integers, consisting of certain higher differential operators in the free (Fox) calculus. Bases for Hom $(Q_n(F), Z)$ and, simultaneously, for $Q_n(F)$ are derived from combinatorial results on lexicographical order and the operation of "shuffling" two sequences. By-products: a direct proof of the Magnus-Witt theorem that F_n is isomorphic with the nth dimension group [W. Magnus, J. Reine Angew. Math. vol. 177 (1937) pp. 105–115, E. Witt, ibid. pp. 152–160]; determination of all finitary relations that hold identically among the coefficients in a Magnus power series of a group element. (Received March 3, 1953.)

333. Sarvadaman Chowla (p) and W. E. Mientka: *The rational points on a cubic curve.*

It is proved that the curve $y^2 = x^3 - Ax - B$ has no "exceptional" points if $A = 0$ (2), $B = 2$ (4). (Received March 9, 1953.)

334t. Harvey Cohn: *Density of Abelian cubics.*

The author shows that the number of *abelian* cubic fields of discriminant $\leqq x$ is $\sim$ const. $x^{1/2}$, so that, with repetitions counted in proper multiplicity, the discriminants are about as dense as perfect squares. This result is shown by the use of conventional "zeta-function methods," applied to $\prod(1-2p^{-s}) = f(s)$ (over $p = 1$ mod 3). It complements unpublished results kindly communicated by Dr. H. Davenport, to the effect that the number of *all* cubic fields of absolute discriminant $\leqq x$ is $O(x)$, and by Dr. H. Heilbronn to the effect that the number of *real* non-abelian cubics of discriminant $\leqq x$ is at least as large as const. x. (Research sponsored by the Army Office of Ordnance Research.) (Received March 9, 1953.)

335t. D. W. Dubois: *Primitivity, strict reality and the condition of Clifford in partly ordered fields.*

Let F be partly ordered with positive cone P. F is said to be (a) ***primitive*** if no proper subfield of F contains P; (b) ***strictly real*** if $x^2 \geqq 0$ holds for all $x \in F$; (c) ***Archimedean in the sense of Clifford*** (p.o.a.c.) if $nx < b$, $n = 1, 2, \cdots$, implies $x \leqq 0$. Among others, the following theorems are proved. (1) F is primitive if and only if it is the field of quotients of its ring B where B is the set of all $x \in F$ for which there are integers m, n with $m < x < n$; (2) F is strictly real if and only if its cone P is the intersection of a nonempty family $\mathfrak{P}$ of positive cones belonging to simple orderings of F; (3) F is primitive and p.o.a.c. if and only if F is strictly real with a family $\mathfrak{P}$ having the property that if x belongs to any $P' \in \mathfrak{P}$ then there is a positive rational $\gamma = \gamma(x)$ such that $x - \gamma$ belongs to some $P'' \in \mathfrak{P}$. Corollaries to (2) are: (2a) Every formally real field G has a simple ordering; (2b) In a formally real field G, the totally positive elements are just those expressible as sums of squares. (Received February 23, 1953.)

336t. Trevor Evans: *An embedding theorem for semigroup with cancellation.*

This note is concerned with the following question. Is there a finite integer n such that any countable cancellation semigroup can be embedded in a cancellation semigroup generated by n elements? From previous theorems of this type one is led to expect that $n = 2$ but this is not the case. An example is given of a countable cancellation semigroup which cannot be embedded in a two-generator cancellation semigroup. However, the following theorem can be proved. Any countable cancellation semigroup not possessing subsemigroups which are groups (apart from a possible unit element) can be embedded in a cancellation semigroup generated by two elements. A plausible conjecture for the case of countable cancellation semigroups containing subgroups is that such semigroups can be embedded in cancellation semigroups generated by six elements. The remainder of the paper is concerned with the embedding of countable rings in rings generated by two elements and some remarks on the manner in which loops can be embedded in monogenic loops. (Received March 9, 1953.)

337. Roland Fraissé: *On certain relations which generalise ordering relations of type η.*

Let $A(x_1, \cdots, x_n)$ be an n-ary relation of base E (a function defined on E^n assuming two values). We say that A is ***homogeneous*** if, given any two restrictions A_1, A_2

of A, each with finite base, any isomorphism of A_1 onto A_2 can be extended to an automorphism of A. Write $A>B$ or $B<A$ if B is isomorphic to a restriction of A. Let Γ_A be the family of relations R with finite base such that $R<A$. All relations considered are assumed to have finite or denumerable bases. The following are proved: (1) There exists a homogeneous n-ary relation H with $H>R$ for every n-ary relation R; H is unique up to isomorphism. (2) If H is homogeneous and $\Gamma_A \subseteq \Gamma_H$, then $A<H$. (3) If H and K are homogeneous and $\Gamma_H=\Gamma_K$, then H and K are isomorphic. (4) A family Γ of relations with finite bases is a Γ_H (with H homogeneous) iff (i): if $A\in\Gamma$ and $B<A$, then $B\in\Gamma$, and (ii): if A, $B\in\Gamma$ and have the same restriction to the intersection I of their bases, then there exist $C\in\Gamma$ and isomorphisms ϕ and ϕ' of A and B resp. onto restrictions of C, such that $\phi(x)=\phi'(x)$ for $x\in I$. (Received March 11, 1953).

338. Emil Grosswald: *On diophantine approximations.*

Let α be an irrational number, $\alpha=[\theta_1, \theta_2, \cdots, \theta_k, \cdots]$ its expansion in a hermitian continued fraction, p_k/q_k the convergent preceding θ_k; let F_k be the set of fundamental regions R_{kj} of the modular group, of common cusp p_k/q_k. The line $L: x=\alpha$ meets infinitely many F_k and in each of them $n_k=\theta_k+1$ fundamental regions R_{kj}. Let $V_{kj}\{z\}=(q_j'z-p_j')/(q_kz-p_k)$ be the modular transformation that maps R_{kj} onto $R_0(|z|\geq 1, |\Re z|\leq 1/2)$, L_{kj} the segment of L in R_{kj}, $z=\alpha+iy_{kj}$ a point of L_{kj}. Then $V_{kj}\{L_{kj}\}=\gamma_{kj}$, an arc of circle in R_0, and $V_{kj}\{z\}=x'_{kj}+iy'_{kj}$ with $y'_{kj}=y_{kj}/Q$, $Q=(q_k\alpha-p_k)^2+q_k^2y_{kj}^2 \geq 2|y_{kj}q_k(q_k\alpha-p_k)|$ so that (1) $|\alpha-p_k/q_k|\leq 1/2q_k^2y'_{kj}$. Using properties of V_{kj} it follows that for proper choice of z, (2) $y'_{kj}\geq 2^{-1}\{(\theta_k-1)^2+m^2\}^{1/2}$, where $m\geq 3^{1/2}$ and, under certain conditions, $m\geq 2$. From (1) and (2) follows: (3) $|\alpha-p_k/q_k|\leq q_k^{-2}\{(\theta_k-1)^2+m^2\}^{-1/2}$. Observing that (a) $\theta_k\geq 1$ always holds; (b) $\theta_k\geq 2$ infinitely often; (c) $\theta_k\geq 3$ infinitely often, unless $\alpha\sim(-1+5^{1/2})/2$, (3) with $m=2$ yields (a) the characteristic property of hermitian convergents; (b) and (c) two theorems of Hurwitz (Math. Ann. vol. 39 (1891) pp. 279–284). In the general case, (3) is the analogue for hermitian continued fractions of a classical inequality for regular continued fractions. It is conjectured that $m=2$ infinitely often for every fixed, infinitely repeated θ_k of an irrational α. (Received March 2, 1953.)

339*t*. D. G. Higman: *Indecomposable representations at characteristic* *p*. Preliminary report.

Let S be a subgroup of the finite group G, Ω a set. By M_S we denote the S-Ω-module induced by a given G-Ω-module M, and by M_S^G the G-Ω-module induced by M_S. The elements of M_S^G are the formal linear combinations $\sum_{x\in L} x\cdot u_x$, where L is a set of left representatives for G over S, and u_x is in M. The mapping α: $\sum x\ u_x \rightarrow \sum xu_x$ is a G-Ω-homomorphism of M_S^G onto M, whereas β: $u\rightarrow\sum x\cdot x^{-1}u$ for u in M is a G-Ω-isomorphism of M onto a G-Ω-submodule $\tilde{M}$ of M_S^G. Furthermore, $u\beta\alpha=(\sum x\cdot x^{-1}u)=\sum u=[G:S]u$ [cf. B. Eckmann, Bull. Amer. Math. Soc. Abstract 58-3-278]. If S contains a p-Sylow subgroup of G, Ω is a field of characteristic p, and M has finite dimension over Ω, then $\alpha(\beta\alpha)^{-1}\beta$ is a decomposition operator of M_S^G, giving $M_S^G\simeq M\oplus\{M_S^G-\tilde{M}\}$. Hence, by the Remak theorem, each indecomposable representation of G over Ω is a component of a representation induced by an indecomposable representation of a p-Sylow subgroup of G. Corollaries: If G has cyclic p-Sylow subgroups, then the number of inequivalent indecomposable representations of G over Ω is finite. If G has non-cyclic p-Sylow subgroups and Ω is infinite, then G has infinitely many inequivalent indecomposable representations. The degrees of the indecomposable representations of G are bounded if and only if the p-Sylow subgroups of G have this property. (Received March 12, 1953.)

340. L. Aileen Hostinsky: *Loewy chains and uniform splitting of lattices.*

An ascending Loewy η-chain is defined for an endomorphism η and a complete modular lattice L satisfying the additional property (*): $a\sum_\alpha p_\alpha = \sum_\alpha ap_\alpha$ for a and the ascending chain p_α elements of L. The following theorem is proved: The endomorphism η of $p/0$ is a uniformly splitting endomorphism if there exists an ascending Loewy η-chain the sum of which is p. In the method of proof much use is made of the modular law, property (*), and complete induction. (Received February 25, 1953.)

341. Bjarni Jónsson: *Modular lattices and normal subgroups.* Preliminary report.

The following proposition holds in every lattice of normal subgroups of a group and, more generally, in every lattice of commuting equivalence relations: If a_0, a_1, a_2, x are any elements, $y=(a_1+x)(a_2+x)$, $b_0=(a_0+y)(a_1+a_2)$ and cyclically, $c_0=(a_1+a_2)\cdot(b_1+b_2)$ and cyclically, then $c_0+c_1+c_2=c_1+c_2$. Applied to the lattice of a projective space this condition implies a special case of Desargues Theorem, and is known to fail in certain projective planes. This solves Problem 27 of G. Birkhoff's *Lattice theory* (rev. ed., 1948). It also shows that a free modular lattice with four generators is not isomorphic to a lattice of normal subgroups or of commuting equivalence relations. (Received March 10, 1953.)

342t. R. C. Lyndon: *On Burnside's problem.* Preliminary report.

A method of K. T. Chen, R. Fox, and the author is applied to the problem of Burnside. For $q \geqq 2$, p prime, let B be the group on q generators defined by the identical relation $w^p=1$, and let $p^{\beta(n)}$ be the order of the lower central quotient B_n/B_{n+1}. Results: (1) for $n<p$, $\beta(n)=\psi(n)$ where [E. Witt] $\psi(n)=n^{-1}\sum \mu(n/d)q^d$ for d/n; (2) $\beta(p)=\psi(p)-C_{p+q-1,q-1}+q$; (3) for $q=2$ henceforth, $\beta(p+1)=\psi(p+1)-p$; (4) for $p=5$, $\beta(7)=4$, and for $p \geqq 5$, $\beta(p+2) \geqq \psi(p+2)-7p+13$; (5) for $n \leqq 2p-5$, $\beta(n)>0$. [(2), (3), (4) are contained in an unpublished work of P. Hall, and (5) in a result of J. A. Green.] (Received March 11, 1953.)

343. G. de B. Robinson: *On the modular representations of the symmetric group.*

The modular representation theory of the symmetric group has been the subject of much study in recent years. J. H. Chung suggested a method for determining the D-matrix. The p-graph of a Young diagram, due to D. E. Littlewood and the consequent notion of r-inducing (Robinson, Proc. Nat. Acad. Sci. U.S.A. vol. 38 (1952) pp. 129–133, 424–426) has made it possible to limit and keep track of the processes involved. Chung's linkages can be interpreted as determining a partial order, so that an indecomposable representation of the regular representation now appears as a *lattice* of its ordinary irreducible representations. The 0, 1 elements of the lattice are easily recognized, and the indecomposable is uniquely determined when either one of these two representations is given. (Received March 26, 1953.)

344. A. L. Whiteman: *The 16th power residue character of* 2.

On the basis of extensive tables Cunningham [Proc. London Math. Soc. (1) vol. 27 (1895) pp. 85–122] conjectured the following criterion for the 16th power residue character of 2. Let $p=a^2+b^2=c^2+2d^2$, a and c odd, be a prime of the form $16n+1$. If $2^{(p-1)/8}\equiv 1 \pmod{p}$, then $2^{(p-1)/16}\equiv(-1)^{b/16+d/4} \pmod{p}$. The first proof of this result was given by Aigner [Deutsche Mathematik vol. 4 (1939) pp. 44–52]. Aigner's method employs class-field theory. In the present paper a second proof, based upon the theory of cyclotomy, is given. An alternative formulation of Cunningham's criterion is as follows. Let u and v denote odd numbers. The number 2 is a 16th power residue of a prime p of the form $16n+1$ if and only if p is simultaneously representable in the forms $p=a^2+1024b^2=c^2+128d^2$, or in the forms $p=a^2+256u^2=c^2+32v^2$. (Received February 2, 1953.)

345t. K. G. Wolfson: *The alegbra of bounded operators on Hilbert space.*

Let K be a B^*-algebra which contains an identity element and minimal right ideals. A left ideal of K is called a left annulet if it is the totality of left annihilators of a subset of K. Then, there exists a unique Hilbert space H, such that K is isomorphic (in a norm and * preserving manner) to the algebra $B(H)$ of *all* bounded operators on H, if and only if (1) K contains a smallest closed two-sided ideal (not the zero ideal). (2) If J_1 and J_2 are left annulets which satisfy $J_1J_2^*=0$, then J_1+J_2 is a left annulet. The proof depends on the existence of a lattice isomorphism between the totality of left annulets of $B(H)$ and the totality of (closed) subspaces of H. (Received March 9, 1953.)

ANALYSIS

346. Shmuel Agmon, Louis Nirenberg (p), and M. H. Protter: *A maximum principle for a class of hyperbolic equations.*

A maximum principle is proved for solutions of certain linear hyperbolic equations in second order in two independent variables. It contains as a special case the maximum principle for Tricomi's equation due to Germau and Rader. Applications to boundary value problems are made. The methods are elementary. (Received March 10, 1953.)

347t. Joseph Andrushkiw: *On the power series whose partial sums have zeros in a sector of the complex plane.*

Let $f_n(z)=1+\sum_{k=1}^{n} a_kz^k$ be the nth partial sum of the power series $f(z)=1+\sum_{k=1}^{\infty} a_kz^k$ (a_k complex numbers) and $z_j=\alpha_j+i\beta_j$ $(j=1, 2, \cdots, n)$ its zeros. If $\theta_1 \leq \arctan \beta_j/\alpha_j \leq \theta_2$, $\theta_2-\theta_1=(\pi/2)-2\eta$, $0<\eta\leq\pi/4$, the inequality $|a_n|^{2/n}=(c/2)\cdot((A_1+\bar{A}_1)^2-2(A_2+\bar{A}_2+A_1\bar{A}_1))n^{-1}$ holds. There is $c=\csc 2\eta$, $A_1=a_1e^{i\vartheta}$, $A_2=a_2e^{2i\vartheta}$, $\bar{A}_1$ and $\bar{A}_2$ conjugate of A_1 and A_2 and $\vartheta=\theta_1+(\theta_2-\theta_1)/2$. It follows from the above inequality that $f(z)$ represents an integral function of the order not greater than 2. The example $g(z)=e^{z^2}(z^2-1)$ shows that the converse theorem is not true. (Received February 25, 1953.)

348t. H. A. Antosiewicz: *A boundedness theorem for a nonlinear differential equation.*

The differential equation $\ddot{x}+(f(x)+g(x)\dot{x})\dot{x}+h(x)=e(t)$, which can be written as the system $\dot{x}=(y-b(x))/a(x)$, $\dot{y}=-a(x)(h(x)-e(t))$ where $a(x)=\exp\left(\int_0^x g(u)du\right)$, $b(x)=\int_0^x a(u)f(u)du$, is considered under the assumptions that $f(x)$, $g(x)$, $h(x)$ are continuous and satisfy a Lipschitz condition for all x and $e(t)$ is continuous and bounded for $t\geq 0$. It is shown that if (i) $f(x)\geq 0$ for all x; $xh(x)>0$ for $x\neq 0$, $|h(x)|\to\infty$ and $\int_0^x a^2(u)h(u)du\to\infty$ with $|x|$; (ii) given any constant $M>0$, there exist constants $X(M)$

>0, $m>0$ such that $b^2(x)-M\left|\int_0^x a^2(u)du\right| \geqq m$ for $|x|>X(M)$; then every solution ultimately satisfies $|x(t)|<B_1$, $|\dot{x}(t)|<B_2$ where B_1, B_2 are positive constants independent of the particular solution considered. This result is not included in previous investigations of nonlinear differential equations of the second order. (Received March 4, 1953.)

349. Jerome Blackman: *The inversion of the generalised Fourier transform by Abelian summability.*

The space L_p^k, $1\leqq p<\infty$, consists of those functions for which the norm $\left(\int_{-1}^1|\phi(x)|^p dx+\int_{|x|>1}|\phi(x)/x^k|^p dx\right)^{1/p}=\|\phi\|_{k,p}<\infty$. For $\phi\in L_1^k$ or $\phi\in L_p^{k-1}$, $p>1$, the generalized Fourier transform of kth order, $E(k, t)$, has been defined by Bochner. Using the notation of Bochner let $\phi(x)=(1/(2\pi)^{1/2})\int_0^\infty e^{-ixt}e^{-\epsilon t}d^kE(k, t)+(1/(2\pi)^{1/2})\cdot\int_{-\infty}^0 e^{-ixt}e^{\epsilon t}d^kE(k, t)$. If $\phi\in L_1^k$, $\lim_{\epsilon\to 0}\|\phi_\epsilon(x)-\phi(x)\|_{k,p}=0$; $\lim_{\epsilon\to 0}\|\phi_\epsilon(x)-\phi(x)\|_{k-1,p}=0$ if $\phi\in L_p^{k-1}$. Similar results hold in the case corresponding to $p=\infty$. If $1<p\leqq 2$ and $\phi\in L_p^k$, the kth transform $E(k, t)$ may be defined by mean convergence and exists in a normed space with norm $\|E(k, t)\|^{k,p'}$ such that $\|E(k, t)\|^{k,p'}<C_p\|\phi\|_{k,p}$ where $1/p+1/p'=1$. (Received March 6, 1953.)

350*t*. E. K. Blum: *A uniqueness theorem for the Euler-Poisson-Darboux equation.*

We consider the equation $E(k)$: $u_{xx}-u_{tt}-(k/t)u_t=0$ for $k=-(2n+1)$, n a positive integer or zero. Let $f(x)$ be any function having a continuous derivative of order $n+3$ on an interval (C, B) of the x-axis. Let T be the characteristic triangle having (C, B) as base. A function $u^{[k]}(x, t)$ is said to be a solution of the Cauchy problem for $E(k)$ if (1) $u^{[k]}(x, t)$ is a regular solution of $E(k)$ in the interior of T, (2) $\lim_{t\to 0}u^{[k]}(x, t)=f(x)$, and (3) $\lim_{t\to 0}\partial u^{[k]}/\partial t=0$. We prove the following theorem: Let $\psi(x, t)=\int_0^1 f[x+(1-2s)t]s^{-1/2}(1-s)^{-1/2}\log[ts(1-s)]ds$. Then every solution of the Cauchy problem for $k=-(2n+1)$ is of the form $u^{[k]}(x, t)=\sum_{r=1}^{n+1}b_{rn}t^r\partial^r\psi/\partial t^r+t^{2n+2}u^{[2n+3]}$ where the b_{rn} are constants determined by the recursion formula $b_{r,n+1}=b_{r-1,n}-b_{rn}(2n+2-r)$, $b_{0n}=0$, $b_{n+1,n}=1$ and $u^{[2n+3]}$ denotes a solution of $E(2n+3)$ which is regular in the interior of T and such that $\lim_{t\to 0}t^n u^{[2n+3]}=0$ and $\lim_{t\to 0}t^{n+1}\partial u^{[2n+3]}/\partial t=0$. (Sponsored by O.N.R.) (Received March 2, 1953.)

351. F. E. Browder: *The asymptotic distribution of the eigenvalues and eigenfunctions of the self-adjoint elliptic differential operator with variable coefficients.*

Let K be a self-adjoint linear elliptic differential operator of order $2m$ with suitably differentiable coefficients on the bounded domain D of Euclidean n-space. Suppose that $2m>n$. Let $\{\phi_n\}$ be a complete sequence of complex-valued eigenfunctions of $(-1)^m K$, orthonormalized in $L_2(D)$ and arranged in nondecreasing order of eigenvalues $\{\lambda_i\}$. If $a(x, \xi)$ is the characteristic form for K at x in D, let $\rho(x)=\int_{a(x,\xi)<1}d\xi$. Then as $t\to\infty$, the number of eigenvalues λ_i less than t is asymptotic to $(2\pi)^{-n}t^{n/2m}\int_D\rho(x)dx$; $\sum_{\lambda_j\leqq t}|\phi_j(x)|^2=(2\pi)^{-n}\rho(x)t^{n/2m}(1+o(1))$; $\sum_{\lambda_j\leqq t}\phi_j(x)\bar{\phi}_j(y)=o(t^{n/2m})$ for $x, y\in D$. The proof uses the method of Carleman. The theorem is a generalization of results of Courant and Pleijel for the vibrating plate and Gårding for elliptic operators with constant coefficients. (Received March 11, 1953.)

352*t*. P. L. Butzer and Waclaw Kozakiewicz: *On the Riemann derivatives for integrable functions.*

The central difference of order s, $\nabla^s_{2h}f(x)$, corresponding to a number $h>0$, is defined inductively by the relations $\nabla^1_{2h}f(x)=f(x+h)-f(x-h)$, $\nabla^{s+1}_{2h}f(x)=\nabla^1_{2h}\nabla^s_{2h}f(x)$. It is established that if $f(x)\in L\{a, b\}$ [i.e. $f(x)$ is Lebesgue integrable in every closed subinterval of the open interval (a, b)] and if there exists a sequence of positive numbers $\{h_n\}$ converging to zero and a function $\sigma(x)\in L\{a, b\}$ such that $\lim_{n\to\infty}\sigma_n(x)=\lim_{n\to\infty}[\nabla^{l+1}_{2h_n}f(x)]/(2h_n)^{l+1}=0$ almost everywhere in (a, b) with $\sup_n|\sigma_n(x)|\leq\sigma(x)$ for $a<x-(l+1)h_n<x+(l+1)h_n<b$, then there exists a polynomial $P_l(x)$ of degree l in x such that $f(x)=P_l(x)$ for almost every x in (a, b). A corresponding result can be stated for the left and also the right differences [see also the authors' paper entitled *A theorem on the generalised derivatives*, Bull. Amer. Math. Soc. Abstract 59-3-284]. (Received March 11, 1953.]

353. Sarvadaman Chowla and Robert Osserman (p): *The e-rank of polynomials.*

Let $P(x)$ be a polynomial of degree n. Let $e^{P(x)}=\sum_{n=0}^{\infty}a_nx^n$. Consider the infinite matrix (a_{i-j}) where $a_n=0$ if $n<0$. By a theorem of Edrei this matrix must have negative minors of some order if $n\geq 2$. Define the e-rank of $P(x)$ as the greatest integer r such that all minors of order less than or equal to r are non-negative. The question arises whether for all polynomials of given degree n, there exists a maximum value $r_0=r_0(n)$ of r. It is proved that $r_0(2)=2$. The conjecture $r_0(n)=n$ is unanswered $(n>2)$. (Received March 9, 1953.)

354. M. H. Clarkson (p) and H. J. Ettlinger: *On obtaining solutions to the nonhomogeneous wave equation which satisfy boundary conditions of mixed type.*

Use is made of the Riesz Transform to find solutions of the nonhomogeneous wave equation in two and three dimensions which satisfy boundary conditions of mixed type. (Received March 11, 1953.)

355. Ruth M. Davis: *On the Cauchy problem for the Euler-Poisson-Darboux equation.*

Let $u(x, t)$ denote $u(x_1, x_2, \cdots, x_m, t)$. A solution to the equation (1) $\Delta u=u_{tt}+kt^{-1}u_t$ with given $u(x, 0)$ and $u_t(x, 0)=0$, where k is a real parameter, has been given by A. Weinstein in C. R. Acad. Sci. Paris vol. 234 (1952) pp. 2584–2585. For $k<m-1$, $k\neq -1, -3, -5, \cdots$, a new explicit form of the solution is obtained in the present paper. Let $\omega_s=2(\pi^{1/2})^s[\Gamma(s/2)]^{-1}$, let $m-k$ not be an odd positive integer, and let $L(x, t)=(\omega_{s+m})(\omega_{s+m+1})^{-1}\cdot\int\cdots\int_\Sigma f(x_i+\alpha_it)(1-\alpha_i^2)^s d\alpha_1\cdots d\alpha_m$, where $2s=(k+2n-1-m)$ and $\alpha_1^2+\cdots+\alpha_m^2\leq 1$. Then the solution to (1) may be written as (2) $u(x, t)=[(k+1)(k+3)\cdots(k+2n-1)]^{-1}\sum_{j=0}^{n}B_{nj}t^{n-j}L_{(n-j)t}$ where $B_{nj}=(-1)^{n-j}2^n[(n-j)!]^{-1}\sum_{r=0}^{n-j}(-1)^rC_{n-j,r}\Gamma[(k+2n+1+r)/2]\{\Gamma[(k+1+r)/2]\}^{-1}$. If $m-k$ is an odd positive integer, $L(x, t)$ must be replaced by Poisson's mean value M. In this case (2) becomes the solution already obtained by J. B. Diaz and H. F. Weinberger, Bull. Amer. Math. Soc. Abstract 59-1-12. This paper was sponsored by the Office of Naval Research. (Received March 9, 1953.)

356. J. B. Diaz (p) and H. F. Weinberger: *The exceptional cases in the Euler-Poisson-Darboux equation.*

Let x denote $(x_1\cdots x_m)$. A solution of the singular Cauchy problem (*)$\Delta u-u_{tt}-Kt^{-1}u_t=0$, with $u(x, 0)=f(x)$, $u_t(x, 0)=0$, is found for the exceptional values

$K=-1, -3, \cdots$, excluded in a previous abstract (Bull. Amer. Math. Soc. Abstract 59-1-12). It is found that in the solution $u(x, t)$ previously obtained for all other values of K, the singularities occurring at these exceptional values can be removed by subtracting an appropriate function of the form $t^{1-K}V(x, t)$, where v satisfies (*) with K replaced by $2-K$. The initial value of v is a multiple of an iterated Laplacian of f, and so v can be found by the previous methods. Passage to the limit provides a solution of (*) for K an odd negative integer. The $(1-K)$th derivative of u is logarithmic in t at $t=0$ except when f is polyharmonic of order $(1-K)/2$. This work was sponsored by the Office of Naval Research. (Received February 13, 1953.)

357. Avron Douglis: *Hadamard's conjecture.*

A complete, relatively simple proof is given of a conjecture of Hadamard that the linear hyperbolic, partial differential equations of second order in an even number of independent variables for which Huygens' Principle is valid are just those which result from the wave equation after arbitrary transformations of three possible kinds: change of independent variables, multiplication of the dependent variable by a variable factor, and multiplication by another variable factor of each term of the equation. The proof is based on the author's method of approach to Cauchy's problem for hyperbolic equations of the second order which has been reported in Bull. Amer. Math. Soc. Abstract 58-4-354. (Received March 17, 1953.)

358. R. J. Duffin: *Continuation of biharmonic functions by reflection.*

Let R be a region contained in the right half-plane and bounded on one side by a segment of the y axis. Let the function $w(x, y)$ be biharmonic in R; that is, $\Delta\Delta w=0$. On the segment of the y axis let w satisfy the clamped plate boundary conditions; that is, $w=\partial w/\partial x=0$. Then it is shown that the formula $w(-x, y)=w-x^2\Delta(w/x)$ continues w so as to be biharmonic in the region obtained by reflecting R in the y axis. The same formula applies to the analogous three-dimensional problem. Other continuation formulae are found for circular boundaries and various other boundary conditions. (Received March 11, 1953.)

359. W. F. Eberlein: *A higher order connection formula.*

The methods of Langer applied to the equation $u''+k^2Pu=0$, where P has a zero of order n at $x=0$ and k is a large parameter, yield explicit approximate solutions w with errors roughly $O(k^{-1})$. When $n=1$, Langer [Trans. Amer. Math. Soc. vol. 67 (1949) pp. 461–490] has extended the procedure to obtain w's approximating the u's to higher powers of k^{-1}, but the new method breaks down in general if $n>1$. In this paper a new approximating equation is introduced in the case $n>1$ to obtain approximations v to the u's with errors essentially $O(k^{-2})$ in the complement of an arbitrary neighborhood of $x=0$. (Received March 11, 1953.)

360. Albert Edrei: *Meromorphic functions with three radially distributed values.*

Let $f(z)$ be a meromorphic function of the complex variable $z=r\exp(i\theta)$. Assume that all but a finite number of the roots of the three equations (1) $f(z)=0$; (2) $f(z)=\infty$; (3) $f^{(l)}(z)=1$ $[l\geq 0, f^{(0)}=f]$ are distributed on the q radii $r\exp(i\theta_1)$, $r\exp(i\theta_2)$, $\cdots$, $r\exp(i\theta_q)$ $[r\geq 0; 0\leq\theta_1<\theta_2<\cdots<\theta_q<2\pi]$. Denote by $\delta^{(l)}(a)$ the defect of the a's of $f^{(l)}(z)$ and assume $\delta(0)+\delta^{(l)}(1)+\delta(\infty)>0\,[\delta^{(0)}=\delta]$. Then, the order ρ, of (z), is necessarily finite and $\rho\leq\sup\{\pi/(\theta_2-\theta_1), \pi/(\theta_3-\theta_2), \cdots, \pi/(\theta_{q+1}-\theta_q)\}$ where $\theta_{q+1}=2\pi+\theta_1$. This result lends itself to the following application. Consider the entire function $g(z)$

$=P(z)\exp(Q(z))$ where $Q(z)$ is any entire function and $P(z)$ an entire function of finite order ρ. Assume that all but a finite number of the roots of the three equations $g(z)=0$, $g'(z)=0$, $g''(z)=0$ are real. Then the order ρ_1, of $Q(z)$, is necessarily finite. Furthermore, either $\rho_1 \leqq \rho$ or else $\rho < \rho_1 \leqq 1$. This proposition easily yields extensions of known results of Pólya and Saxer. (Received March 11, 1953.)

361. H. J. Ettlinger and C. E. Abraham (p): *On the Holmgren-Riesz transform.*

In one dimension, Riemann defined the transform $F[\alpha;\, a,\, x|f(t)]$, where $f(t)$ has continuous derivatives and α is a real parameter [*Versuch einer allgemeinen Auffassung der Integration und Differentiation*, Collected Works, Leipzig, 1876, pp. 331–344]. M. Riesz generalized the transform to complex values of α and to higher dimensions [*L'integral de Riemann-Liouville et le problème de Cauchy*, Acta Math. vol. 81 (1949) pp. 1–223]. H. J. Holmgren independently developed in one dimension an equivalent transform by means of a different definition [*Om differentialkalkylen med indices of hvilken natur som helst*, Kongl. Svenska Vetenskaps-Akademies Hedlinger, vol. 5 (1865) pp. 1–83]. The properties of this transform are developed systematically under general conditions. (Received March 11, 1953.)

362t. Herbert Federer: *The fundamental inequality between the Lebesgue areas of a surface and its projections.*

Consider a finitely triangulable subset X of the plane and a continuous mapping f of X into Euclidean n-space E_n, with $n \geqq 2$. For any two integers i and j such that $1 \leqq i < j \leqq n$, let $P_{i,j}$ be the projection of E_n onto E_2 which maps any $x=(x_1, \cdots, x_n) \in E_n$ onto $(x_i, x_j) \in E_2$. It is proved that $L(f) \leqq \sum_{1 \leqq i < j \leqq n} L(P_{i,j}) \circ f)$, where L is the 2-dimensional Lebesgue area. For the special case in which $n \leqq 3$ and X is a 2-cell, this inequality was first established by L. Cesari [Annali della R. Scuola Normale Superiore di Pisa (2) vol. 10–11 (1941–42)]. The method of the present paper is new even when applied to the previously known special case. As a consequence of the general inequality now proved, the whole theory of 2-dimensional Lebesgue area becomes equally as extensive for surfaces in n-space as for surfaces in 3-space. (Received March 3, 1953.)

363t. Herbert Federer: *A new formula for Lebesgue area.*

Assuming that X is a compact, locally connected, finitely connected subset of the plane, let f be a mapping of X into E_n with monotone-light factorization $f=l \circ m$ and middle space M. Consider the set P of all $n-1$ dimensional planes of E_n (a factor space of the group of all rigid motions of E_n) and suppose μ is a Haar measure over P. For each $p \in P$ let $S(p)=M \cap \{\alpha | l(\alpha)=p\}$ and let $T(p)$ be the set of all those points of $S(p)$ at which $S(p)$ has positive topological dimension. For $p \in P$ and $y \in p$, let $N(p, y)$ be the number (possibly ∞) of points $\alpha \in T(p)$ such that $l(p)=y$. Furthermore suppose that $\mathcal{H}^1_n$ and $\mathcal{J}^1_n$ are the 1-dimensional Hausdorff and integralgeometric measures over E_n and that H^1_f is the 1-dimensional Hausdorff measure over M (see *Measure and area*, Bull. Amer. Math. Soc. vol. 58 (1952) pp. 306–378). Then the three integrals $\int_P \int_p N(p, y)\, d\mathcal{H}^1_n y\, d\mu p$, $\int_P \int_p N(p, y)\, d\mathcal{J}^1_n y\, d\mu p$, $\int_P H^1_f[T(p)]\, d\mu p$ are equal to each other and to c times the 2-dimensional integralgeometric stable area of f (loc. cit.), where c depends only on n and μ. [The proofs given in the paper use a particular Haar measure μ and yield the corresponding value of c explicitly in terms of n.] Moreover if X is a 2-cell then the integralgeometric stable area of f is equal to the Lebesgue area of f; thus results a formula expressing the Lebesgue area of a surface in terms of the lengths of the positive-dimensional parts of its sections by $n-1$ planes. (Received March 3, 1953.)

364. F. A. Ficken: *Bounded linear adjoints for certain nonlinear transformations between Banach spaces.* Preliminary report.

Let X be a complex Banach space with elements x and norm $\|x\|$. Let Φ contain all real-valued continuous isotonic $\phi(r)$ $(r \geqq 0)$ with $\phi(0) > 0$. With $\phi \in \Phi$, let $X^{\ddagger}_{\phi}$ be the Banach space of complex-valued functionals $x^*(x)$ that are continuous on X and have $\|x^*\|_{\phi} = \sup_x [|x^*(x)|/\phi(\|x\|)] < \infty$. Let Y be a complex Banach space and define $Y^{\ddagger}_{\psi}$ similarly with $\psi \in \Phi$. Let $x \to y = Tx$ be a continuous closed mapping of X into Y for which there exist ϕ and ψ (henceforth fixed) such that $\beta = \sup_x [\psi(\|Tx\|)/\phi(\|x\|)] < \infty$, and say then that T is ϕ, ψ-bounded. For each $y^* \in Y^{\ddagger}_{\psi}$ and $x \in X$ define $x^*(x) = T^*y^*(x) = y^*(Tx)$. Then T^* maps $Y^{\ddagger}_{\psi}$ linearly into $X^{\ddagger}_{\phi}$ and $\|T^*y^*\|_{\phi} \leqq \beta\|y^*\|_{\psi}$. Several theorems on the adjoint of a linear transformation (as in Banach's book) carry over to T^*. Example: If $x^* = T^*y^*$ has a solution $y^* \in Y^{\ddagger}_{\psi}$ for each $x^* \in X^{\ddagger}_{\phi}$, then T has an inverse T^{-1}; conversely, if T^{-1} exists and is ψ, ϕ-bounded, then T^* maps $Y^{\ddagger}_{\psi}$ onto $X^{\ddagger}_{\phi}$. If X and Y are complete metric spaces with distances $d(x, x')$ and $\delta(y, y')$, one can retain much of the argument by replacing $\phi(\|x\|)$ and $\psi(\|y\|)$ by $\phi(d(p, x))$ and $\phi(\delta(q, y))$ with $p \in X$ and $q \in Y$ fixed arbitrarily. (Received March 9, 1953.)

365. B. A. Fleishman: *Periodic solutions of a nonlinear wave equation.*

Consider the nonlinear wave equation of Duffing type $u_{xx} = u_{tt} + 2\kappa u_t + \alpha u + \epsilon u^3 + B(x, t)$ $(0 < x < L, t > 0)$ with positive damping $(\kappa > 0)$ and p-periodic forcing term: $B(x, t+p) = B(x, t)$ and $B(0, t) = 0 = B(L, t)$. Conditions are obtained under which this equation has at least one solution $u(x, t)$ that meets the boundary conditions $u(0, t) = 0 = u(L, t)$ and is p-periodic in t. The author first finds, by solving an equivalent integral equation, the solution $u(x, t; f, g)$ of the above differential equation and boundary conditions with initial data $u(x, 0; f, g) = f(x)$, $u_t(x, 0; f, g) = g(x)$. Then, under suitable assumptions, the functional equations expressing the periodicity conditions $u(x, p; f, g) = f(x)$, $u_t(x, p; f, g) = g(x)$ are solved by an iterative method in a suitable Banach space, yielding unique (appropriately bounded) initial data $f(x)$, $g(x)$ such that $u(x, t; f, g)$ is p-periodic in t. The restrictions on the parameters are: $e^{-p\kappa} \max [Q_1(\kappa), Q_2(\kappa)] < 1$, where $Q_1(\kappa)$ and $Q_2(\kappa)$ are quadratics with positive coefficients depending only on L; $\eta = pL|\alpha - \kappa^2| e^{p\kappa} < 1/2$; and, if $\beta = \|B(x, t)\|$ and R(fixed) $> 3Lp\beta e^{p\kappa}/2(1-\eta)$, then $3\epsilon R^2 \leqq |\alpha - \kappa^2|$. (Received February 6, 1953.)

366t. M. P. Gaffney, Jr.: *A special Stokes's theorem for complete Riemannian manifolds.*

Let M be an orientable, complete, Riemannian manifold. The theorem stated in vector analysis terminology is: If v is a C^1 vector field on M such that both $|v|$ and div v are integrable functions, then $\int_M \operatorname{div} v \, dV = 0$. The equivalent statement for differential forms is: Let α be an $n-1$ form of class C^1 such that both α and $d\alpha$ are in L_1. Then $\int_M d\alpha = 0$. (β is in L_1 if $\int(*(\beta * \beta))^{1/2} * 1$ is finite.) This generalizes Stokes's theorem for exact n forms on a compact manifold. Let d be restricted to C^1 forms α for which $\|\alpha\|$ and $\|d\alpha\|$ are finite; similarly δ $(\|\beta\|^2 = \int \beta * \beta_1)$. The equations $0 = \int_M d(\alpha * \beta) = (d\alpha, \beta) - (\alpha, \delta\beta)$ yield the corollary $(d\alpha, \beta) = (\alpha, \delta\beta)$. Thus M has negligible boundary. Let $\Delta = d\delta + \delta d$ be restricted to α for which the norms of α, $d\alpha$, $\delta\alpha$, $\delta d\alpha$, and $d\delta\alpha$ are all finite (the natural Hilbert space definition of Δ). Then Δ is symmetric, and hence of Weyl's limit point type on M. Thus the formalism of the compact case is valid on M. A previous result is that for negligible boundary the closure of Δ is self-adjoint ($=\Delta^*$). (Received April 20, 1953.)

367. F. W. Gehring: *A study of α-variation.*

A function $f(x)$ is said to have bounded α-variation over the interval $a\leqq x\leqq b$ and belong to W_α if the sums $\{\sum_{n=1}^{N}|f(x_n)-f(x_{n-1})|^{1/\alpha}\}^\alpha$, $0\leqq\alpha\leqq 1$, are bounded for every subdivision $a=x_0<\cdots<x_N=b$. Functions in W_1 have bounded variation. L. C. Young (Acta Math. vol. 67) has proved that a Stieltjes integral $\int_a^b f(x)dg(x)$ can be defined if $f(x)$ and $g(x)$ are in W_α and W_β with $\alpha+\beta>1$. A moment problem is discussed for the class W_α $(0<\alpha\leqq 1)$ and necessary and sufficient conditions are obtained for a sequence, $\{a_n\}$, to be represented $a_n=\int_0^1 x^n dg(x)$ where $g(x)$ is in W_α. The concept of α-variation can be applied to the study of infinite series. An absolutely convergent series is one whose partial sums have bounded variation in n. A scale for Cesaro and Abel summability is discussed along with consistency results and a Tauberian theorem for absolute Abel summability. (Received March 16, 1953.)

368. J. J. Gergen and F. G. Dressel (p): *Mapping for elliptic equations.*

The following extension of the Riemann mapping theorem is obtained. Let D and T be finite plane domains whose boundaries D^* and T^* are simple, closed Jordan curves. Let α, β, γ, δ be real, bounded functions of (x, y) of class C' on D such that $0<\alpha$, $0<m<\alpha\delta-(\beta+\gamma)^2/4$ on D, where m is a constant. Let $z^{(j)}$ $(j=1, 2, 3)$ be distinct points on D^*, and let $Z^{(j)}$ $(j=1, 2, 3)$ be distinct points on T^* in the same order on T^* as the points $z^{(j)}$ on D^*. Then there exists a 1:1 and continuous mapping $X=u(x, y)$, $Y=v(x, y)$ of $D+D^*$ into $T+T^*$ which carries D, D^*, $z^{(j)}$ into T, T^*, $Z^{(j)}$ and is such that u, v are of class C' and satisfy $\alpha u_x+\beta u_y=v_y$, $\gamma u_x+\delta u_y=-v_x$, $0<u_xv_y-u_yv_x$ on D. The analysis is based on the work of Douglas, Courant, Morrey, and earlier results of the authors. (Received January 9, 1953.)

369. Murray Gerstenhaber: *A characterization of the modular group and certain similar groups.*

Two realizations G' and G'' of an abstract group as a group of properly discontinuous transformations of the upper half-plane onto itself will be considered not essentially distinct, if G'' is conjugate to G' in the full group of conformal transformations of the upper half-plane onto itself. Let $Z_2 * Z_n$ (n finite) denote the free product of a cyclic group of order two and one of order n. Then for $n>2$, $Z_2 * Z_n$ can be realized in one and only one way as a properly discontinuous group of conformal transformations of the upper half-plane onto itself with a fundamental domain having finite hyperbolic area; $Z_2 * Z_2$ can not be realized in such a way at all. As the modular group is algebraically isomorphic to $Z_2 * Z_3$, this gives a characterization of the modular group. The theorem is proved by showing that the commutator subgroup of $Z_2 * Z_n$, acting as a group of conformal transformations of the upper half-plane, is the fundamental group of a Riemann surface possessing certain conformal transformations onto itself. These characterize the surface in question and it determines the realization of $Z_2 * Z_n$. (Received February 13, 1953.)

370. Samuel Goldberg: *Infinite series solutions of a singular diffusion equation.*

The partial differential equation $u_t=(au)_{xx}-(bu)_x$, $0<x<1$, $t>0$, for a probability density function $u=u(t, x)$ is studied, where $a=x(1-x)$, $b=\beta-(\alpha+\beta)x$, α and β are positive constants. One is led to an eigenvalue problem for a modified hypergeometric differential equation which, transformed to a corresponding integral equation, reduces the problem to that of finding the Green's functions for $L(X)=0$ where $L(X)=d[x^{1-\beta}(1-x)^{1-\alpha}X']/dx$. Boundary conditions with X and X' finite or infinite at $x=0$ and $x=1$ are considered. In the unbounded case, the boundary condition is $x^pX'/X\to k(0<p<1)$ as $x\to 0$. In each of these cases we have found the Green's functions and the eigenfunctions of the integral equation. These are combinations of hypergeometric functions. The solutions of the diffusion p.d.e. are obtained as infinite series expansions in terms of these functions. (Received March 11, 1953.)

371. Wilfred Kaplan: *Close-to-convex schlicht functions.*

It follows from known theorems that if $w=\phi(z)$ is a schlicht mapping of $|z|<R$ onto a convex domain and $w=f(z)$ is an analytic function for $|z|<R$ such that $f'(z)/\phi'(z)$ has positive real part, then $f(z)$ is schlicht. The schlicht functions $f(z)$ obtainable in this way are called close-to-convex. It is shown that $f(z)$ is close-to-convex if and only if $\arg f'(re^{i\alpha})-\arg f'(re^{i\beta})$ is less than $\pi+\beta-\alpha$ for $\alpha<\beta$; from this criterion a geometric description of $f(z)$ is obtained. It is shown that the class of close-to-convex functions includes the class of mappings of $|z|<R$ onto a star-shaped domain and includes those f for which, for some real constants α and β, $(z-Re^{i\alpha})(z-Re^{i\beta})f'(z)$ has positive real part. It is shown that a function $f(z)$ having a Poisson integral representation in terms of a function $h(\theta)$ which is monotone increasing for $0\leqq\theta\leqq\pi$ and monotone decreasing for $\pi\leqq\theta\leqq 2\pi$ is necessarily close-to-convex. (Received March 9, 1953.)

372*t*. Hyman Kaufman and R. L. Sternberg: *A two-point boundary problem for ordinary self-adjoint differential equations of even order.* I.

Consider the boundary problem: (1) $[p_0(x)u^{(n)}]^{(n)}-[p_1(x)u^{(n-1)}]^{(n-1)}+\cdots+(-1)^np_n(x)u=0$, (2) $u(x_r)=u'(x_r)=\cdots=u^{(n-1)}(x_r)=0$ $(r=1, 2)$, where $n\geqq 2$, $p_0(x)>0$, $p_i(x)\in C^{n-i}$ on an interval $[a, \infty)$ and all elements are real. Call $u=u(x)\in C^n$ a solution of (1)–(2) for $a\leqq x_1<x_2<\infty$ if $p_i(x)u^{(n-i)}\in C^{n-i}$ on $[a, \infty)$ and $u(x)$ satisfies (1)–(2) with (1) an identity on $[a, \infty)$. If $a>0$ and $p_0(x)\geqq k_0x^{2n-2}$ on $[a, \infty)$ for some $k_0>0$ and if the integrals $\int_a^\infty x^{2n-1}p_i(x)dx$ $(i=1, 2, \cdots, n)$ converge, then there exists an $a_0\geqq a$ such that for $a_0\leqq x_1<x_2<\infty$ the only solution $u=u(x)$ of (1)–(2) is the trivial solution $u(x)\equiv 0$ on $[a, \infty)$. The proof is obtained with the aid of an application of Theorem 5.3 of Sternberg (Duke Math. J. vol. 19 (1952) pp. 311–322). (Received March 11, 1953.)

373. I. I. Kolodner: *A method for solving the heat equation with moving boundary condition.*

Problem: find $\rho(t)$, $u(x, t)\in C^1$ $(D: t>0, x>\rho(t))$ such that (1) $u_{xx}=u_t$ in D, (2) $u(\rho, t)=f(t)$, $u_x(\rho, t)=g(t)$, $t>0$, (3) $u(x, 0)=0$, $x\geqq\rho(0)=0$, $u(\infty, t)=0$. Although a formal expression for u involving f, g, and ρ can be easily derived, it is in general of little use in proving the existence and uniqueness of solutions. Let $R(t)\in C^1$ $(t>0)$, $R(0)=0$, $\dot R(t)\neq 0$ $(\cdot=d/dt)$, and assume that $f\in C^1$, $g\in C$, and $|f|+|g|+|f\dot R|\leqq At^{-1+\epsilon}$, $\epsilon>0$. We show: i. There exists a unique square integrable solution $v^R(x, t)\in C^1$ $(D^R: t>0, r\neq R(t))$ of the problem (1) $v_{xx}=v_t$ in D^R, (2) $v(R+, t)-v(R-, t)=f(t)$, $v_x(R+, t)-v_x(R-, t)=g(t)$, $t>0$, (3) $v(x, 0)=0$, $|x|<\infty$, $v(\pm\infty, t)=0$. ii. If

$v^R(R-, t) = 0$ has a (unique) solution R^* satisfying the conditions on R, then $v^{R^*} = 0$ for $r < R^*$, and $\rho = R^*$, $u = v^{R^*}$, $x > R^*$ form the desired (unique) solution of the original problem. iii. If $v_*^R(R-, t) = 0$ has a unique solution R^{**}, and $\dot{R}^{**} < 0$, conclusions in ii follow. If R^{**} is unique, so is R^* and $R^* = R^{**}$. Tools used were: Laplace transforms to derive a formal expression for v^R, and energy integral methods in proving uniqueness. An application arises in the determination of evaporation of a liquid sphere in presence of its vapor. In this case $f(t) = \rho(t) + 1$, $g(t) = (\rho + 1)(\alpha \dot{\rho} + 1)$, $\alpha > 0$, and application of iii leads to an integro-differential equation for ρ which can be handled very efficiently. The method applies also to cases when the boundary condition (3) is more complicated. (Received March 11, 1953.)

374. C. N. Moore: *On convergence factors for double series summable (N) whose partial sums do not remain bounded.*

In the author's colloquium volume (Amer. Math. Soc. Colloquium Publications, vol. 22) necessary and sufficient conditions were derived for convergence factors in double series, whose partial sums remained bounded, and which are summable by Norlund means. It is not difficult to modify the treatment there given to cover the case where the partial sums become infinite in some definite manner. The conditions thus obtained enable one to infer summability (A) of the restricted type for the series in question. In several recent articles published in Doklady Akad. Nauk some of the Russian mathematicians have treated the case where the summability considered was confined to the case $(C, 1)$. Presumably these writers had some reason not to take account of the wide generality obtained in the modern treatment of convergence factor theorems. (Received April 25, 1953.)

375. Cathleen S. Morawetz: *A uniqueness theorem for an equation of mixed type.*

Consider the equation (*) $K(y)u_{xx} + u_{yy} = 0$, with $K'(y) \geqq 0$, $K(0) = 0$. Let D be a domain bounded by five curves C_1, γ_1, γ_2, C_2, and C_0. γ_1 and γ_2 are two characteristic arcs issuing from the origin. C_0 is a smooth arc in $y \geqq 0$ joining two points $(a_1, 0)$ and $(a_2, 0)$, $a_1 \leqq 0 \leqq a_2$. C_1 and C_2 are smooth arcs in $y \leqq 0$ issuing from $(a_1, 0)$ and $(a_2, 0)$ respectively and on which $dx^2 + K dy^2 \geqq 0$. C_0 is assumed to be star-shaped with respect to the origin. Theorem: A solution of (*) vanishing on $C_0 + C_1 + C_2$ vanishes identically. The following limiting cases are included: $a_1 = 0$; C_1 is a characteristic; C_1 and C_2 are characteristics. (Received March 10, 1953.)

376t. Harry Pollard: *Solution of Bernstein's approximation problem.*

Let $K(u)$ be a function continuous on $(-\infty, \infty)$ and vanishing more rapidly than any power of u as $|u| \to \infty$. The Bernstein problem is to decide when $u^n K(u)$, $n = 0, 1, \cdots$, is fundamental in the space C_0 of functions continuous on (∞, ∞) and vanishing at $\pm\infty$. A survey of the literature can be found in Carleson (Proc. Amer. Math. Soc. vol. 2 (1951) pp. 953–966). The present paper gives the first complete solution, as follows. It is necessary and sufficient that (i) $K(u) \neq 0$, (ii) $\int (1 + u^2)^{-1} \log |K(u)| du = -\infty$, and (iii) there exists a set of polynomials $p_n(u)$ such that $p_n(u)K(u) \to 1$ and $|p_n(u)K(u)| \leqq C$. The proof depends on Loomis' work on Hilbert-Stieltjes transforms (Bull. Amer. Math. Soc. vol. 52 (1946) pp. 1082–1086.) (Received February 4, 1953.)

377. M. H. Protter: *An existence theorem for the generalised Tricomi problem.*

A boundary value problem is treated for the equation (*) $K(y)u_{xx}+u_{yy}=0$ when $K(y)$ is a monotone increasing function of y possessing a continuous third derivative with $K(0)=0$, $K'(0)>0$. The domain considered is the following: let Γ be a simple rectifiable arc lying in the upper half-plane with end points on the x-axis at A and B $(A<B)$. Let Γ_1 and Γ_2 be real characteristics of equation (*) emanating from A and B respectively and which intersect. Suppose that γ is an arc lying in the triangle bounded by Γ_1, Γ_2 and the segment AB. It is further assumed that (i) Γ contains the "normal curve" in its interior and in some neighborhood of A and B coincides with this curve. (ii) γ passes through the point A, has a positive slope, intersects Γ_2, and in some (arbitrarily small) neighborhood of A coincides with Γ_1. Let D be the domain bounded by Γ, γ, and Γ_2. We consider the boundary value problem for (*) in which values are prescribed along Γ and γ. For this problem it is shown that the Fredholm alternative holds. This generalizes a recent result of Frankl. If γ coincides with Γ_1, then the problem reduces to the Tricomi problem and generalizes results of Tricomi and Gellerstedt. (Received March 11, 1953.)

378. Walter Rudin: *H_p classes in general domains.*

In the unit circle, the class H_p $(p>0)$ of analytic functions f for which $\int_0^{2\pi}|f(re^{i\theta})|^p d\theta$ is bounded as $r\rightarrow 1$ has been extensively studied. The author defines a conformally invariant extension for arbitrary plane domains D: $f\in H_p$ in D if f is analytic and single-valued in D, and if $|f|^p$ has a harmonic majorant in D. The following results are obtained. (1) If Z is a closed set of capacity zero, and if $f\in H_p$ in $D-Z$, then f can be defined on Z so that $f\in H_p$ in D. (2) Fixing a point $t\in D$, let $\|f\|_p=\{u(t)\}^{1/p}$, where u is the least harmonic majorant of $|f|^p$. If $p\geqq 1$, this makes H_p into a Banach space (H_2 is a Hilbert space). Although the norm depends on the choice of t, the induced topology does not. (3) If T is a linear functional of norm ρ on H_p and $p>1$, there is a unique $F\in H_p$ such that $\|F\|_p=1$ and $TF=\rho$. (4) If the boundary C of D is analytic, the boundary behavior of the functions under consideration is as in the unit circle. In particular, H_p can then be regarded as a subspace of L_p, the space of complex functions f on C, normed by $\|f\|_p=\{(1/2\pi)\int_C|f(z)|^p\partial G/\partial n ds\}^{1/p}$, where G is the Green's function of D with pole at t; if $p\geqq 1$, the Cauchy formula holds with C as path of integration. (Received March 17, 1953.)

379t. Walter Rudin: *Schwarz's lemma in H_1.*

Let D be a domain whose boundary C consists of n simple closed nonintersecting analytic curves. Fixing a point $t\in D$, let H^* be the set of functions $f\in H_1$ for which $f(t)=0$ and $\|f\|_1\leqq 1$, where the norm is defined as in the preceding abstract. Let $\rho=\sup|f'(t)|$ for $f\in H^*$, and let $\alpha=\sup|f'(t)|$ under the stronger restriction $|f(z)|<1$ in D. There always exists at least one function $F\in H^*$ for which $F'(t)=\rho$, and a unique function g of the form $g(z)=(z-t)^{-1}+\sum c_k(z-t_k)^{-1}+w(z)$ (where $t_1, \cdots t_{n-1}$ are the critical points of the Green's function of D with pole at t, and $w(z)$ is analytic and single-valued on $D\cup C$), such that $|g(z)|=\rho$ on C; moreover, $F(z)g(z)\geqq 0$ on C. Two cases are to be distinguished. (1) If the Szegö kernel function $K(z, t)$ has its zeros at $z=t_1, \cdots, t_{n-1}$, then g has n poles and no zeros in D, F has n zeros in D, and $\rho=\alpha$. (2) Otherwise, g has at least one zero in D, F has fewer than n zeros in D, and $\rho>\alpha$. If D is the ring $0<r<|z|<1$, then (1) occurs if $|t|=r^{1/2}$; in this case there are infinitely many extremal functions F; the ratio of any two of them is an elliptic function of $\log z$. If $|t| \neq r^{1/2}$, then (2) occurs, and the extremal function F is unique. (Received March 17, 1953.)

380. Jacob Samoloff: *Mixed means, extensions, and multiple iterative sequences.*

An investigation is presented of convergence properties in the case of multiple iteration processes of the form $x_{k,n+1} = f_k(x_{1,n}, \cdots, x_{p,n})$, $k = 1, 2, \cdots, p$; $n = 0, 1, \cdots$, where the f_k are means, i.e., they satisfy the inequalities $\max (x_i) \geqq f(x_1, \cdots, x_p) \geqq \min (x_i)$, for $a \geqq$ all $x_i \geqq b$. Sufficient conditions for uniform convergence are given, independently of continuity or differentiability of the functions. Elementary methods are used. In all cases under discussion the sequences $\{x_{k,n}\}$, $k = 1, 2, \cdots, p$, converge to a common limit $F(x_1, \cdots, x_p)$, where $x_k = x_{k,0}$, and F is found to be also a mean, which is uniquely defined by the characteristic property $F(x_1, \cdots, x_p) = F[f_1(x_1, \cdots, x_p), \cdots, f_p(x_1, \cdots, x_p)]$. Applications are to mixed means (e.g. arithmetic-geometric mean of Gauss) and to a method of defining a mean of $p+1$ variables corresponding to a given mean of p variables (called extension). Explicit expressions are found for F in special cases. (Received March 9, 1953.)

381. Samuel Schecter: *On a class of harmonic mappings.*

A harmonic mapping of a domain D in the z-plane is defined by a pair of real-valued functions $u(z)$, $v(z)$ harmonic in D. Let a one-to-one mapping of an annulus $0 < q < |z| < 1$ onto a convex domain with the origin deleted be given by $w = \int ce^{-h(z)}dz - \overline{\int ce^{h(z)}z^{-2}dz}$ where $h(z)$ is analytic in the annulus. Estimates are obtained for $\arg w'(\theta) = \phi(\theta)$ on $|z| = 1$ when suitable conditions are prescribed for $\text{Re}\,\{h(e^{i\theta})\} = A(\theta)$. It is shown for instance that if $A \in L_p(0, 2\pi)$, $p > 1$, then $\phi \in L_p(0, 2\pi)$ and inequalities relating their p-norms are obtained. If $A \in \text{Lip}\,\alpha$, $0 < \alpha < 1$, then $\phi \in \text{Lip}\,\alpha$. The method of proof depends on an extension of the theorems of M. Riesz, Privaloff, and Warschawski on conjugate functions to harmonic functions defined in an annulus. Results in this direction have been obtained by C. Saltzer (Bull. Amer. Math. Soc. Abstract 58-2-195). The following has been shown for harmonic mappings of the unit disk onto itself: Let $\phi(\theta)$ define a homeomorphism of $z = e^{i\theta}$ onto itself and let $\phi(\theta) - \theta = 0$ for $0 \leqq \alpha \leqq \theta \leqq \beta < 2\pi$, $\phi(\theta) - \theta \neq 0$ for $\beta < \theta < 2\pi$, $\phi(2\pi) - \phi(0) = 2\pi$. Then the one-to-one harmonic mapping extension of this homeomorphism into $|z| < 1$ cannot leave the origin fixed. (Received March 11, 1953.)

382. Albert Schild: *On a class of functions, schlicht in the unit circle.*

Let S_p be the class of functions having $|z| = 1$ as radius of schlichtness, and let $f_p(z) = z - \sum_{n=2}^{N} a_n z^n$, having all a_n real and non-negative for $n = 2, 3, \cdots, N$, $N \geqq 2$. It is shown that a necessary and sufficient condition for $f_p(z) \in S_p$ is that $1 - \sum_{n=2}^{N} n a_n = 0$. Certain other properties of this class of functions are studied. Also new estimates of the bound of convexity, of the shortest distance of the boundary of the map of $|z| = 1$ by $f_p(z)$ from origin, and of other quantities connected with the conformal mapping of schlicht functions are obtained. The methods used are of an elementary nature. (Received March 5, 1953.)

383*t*. H. M. Schwartz: *A class of integrals arising in some problems of quantum mechanics.*

Special cases of the integrals $\int_0^\infty e^{-kx} x^p T_m^n(ax) T_\mu^\nu(bx) dx$ $[T_m^n(x) = (-1)^{n+m}(1/(n+m)!^2)$

$L^m_{n+m}(x)$, see H. Bateman, *Partial differential equations*, p. 450] have been discussed in the literature [for example, H. Bateman, loc. cit., E. Schrödinger, *Collected papers on wave mechanics*, p. 100, W. Gordon, Ann. d. Phys. vol. 2 (1929) p. 1031]. The following general result has been obtained by two alternative very simple methods: These integrals, with obvious restrictions on the parameters, equal $a^n b^\nu/(n+m)!(\nu+\mu)!k^{n+\nu+p+1} \cdot \sum_{i=0}^{n} \sum_{j=0}^{\nu}(-1)^{i+j}(n+\nu+p-i-j)!(C_{n+m,i})(C_{\nu+\mu,j}) \cdot k^{i+j}/(n-i)!(\nu-j)!\; a^i b^j$. More compact expressions are also obtained. It is expected to submit this note for publication in the American Mathematical Monthly. This work was supported in part by the United States Atomic Energy Commission. (Received March 10, 1953.)

384*t*. H. M. Schwartz: *On the construction of potential functions for given asymptotic phase in a Schrödinger wave equation.*

It is known that $u''(x)+(k^2-U(x))u(x)=0$, $u(0)=0$ imply the existence of the finite limit of $u(x)/\sin(kx-d(k))$ as $x\to\infty$ for sufficiently rapid decrease of $U(x)$ for $x\to\infty$. Prompted by some interest in connection with quantum mechanical interpretation of the differential equation and of the function $d(k)$ in scattering theory, the question of the determination of the function $U(x)$ over $(0, \infty)$ when $d(k)$ is assigned for $0\leqq k\leqq\infty$ has been subject to some recent discussion [N. Levinson, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. vol. 25, no. 9 (1949), C. E. Fröberg, Arkiv Mat. Astron. Fys. vol. 34A, no. 28 (1948); vol. 36A, no. 11 (1949); Arkiv Fys. vol. 3, no. 1 (1951), R. Jost and W. Kohn, Phys. Rev. vol. 87 (1952) p. 977] and preliminary results by the author were presented to the Arkansas Academy of Science [Proc. Ark. Acad. of Sci. (1951)]. In the present paper the construction of $U(x)$ is carried out by two methods both involving an iteration process in which occur nonlinear integrals of the first kind over the interval $(0, \infty)$. One of these methods is based on an explicit expression for $d(k)$ [see Courant and Hilbert, *Methoden der mathematischen Physik*, vol. I, p. 287]. The convergence of the process and uniqueness of solution is discussed under a number of restrictive conditions. It is expected to submit this paper for publication in the Proceedings of the American Mathematical Society. This work was supported in part by the U. S. Atomic Energy Commission. (Received March 10, 1953.)

385*t*. Helen M. Sternberg and R. L. Sternberg: *A two-point boundary problem for ordinary self-adjoint differential equations of even order*. II. Preliminary report.

For the preliminary case $n=2$: $(1_1)\,[p_0(x)u'']''-[p_1(x)u']'+p_2(x)u=0$, $(2_1)\,u(x_r)=u'(x_r)=0$ $(r=1, 2)$ of the boundary problem (1)–(2) considered in (I) (Bull. Amer. Math. Soc. Abstract 59-4-372) the following partial converse is obtained. If $p_0(x)\leqq k_0$, $p_i(x)\leqq 0$ $(i=1, 2)$ on $[a, \infty)$ and if there exists an $a_0\geqq a$ such that for $a_0\leqq x_1<x_2<\infty$ the only solution $u=u(x)$ of (1_1)–(2_1) is the trivial solution $u(x)\equiv 0$ on $[a, \infty)$, then the integrals $\int_a^\infty x^{2i-1}p_i(x)dx$ converge and $\limsup_{x\to\infty}[x\int_x^\infty t^{2i-2}|p_i(t)|dt]\leqq 2^{2i-1}k_0$ $(i=1, 2)$. The proof is similar but not equivalent to that of Theorem 5.2 of the paper of Sternberg referred to in (I). (Received March 11, 1953.)

386*t*. Walter Strodt: *A lemma on homogeneous and isobaric differential polynomials with constant coefficients.*

Let $P(y(x), y'(x), \cdots, y^{(n)}(x))$ be a homogeneous isobaric differential polynomial, of degree D and positive weight W. Let $m=W/D$. Then the differential polynomial $Q(v(u), v'(u), \cdots, v^{(n)}(u))$ which is obtained from P by the substitution

$y(x)=v(u)e^{mu}$, $x=e^u$ effectively involves terms of weight less than W. This lemma is essential in showing the existence of logarithmic monomials which are approximate solutions of differential equations. Its proof is obtained by noting that the denial of the conclusion implies that $P(t_0, t_1, \cdots, t_n)=Q(t_0, t_1, \cdots, t_n)$, so that whenever $f(x)$ is a solution of $P=0$ so is $x^m f(\log x)$. But the constancy of the coefficients implies that if $f(x)$ is a solution so is $f(x+a)$. From these two remarks it follows easily that for every positive integer k and every set of complex numbers $a_0, a_1, \cdots, a_k$ the function $[H_0(x)H_1(x)\cdots H_k(x)]^m$ is a solution of $P=0$, where $H_0(x)=a_0+x$, and $H_{j+1}(x)=a_{j+1}+\log H_j(x)$. Thus the manifold of $P=0$ involves $k+1$ arbitrary constants which are easily seen to be essential, and this is absurd since k is arbitrary. The result remains valid, and the method of proof changes only slightly, if m is any nonzero complex number. (Received January 22, 1953.)

387t. Walter Strodt: *Logarithmic monomials which are approximate solutions of algebraic differential equations with constant coefficients.*

Let $P_0(y(x), y'(x), \cdots, y^{(n)}(x))$ be a differential polynomial with constant coefficients, effectively of positive degree, and such that $P_0(0, 0, \cdots, 0)\neq 0$. Then there exists a logarithmic monomial $M(x)=Cx^{r_0}(\log x)^{r_1}(\log\log x)^{r_2}\cdots(\log_s x)^{r_s}$, where C is complex and the r_j are real, such that $P_0(M(x), M'(x), \cdots, M^{(n)}(x))\to 0$ as $x\to\infty$ (in any sector of the complex plane in which M is analytic). The method of proof provides an alogorithm for finding M. For each term effectively present in P_0 the ratio of weight to degree is calculated and r_0 is the minimum of these ratios. The substitution $y=ve^{r_0u}$, $x=e^u$ transforms P_0 into $P_1(v(u), v'(u), \cdots, v^{(n)}(u))+Q_1(u, v(u), v'(u), \cdots, v^{(n)}(u))$ where every coefficient in Q_1 tends to zero as $u\to\infty$. P_1 is treated the way P_0 was treated, and thereby r_1 is determined. Repetition of this procedure leads to a finite sequence of differential polynomials $P_0, P_1, \cdots, P_s$, the terminal polynomial P_s being characterized by the property that it effectively involves terms of weight zero. The coefficient C in M is then any one of the obviously existing constants which satisfy the equation $P_s=0$. Assurance that the process terminates in finitely many steps, and a bound for the number of steps required, is given by the author's lemma on homogeneous isobaric differential polynomials, stated in the foregoing abstract. (Received January 22, 1953.)

388t. Walter Strodt: *Principal logarithmic monomials for algebraic differential equations.*

Let a system of simply-connected neighborhoods of ∞ in the complex domain be given. The notation $f(x)<1$ is to mean that for any non-negative integers j and k, and any positive ϵ, there is a neighborhood of ∞, $W(j, k, \epsilon)$, such that $|d^jf(e_k(x))/dx^j|<\epsilon$ in $W(j, k, \epsilon)$, where $e_0(x)=x$ and $e_{j+1}(x)=\exp(e_j(x))$. As usual $f\prec g$ means $f/g\prec 1$ and $f\sim g$ means $f-g\prec g$. A function f is called trivial if $f\prec x^{-N}$ for every positive N. A function f is called logarithmic of rank s if f is trivial or if $f\sim$ some logarithmic monomial $Cx^{r_0}\cdots(\log_s x)^{r_s}$. A set E of functions is called a logarithmic domain of rank s if every sum $M_1f_1+M_2f_2+\cdots+M_kf_k$, in which the M_j are logarithmic monomials of rank s and the f_j belong to E, is logarithmoid of rank s. If $P(x, y, y', \cdots, y^{(n)})$ is a differential polynomial with coefficients in a logarithmic domain, and if $P(x, 0, \cdots, 0)$ and at least one other coefficient are nontrivial, then there is a logarithmic monomial M, obtainable by an algorithm like the one in the author's foregoing abstract, and called a principal (logarithmic) monomial for P, such that $P(x, M, M', \cdots, M^{(n)})\prec P(x, 0, \cdots, 0)$. Equations with rational coefficients are included as a very special case. (Received January 22, 1953.)

389t. Walter Strodt: *Principal solutions of algebraic differential equations of first order.*

A solution of $P(x, y, y', \cdots, y^{(n)})=0$ is called principal if it is asymptotically equivalent to a principal monomial of P, as defined in the author's foregoing abstract. Let M, L, and K be principal monomials for $P(x, y, y')$, $\partial P(x, y, y')/\partial y'$, and $P(x, y, y')-y\partial P(x, 0, 0)/\partial y-y'\partial P(x, 0, 0)/\partial y'$, respectively. If $M \prec L$ and $M \prec K$ then P is called quasi-linear, and $M[\partial P(x, 0, 0)/\partial y']/P(x, 0, 0) \sim$ some logarithmic monomial $Tx^{l_0}(\log x)^{l_1} \cdots (\log_q x)^{l_q}$. Let the coefficients of P belong to a logarithmic domain where the neighborhoods of ∞ fit a sector $\mathfrak{S}$ of the complex x-plane. $\mathfrak{S}$ is divided into sectors $\mathcal{U}_1, \cdots, \mathcal{U}_k$ of uniqueness, and sectors $\mathcal{N}_1, \cdots, \mathcal{N}_k$ of non-uniqueness, according to the negativity and positivity of $\mathfrak{R}[T \exp (il_0\theta-i\theta)]$ $(\theta=\arg x)$ (unless $l_0=1$ and $\mathfrak{R}(T)=0$). For each $\mathcal{U}_j$ there exists a unique solution $y(x, \mathcal{U}_j)$ of $P=0$ which is principal if neighborhoods of ∞ are redefined to fit $\mathcal{U}_j$. If $j\neq k$, $y(x, \mathcal{U}_j)$ is sometimes, but not always, different from $y(x, \mathcal{U}_k)$. For each $\mathcal{N}_j$ there exists a one-parameter family of principal solutions in $\mathcal{N}_j$. The proof is by successive approximations, after detailed preliminary reductions utilizing the theory of principal monomials. (Received January 22, 1953.)

390t. C. T. Taam: *On the solutions of second order self-adjoint linear differential equations.*

Let $P(x)$ and $Q(x)$ be complex-valued, $r(x)(>0)$ and $f(x)$ real-valued functions defined for $x\geqq 0$, the functions $1/P$, Q, $1/r$, and f belonging to $L(0, R)$ for every positive R. The differential equation (A): $(PW')'+QW=0$ is called disconjugate on an interval I if no solution of (A) other than the trivial solution $(\equiv 0)$ possesses more than one zero on I. Let (B) denote what results in (A) if P and Q are replaced by r and f respectively. Write $P=p_1+ip_2$, $Q=q_1+iq_2$, where p_1, p_2, q_1, and q_2 are real. In this note a general criterion for (A) disconjugate on I is obtained and from which the following comparison theorem is proved: Suppose that (B) is disconjugate on a closed or open interval I_0. If $jp_1+kp_2\geqq r$, $jq_1+kq_2\leqq f$ almost everywhere on I_0 for some real constants j and k, then (A) is disconjugate on I_0. These results generalize those obtained previously by the author for the case $P=1$. (Received March 9, 1953.)

391. E. W. Titt: *On a theory of the linear second order partial differential equation—integration of potentials for hyperbolic exterior with n independent variables.*

In a previous paper [Bull. Amer. Math. Soc. Abstract 58-6-663] integrating factors for hyperbolic interior were developed from a weighted non-Euclidean area of a portion of a fixed noncharacteristic splitting plane, the weight function being a new type of potential called retarded. This multiple integral, when reduced to a single integral, takes on the nature of a transform of the original potential and the study of the kernel involved in the transform has proved to be an important key in the theory. The integrating factor having a jump discontinuity is obtained by applying a certain number of differential operations to the transform and it turns out that in even dimensions the same integrating factor can be used in both interior and exterior, proper attention being paid to the difference in geometrical meaning of the quantities involved. An instance of our integrating factor was used by d'Adhémar in $n=4$ exterior. In odd dimensions an alteration in the kernel, suggested by previous work, proves necessary to obtain the exterior theory. After the Green's theorem stage further differential operations would yield a formula based on logarithmetic quantities which extend quantities appearing in the work of Volterra and Tedone. (Received March 12, 1953.)

392t. J. L. Walsh: *An interpolation problem for harmonic functions.*

Complex-variable methods of orthogonal functions developed by Kakeya, Takenaka, Malmquist, Walsh, and Lokki yield a solution of the interpolation problem for functions $u(z)$ of class H, namely harmonic in B: $|z|<1$ with $\int[u(re^{i\theta})]^2d\theta$ bounded in B and norm defined by $\|u\|^2=\int[u(e^{i\theta})]^2d\theta$. The function $u_\alpha(z)$ of H of minimum norm with $u_\alpha(\alpha)=1$, $|\alpha|<1$, is (*) $[(1-|\alpha|^2)/(1+|\alpha|^2)]$ Re $[(1+\bar{\alpha}z)/(1-\bar{\alpha}z)]$. Given distinct points $\alpha_0, \alpha_1, \cdots$, in B, a suitable linear combination $\psi_k(z)$ of the functions (*) corresponding to $\alpha_0, \alpha_1, \cdots, \alpha_k$ has the property $\psi_k(\alpha_0)=\cdots=\psi_k(\alpha_{k-1})=0$, $\psi_k(\alpha_k)\neq 0$; the $\psi_k(z)$ are mutually orthogonal on C: $|z|=1$; every $u(z)$ of H possesses a unique formal expansion $\sum a_k\psi_k(z)$ found by interpolation to $u(z)$ in the α_k or alternately by the usual orthogonal-function method on C. This expansion represents in B the function of H of least norm equal to $u(\alpha_k)$ in the points α_k. For real α_k with $\alpha_0=0$, the $\psi_k(z)$ are the real parts of the rational functions $\phi_k(z)$ orthogonal on C defining an interpolation series $\sum c_k\phi_k(z)$ formally expanding an arbitrary function analytic in B by interpolation in the points α_k. (Received April 9, 1953.)

393t. J. L. Walsh and Lawrence Rosenfeld: *On the boundary behavior of a conformal map.*

A function $\phi(u)$ defined for $u_1\leq u<+\infty$ is said to possess *property* B at $u=+\infty$ if, uniformly in every interval $|U|\leq U_0$, we have $\lim_{u\to+\infty}\phi[U\phi(u)+u]/\phi(u)=1$. Here $\phi(u)$ need not be single-valued but we require this relation for any choice of values of $\phi(u)$. Let S be the simply-connected infinite strip in the plane $w=u+iv$ defined by $\phi_-(u)<v<\phi_+(u)$, $u_1<u<\infty$, where $\phi_+(u)$ has property B and where some segment $u_2<u<\infty$, $v=0$ lies in S. Let $w=f(z)$ with $z=x+iy$ map the infinite strip Σ: $|y|<\pi/2$ onto S so that $u(z)\to+\infty$ when z is in Σ and $x\to+\infty$. If x_n is any sequence of real points such that $x_n\to+\infty$ and if $\lim_{u\to+\infty}\phi_-(u)/\phi_+(u)=\lambda$, then $\lim_{n\to\infty}[f(z+x_n)-f(x_n)]/\theta(u(x_n))=z/\pi$ for z in Σ, uniformly on any closed bounded subset of Σ, where $\theta(u)=\phi_+(u)-\phi_-(u)$. The same conclusion is valid if $\phi_-(u)$ has property B and the condition $\lim\phi_-(u)/\phi_+(u)=\lambda$ is replaced by $0<m\leq|\phi_-(u)/\phi_+(u)|\leq M<\infty$. (Received April 25, 1953.)

394. C. Y. Wang: *Polynomial approximations to solutions of variational problems in the complex domain.*

Let C be a closed analytic Jordan curve which bounds a region D. Let $\mathcal{J}$ be the class of analytic functions $f(z)$ in D such that $f(z_0)=0$, $f'(z_0)=1$, $z_0\in D$, and $f'(z)\in H(2, C)$. It is well known that there exists $\phi(z)\in\mathcal{J}$ which minimizes $\{(1/L)\cdot\int_C|\phi'(z)|^2|dz|\}^{1/2}=\|\phi'\|$, where L is the arc length of C. In this paper we approximate $\phi(z)$ by the polynomial $Q_n(z)\in\mathcal{J}$ of degree $\leq n$ which minimizes $\|Q_n'\|$ and get an estimate for this approximation. We make use of the fact that Hilbert space is a uniformly convex Banach space, and a rate of convergence is obtained. By means of this rate we use Szegö polynomials and Féjer-Riesz theorem to obtain the estimate $|\phi(z)-Q_n(z)|\leq M\rho^n$ where $0<\rho<1$, ρ depends on the analyticity of the curve C, and M is some constant. Furthermore the following problem is solved. Let $L_j(f)$, $j=1, 2, \cdots, m$, be the given linear functionals in $H(2, C)$, let $f\in\bar{E}\subset H(2, C)$ such that $L_j(f)=0$ for all j, and let $f_0\in H(2, C)-\bar{E}$ such that not all $L_j(f_0)$ vanish. We find $g\in\bar{E}$ such that $\|f_0-g\|=$ minimum. Then let P_n be the polynomial of degree $\leq n$ in

E such that $L_j(P_n)=0$ for all j and $\|f_n-P_n\|=\text{minimum}$. We show that $P_n\to g$ and obtain an estimate of the rate of convergence. (Received February 20, 1953.)

395t. Alexander Weinstein: *On an elementary property of the Darboux operators and on a theorem of Friedrichs.*

Let $u^\alpha(x_1, \cdots, x_m, t)$, $-\infty<\alpha<\infty$, denote a solution of the equation (*) $L_\alpha u^\alpha = u^\alpha_{tt}+\alpha t^{-1}u^\alpha_t-\Delta u^\alpha=0$. L_α is called a Darboux operator. According to Darboux (*Théorie des surfaces*, vol. 2, p. 63) $t^{-1}u^\alpha_t=u^{\alpha+2}$. Since $L_\alpha u^\beta=L_\beta u^\beta+(\alpha-\beta)t^{-1}u^\beta_t$, it follows immediately that $L_\alpha u^\beta=u^{\beta+2}$. Therefore, (**) $L_{\beta+2m}L_{\alpha_m}\cdots L_{\alpha_2}L_{\alpha_1}u^\beta=0$. The special case in which $m=1$, $\beta=0$, and $\alpha_1=\cdots=\alpha_n=2n$ yields $L_{2n}^{n+1}u^0=0$ and was found by Friedrichs by an elaborate computation (see Courant-Hilbert, vol. 2, p. 416 ff.). This work was sponsored by the Office of Naval Research. (Received March 9, 1953.)

396. František Wolf: *Spectral decomposition of a class of operators in Banach space.*

The generalized spectral decomposition $f(A)=\int_{\sigma(A)}f(\lambda)d^nE(\lambda)$, reported in Bull. Amer. Math. Soc. Abstract 58-4-369, has as its essential features (i) it exhibits that the "support" (the smallest closed set such that $f(\lambda)=0$ on it implies $f(A)=0$) of the mapping $f\to f(A)$ is $\sigma(A)$, (ii) we can speak about the spectrum of an element of the space $\sigma_A(x)\subseteq\sigma(A)$, which is the support of the mapping $f\to f(A)x$, (iii) to an arbitrary closed set S we can construct an f, zero exactly on S and $f(A)x=0$ defines a subspace invariant under A such that $\sigma_A(x)\subseteq S$, (iv) if $S\cap\sigma(A)\neq S'\cap\sigma(A)$, then the corresponding subspaces are different. Under the conditions given in the report $f(A)$ is defined for $f\in C^n$. If A is more general, $f(A)$ will be defined for a more restricted class of f's. If A is too general, the class of f's becomes quasianalytic, and the concept of the support of the mapping evidently collapses. But as long as the class is not quasianalytic, there exist "localizing" functions, and all the results still hold. A condition for this is the existence of $p(t)$ such that $p(-t)=p(t)$, $p(t+\tau)\leqq p(t)+p(\tau)$, $p'(t)\geqq 0$, $\lim_{t\to\infty} tp'(t)=\infty$, $\int_1^\infty(p(t)/t^2)dt<\infty$, $\|T^n\|\leqq e^{p(n)-\log(1+n^2)}$ (cf. Mandelbrojt, *Classes quasianalytiques*, Paris, 1935, p. 78 ff.). With similar conditions J. Wermer shows the existence of proper invariant subspaces (Duke Math. J. vol. 19 (1952) pp. 615–622). (Received March 12, 1953.)

397t. K. G. Wolfson: *On the separation of spectra.*

It is assumed that the differential equation $(px')'+(q+\lambda)x=0$ is of "limit point" type for both $t=-\infty$ and $t=\infty$. A separation theorem is proved relating the spectrum of this boundary value problem with the spectra of the boundary value problems defined on $[0,-\infty)$ and $[0,\infty)$ by the boundary condition $x(0)=0$. The method is based on the spectral theorem for self-adjoint operators on a Hilbert space. (Received March 9, 1953.)

398. D. M. Young and M. L. Juncosa (p): *On the degree of convergence of solutions of difference equations to solutions of the heat equation.* Preliminary report.

Let $u(x,t)=\sum_{n=1}^\infty a_n \sin n\pi x \exp(-n^2\pi^2t)$ be a Fourier series solution of the heat equation $u_{xx}=u_t$ in R: $0<x<1$, $t>0$, with boundary conditions $u(+0,t)=u(1-0,t)=0$, $t>0$, and initial condition $u(x,+0)=f(x)$, $0<x<1$, where $f(x)$ is continuous except for a finite number of finite jumps and has a convergent Fourier series. For each integer M, let $u_M(x, t)$ satisfy the difference equation $u_M(x, t+p)-u_M(x, t) = r\{u_M(x+h, t)+u_M(x-h, t)-2u_M(x, t)\}$ $(h=M^{-1};\ p=rh^2)$, for all points (x, t) of R such that Mx and M^2tr^{-1} are integers. Let $\bar{u}_M(x, t)$ be obtained from $u_M(x, t)$ by bilinear interpolation. If $0<r\leqq 1/2$, then in any closed interior region of R, the convergence of $\bar{u}_M(x, t)$ to $u(x, t)$ is of the respective orders M^{-2}, M^{-1}, $M^{-\alpha}$ according as $f(x)$ has a derivative of bounded variation, or is itself of bounded variation, or satisfies a Lipschitz condition of order α, $0<\alpha\leqq 1$. If $f(x)$ is a step-function, the convergence is at least of the order M^{-1}; if M tends to infinity through a sequence of values $\{M_k\}$ such that, for each k, M_{k+1} is an integral multiple of M_k, and if the abscissa of each point of discontinuity is an integral multiple of M_k^{-1} for some k then the convergence is of the order M^{-2}. (Received March 12, 1953.)

APPLIED MATHEMATICS

399t. Milton Abramowitz: *On regular and irregular Coulomb wave functions in terms of Bessel-Clifford functions.*

Expansions for the regular and irregular solutions of the differential equations $y''+(1-2\eta/\rho-((L+1)/\rho^2))y=0$ are obtained in terms of the functions $t^{n/2}I_n(2t^{1/2})$ and $(-1)^n t^{n/2}K_n(2t^{1/2})$ where $t=2\eta\rho$. The expansions have the important property that the coefficients are the same in both. The expression for the regular solution is shown to be uniformly convergent for all finite values of t. However, the expansion for the irregular solution is shown to be asymptotic in the sense of Poincaré. (Received February 19, 1953.)

400t. Milton Abramowitz: *On the differential equation in problem of heat convection of laminar flow through a tube.*

The solutions of the differential equation $y''+x^{-1}y+\beta^2(1-x^2)y=0$ subject to the boundary condition $y(0)=1$, $y(1)=0$ is solved in terms of the Bessel functions $x^nJ_n(x)$. The first five characteristic values β_s are determined with the aid of this expansion employing numerical methods. (Received February 19, 1953.)

401. P. W. Berg: *The existence of Helmholtz flows of compressible fluids.*

Plane steady barotropic, subsonic potential flows of a gas about a prescribed open obstacle are considered. The question concerns the existence of flows which are uniform at infinity and such that the streamlines leaving the obstacle are free streamlines extending downstream to infinity. It is shown that if the inclination of the obstacle (with respect to the flow direction at infinity) is a Hölder-continuous function of arc length which remains between the bounds 0 and π, then such a flow exists for each prescribed subsonic value of the maximum speed. The problem is reduced successively to a mapping problem and to an integral equation. The existence of a solution of the integral equation is demonstrated using the Leray-Schauder theorem. (Received March 10, 1953.)

402. Lipman Bers: *Existence and uniqueness of a subsonic gas flow past a given profile.*

Let P be a simple closed curve possessing a Hölde-continuously turning tangent except at one point z_T which is a protruding corner or cusp. Theorem: There exists a number $M^*\leqq 1$ such that for every M_∞, $0<M_\infty<M^*$, there is one and only one po-

tential gas flow around P with stream Mach number M_∞ and horizontal velocity vector at infinity. The flow depends continuously on M_∞ and the maximum local Mach number $M_{\max}$ takes on all values between 0 and 1 as M_∞ goes from 0 to M^*. If P is symmetric about the x-axis, $M_{\max}$ is a monotone function of M_∞. The proof uses pseudo-analytic functions, quasi-conformal transformations, and the Schauder-Leray degree. (The case of P smooth with the circulation prescribed was treated by Schiffman, Journal of Rational Mechanics and Analysis, Vol. 1 (1952), by entirely different methods.) (Received March 10, 1953.)

403. Peter Chiarulli: *Diffraction of a sound pulse by a moving obstacle.*

The problem considered is that of a finite flat plate moving parallel to itself at subsonic speeds and encountering at some angle a plane acoustic pulse. G. F. Carrier (unpublished) by means of a Weiner-Hopf approach has found in closed form the resulting diffraction pattern from the leading edge and from the trailing edge. It is of interest to determine the diffraction pattern for those times for which there is an interaction of leading and trailing edge effects and in particular to determine that time for which the effect of the pulse on the obstacle is no longer important; but for these time values the boundary conditions become too complicated to be amenable to the Weiner-Hopf technique. Moving coordinates are introduced and the resulting problem is equivalent to finding the linearized steady supersonic flow over a straight-edged flat wing with one supersonic edge. Standard techniques are available for solving this problem and in particular the Evvard procedure is used to determine the diffraction pattern for those time values which are of interest. (Received March 11, 1953.)

404t. R. C. Di Prima and G. H. Handelman: *Vibrations of rotating beams.*

This paper is concerned with certain aspects of the problem of determining the lowest natural frequency of vibration of a twisted, non-uniform rotating beam. Through an extension of the methods of W. Prager [*Theory of structures*, Brown University lecture notes] for the static case, the equations of a vibrating twisted, nonrotating beam in intrinsic coordinates has been developed. Corresponding minimum and orthogonality principles have been developed. Approximate frequencies have been found by a Rayleigh-Ritz, Galerkin procedure for a uniform beam with constant angle of twist both for the case in which the vibration is restricted to one principle direction and in general. A study of the dependence of frequency upon end restraint and rotation speed has been carried out for the straight, rotating beam. The general vector differential equations and corresponding minimum principle have been established for the non-uniform, twisted, rotating beam with various forms of end restraint. (Received March 11, 1953.)

405. David Gale: *A theory of n-person games with perfect information.*

Let Γ be an n-person game, Σ_i the set of pure strategies σ_i for player i, and P_i the ith player's pay-off function. Then σ_i is called *equivalent* to σ_i' $(\sigma_i \sim \sigma_i')$ if $P_i(\sigma_1, \cdots, \sigma_i, \cdots, \sigma_n) = P_i(\sigma_1, \cdots, \sigma_i', \cdots, \sigma_n)$ for all $(\sigma_1, \cdots, \sigma_{i-1}, \sigma_{i+1}, \cdots, \sigma_n)$, and σ_i *dominates* σ_i' $(\sigma_i \succ \sigma_i')$ if $P_i(\sigma_1, \cdots, \sigma_i, \cdots, \sigma_n) \geqq P_i(\sigma_1, \cdots, \sigma_i', \cdots, \sigma_n)$ for all $(\sigma_1, \cdots, \sigma_{i-1}, \sigma_{i+1}, \cdots, \sigma_n)$ and σ_i not $\sim \sigma_i'$. Let $A(\Gamma)$ be the game obtained from Γ by replacing every maximal set of equivalent pure strategies by a uniform mixed strategy, and let $D(\Gamma)$ be the game obtained from Γ by deleting all dominated strategies. If $\Gamma=\Gamma_0$ we define Γ_i inductively by $\Gamma_i=D(A(\Gamma_{i-1}))$. Assuming Γ_0 finite we must eventually have $\Gamma_i=\Gamma_{i+1}=\Gamma_\infty$ (definition). Theorem: If Γ_0 is the normal form of a game with perfect information, then Γ_∞ contains exactly one strategy for each player. These unique strategies may be considered as mixed strategies for Γ_0 and are then defined to be the optimal strategies for Γ_0. Examples are given to justify the definition. (Received March 4, 1953.)

406. M. M. Gutterman and C. M. Price (p): *On the approximation of product of functions.*

For $i=1,\cdots,n$, let α_i and β_i be real-valued functions defined on some set W_i such that β_i is non-negative and $\lambda_i=\beta_i\circ\alpha_i^{-1}$ is well defined on $\alpha_i(W_i)$. Suppose that each λ_i is concave (convex) on its domain $\alpha_i(W_i)$; that T_i is a subset of W_i and T_i contains two elements y_i and z_i such that $\alpha_i(y_i)\leqq\alpha_i(x_i)\leqq\alpha_i(z_i)$ for all x_i in T_i; and that a is a real number and b a non-negative real number. Then SUP $(INF)_{x_i\in T_i}[a\prod\alpha_i(x_i)-b\prod\beta_i(x_i)]$ $=$ MAX $(\mathrm{MIN})_{u_i=y_i \text{ or } z_i}[a\prod\alpha_i(u_i)-b\prod\beta_i(u_i)]$. Another such result is obtained. Formulas are developed which permit rapid calculation of the SUP and INF in the case, of some interest in practical probability work, when each $\alpha_i(x)=e^{-x}$ and each $\beta_i(x)=1-x$, $0\leqq x<1$. (Received March 9, 1953.)

407. W. S. Jardetzky: *On general solutions of partial differential equations with constant coefficients.*

To write the general solution of (1) $\sum A(\partial^m\phi/\partial x_1^a\partial x_2^b\cdots\partial x_m^g)=0$ ($a+b+\cdots+g=m$, $A=$const.) use can be made of the expression (2) $\chi=\sum_1^m v_i(x_i-\hat{x}_i)$, where v_i and $\hat{x}_i$ are parameters. An arbitrary function $f(\chi)$ satisfies (1) and the equation of characteristics provided (3) $\sum_{a+\cdots+g=m}Av_1^av_2^b\cdots v_m^g=0$. With respect to at least one parameter, say v_m, (3) can be written in the form (4) $\sum_{i=0}^n B_iv_m^i=0$. Thus, there are n expressions $\chi^{(k)}$ and n arbitrary functions $f^{(k)}(x^{(k)})$. Therefore, the general solution of (1) takes the form (5) $\phi=\sum_{(k)}\int(2m-1)\int f^{(k)}(\chi^{(k)};v_1,\cdots,v_{m-1},\hat{x}_1\cdots\hat{x}_m)dv_1\cdots dv_{m-1}d\hat{x}_1\cdots d\hat{x}_m$. The limits of integration have to be adjusted to conditions of a problem. The well known Whittaker's form of solutions of Laplace- and wave equation are particular cases of (5). A generalization of the Bateman's form is also possible. (Received February 26, 1953.)

408. G. S. S. Ludford: *Extensions in the applicability of Riemann's formula.*

The object of this paper is to investigate the Cauchy initial values problem for the linear hyperbolic differential equation $L(u)=u_{xy}+au_x+bu_y+cu=f$, when the initial data has a finite number of discontinuities (even infinite) and when some characteristics meet the initial curve in more than one point, this latter not necessarily implying that there is a tangent characteristic. With the usual continuity conditions on the coefficients in $L(u)$, and a slight generalization of the concept of regularity, it can be shown that there is a unique regular solution of the former problem provided the initial data is absolute integrable, and that there is a unique regular solution of the latter on a certain sheeted surface associated with the original coordinate plane and initial curve. The method used in both cases is a modification of the classical method of Riemann. It shows that if a regular solution exists it is given by Riemann's formula, when that formula is suitably interpreted. Direct verification shows that the formula gives a solution of the desired kind. Problems of this kind occur both in one-dimensional gas dynamics and plasticity. (Received March 5, 1953.)

409. G. W. Morgan (p) and J. L. Fox: *On the stability of some flows of an ideal fluid with free surfaces.*

This work is an extension of an investigation begun by W. D. Hayes and C. M. Ablow (O.N.R. Contract N70nr-35807, NR-062 090; Technical Report No. 1) of small perturbations of flows with free surfaces, the primary object being the determination of the stability of the basic steady state flow. The assumption of plane irrotational motion permits the use of complex functions in the analysis. All relations are linearized with respect to the perturbation variables. The perturbation f of the complex potential is assumed to have the form $f = G_1(w_0) \exp(\lambda t) + G_2(\bar{w}_0) \exp(\bar{\lambda} t)$ where w_0 is the complex velocity of the basic flow. Considerations of analyticity and the application of the boundary conditions lead to complicated eigenvalue problems for the functions G_1, G_2 and the eigenvalues λ. The sign of the real part of all admissible λ determines the stability. The problems treated are the hollow vortex bounded by a circular wall, a class of orifice flows, the flow due to the impinging of two equal and opposite jets, the flow due to the impinging of a jet on a finite plate perpendicular to the jet, and the Helmholtz plate flow. The results indicate that the impinging of opposite jets may represent an unstable flow; all other flows (with one special exception) are neutrally stable or stable. (Received March 11, 1953.)

410. L. E. Payne and H. F. Weinberger (p): *Upper and lower bounds for virtual mass.*

Let $\phi_i(x_1, x_2, x_3)$ be the velocity potential of an infinite incompressible fluid disturbed by the motion of a body with unit velocity in the x_i direction. The virtual mass W_i of the body is defined as the Dirichlet integral of ϕ_i, and is proportional to the kinetic energy of the fluid. Several upper and lower bounds for W_i are obtained. Some of the lower bounds depend on the inequality $W_iP_i \geq V^2$ given by Schiffer and Szegö (Trans. Amer. Math. Soc. vol. 67 (1949) pp. 130–205). Here, V is the volume of the body and P_i its polarization in the x_i direction. Upper bounds for P_i were previously obtained by the authors (Bull. Amer. Math. Soc. Abstract 59-3-310). Upper bounds for W_i depend on the following principle: Let a body A be cut in two by a plane parallel to the x_i-axis. Let B and C be the two symmetric bodies obtained by reflecting the two parts of A in the cutting plane. Then $2W_i(A) \leq W_i(B) + W_i(C)$. This upper bound can, by repeated applications, be reduced to the sum of virtual masses of axially symmetric bodies. These virtual masses are related to polarizations (L. E. Payne, Bull. Amer. Math. Soc. Abstract 59-2-225). This work was sponsored by the Office of Naval Research. (Received March 9, 1953.)

411. F. Virginia Rohde: *Large deflections of a simply supported beam.* Preliminary report.

The large deflections of a simply supported beam with concentrated load off center are considered. This case differs from that in which the load is placed in the middle (Conway, *The large deflection of simply supported beam*, Philosophical Magazine vol. 38 (1947) pp. 905–911) in two respects: the position of maximum deflection is not under the load, and the reactions are not both perpendicular to the elastic curve of the beam. Since the results are in terms of elliptic functions involving the angles at the ends of the beam and under the load, the relations between these angles cannot be found in explicit form. Therefore values of the angles must be assumed, from which the position of the load and ratios of the horizontal projection of the elastic curve and of the maximum deflection to the total length of the beam are found. These ratios, together with that of the maximum deflection to maximum deflection obtained by elementary theory, are plotted against the dimensionless ratio PL^2/EI. The results obtained by small and large deflection theory diverge considerably over most of the range of values considered, elementary theory giving the larger deflections. (Received March 4, 1953.)

412. L. V. Robinson: *Some comparisons between solutions of ordinary and partial differential equations*. Preliminary report.

Assuming the p_k to be differentiable functions of the x_k $(k=1, 2, \cdots, n)$, the inverse of an operator, $\lambda = p_1\,\partial/\partial x_1 + p_2\,\partial/\partial x_2 + \cdots + p_n\,\partial/\partial x_n$, is found in terms of transforms of the general type, $\exp(gD_k)$, where the g are also functions of the x_k and $D_k = \partial/\partial x_k$. It is then shown that λ and λ^{-1} in a generalized calculus bear essentially a one-to-one correspondence to $D=d/dx$ and D^{-1} in ordinary calculus. Similarly, every ordinary differential equation has its counterparts in an infinity of partials, permitting the latter to be solved in much the same manner as the former and leading to a very close parallelism between properties of their solutions. (Received March 9, 1953.)

413. K. M. Siegel and F. B. Sleator (p): *Bessel functions of nearly equal argument and order.*

An upper bound for the function $J_\nu(\nu x)$ has been given by Watson which holds for $0<x\leqq 1$ but becomes infinite as $x\rightarrow 1$. A better bound on the function over a certain range is obtained here which approaches the value 1 as $x\rightarrow 1$. This is an improvement on the previous bound for the range $[1-(2\pi\nu)^{-2}]^{1/2}<x\leqq 1$. Examples are given and corresponding expressions developed for a range of $x\geqq 1$ and for certain modified Bessel functions. (Received March 13, 1953.)

414t. R. L. Sternberg: *A general solution of the two-frequency modulation product problem.* II. Preliminary report.

By replacing the region R: $\cos u + h\cos v \geqq k$ of the square $0\leqq u, v\leqq \pi$ by a suitable polygonal region, formulas are obtained suitable for evaluating to six decimals the integrals $A_{mn}(h, k) = (2/\pi^2)\iint_R(\cos u + h\cos v - k)\cos mu\,du \cos nv\,dv$ $(m, n=0, 1)$, $|h|\leqq 2$, $0<k\leqq 1$, which arise in the problem referred to in the title (see Bull. Amer. Math. Soc. Abstract 58-4-374 and W. R. Bennett, *The biased ideal rectifier*, Bell System Technical Journal vol. 26 (1947) pp. 139–169). Tables of these functions accurate to one unit in the sixth place for $h=-2(0.2)2$ and $k=0.001$, 0.01, 0.1 and 1.0 are near completion. (Received March 3, 1953.)

415t. John Todd: *The condition of the finite segments of the Hilbert matrix.*

The finite segments H_n of the Hilbert matrix $H=((i+j-1)^{-1})$ enjoy a reputation of ill condition. Various measures of the condition of these matrices are estimated and the results fully confirm the bad reputation. These estimates are obtained by using the well known explicit inverse of H_n. The results have been used in the practical evaluation of methods for inversion of matrices on high speed automatic digital computing machines. (Received March 11, 1953.)

GEOMETRY

416. H. S. M. Coxeter: *A simple way to enumerate the five parallelohedra.*

A parallelohedron is a convex polyhedron, in real affine space, which can be repeated by translation to fill the whole space without interstices. It is centrally symmetrical and has centrally symmetrical faces: say f pairs of opposite faces, v pairs of opposite vertices, and n sets of parallel edges. An argument suggested by Voronoi [J. Reine Angew. Math. vol. 133 (1907) p. 107; vol. 134 (1908) p. 278] shows that $f\leqq 7$ and that the faces can be only parallelograms or symmetrical hexagons. Such a polyhedron is represented in the real projective plane by a configuration of n lines intersecting by twos or threes in f points to form v regions [Coxeter, *Regular polytopes*, 1948, pp. 28, 30]. The most complicated case (with $f\leqq 7$) is the complete quadrangle, and every other case can be derived from this by omitting one or more of the six lines. The possible values of v are 4, 6, 7, 9, 12, corresponding to the parallelepiped, hexagonal prism, rhombic dodecahedron, elongated dodecahedron, and truncated octahedron [cf. Fedorov, Mineralogicheskoe Obshchestvo, Leningrad vol. 21.4 (1885)]. (Received March 10, 1953.)

LOGIC AND FOUNDATIONS

417. J. R. Shoenfield: *Enumerations of recursive functions.*

A recursive function enumeration (rfe) is a partial recursive $F(e, x)$ such that for every recursive function $f(x)$, there is an e, called the F-number of f, such that $f(x)=F(e, x)$ for all x. An rfe is effective if for every recursive $G(x, y)$, the F-number of $G(f(x), y)$, as a function of x, is a recursive function of the F-number of $f(x)$ and y. Effective rfe's are known to exist. If F is an rfe, the set of F-numbers of recursive functions is not recursively enumerable; in particular, F cannot be defined for all arguments. If F is an effective rfe, then the class of F-numbers of any recursive function is not recursive. It follows that there is no recursive method of deciding when two F-numbers are F-numbers of the same function, or when they are F-numbers of functions enumerating the same set. The set of F-numbers of functions which enumerate recursive (or finite) sets is not recursive. The set of F-numbers of functions which vanish infinitely often is not recursive. Several other theorems of a similar nature hold. (Received March 11, 1953.)

418t. Alfred Tarski: *Two general theorems on undefinability and undecidability.*

This paper contains a generalization of ideas known from works of Gödel and other authors. See specifically Mostowski, *Sentences undecidable* · · · , Amsterdam, 1952; R. M. Robinson, Proceedings of the International Congress of Mathematicians, 1950, vol. 1; Tarski, Studia Philosophica vol. 1. Assumptions: $\mathfrak{T}$ is any formalized theory; S is a set of $\mathfrak{T}$-formulas including all axioms of predicate calculus with identity and closed under rules of inference; $\Delta_0, \Delta_1, \cdots, \Delta_n, \cdots$ are $\mathfrak{T}$-terms containing no variables; $\sim(\Delta_0=\Delta_1)$ is in S; x, y are fixed $\mathfrak{T}$-variables. A function F on and to the integers is called S-definable (relative to Δ_n) if, for some formula Φ and every integer n, $(x=\Delta_n)\rightarrow[(y=\Delta_{F(n)}\leftrightarrow\Phi]$ is in S. Consider a quite arbitrary one-one correlation between $\mathfrak{T}$-expressions Ψ and integers n; $Nr\,(\Psi)$ is the integer correlated with Ψ, Ω_n is the expression correlated with n. Let $D(n)=Nr[(x=\Delta_n)\rightarrow\Omega_n]$; let $P(n)$ be 0 if Ω_n is in S, and 1 otherwise. Theorem I. *If S is consistent, then functions D and P*

are not both S-definable. New assumptions: $G(n)=Nr(\Delta_n)$ and $H(n, p)=Nr(\Omega_n{}^\frown\Omega_p)$ (where ${}^\frown$ is the concatenation symbol) are general recursive functions. Then Theorem I implies: Theorem II. *If all general recursive functions are S-definable, then S is inconsistent or essentially undecidable.* (Received January 16, 1953.)

STATISTICS AND PROBABILITY

419t. G. B. Kallianpur and Herbert Robbins: *Ergodic property of Brownian motion process.*

Let $X(t)$ denote the linear Brownian motion (Wiener) process, let $f(x)$, $g(x)$ be bounded and summable for $-\infty<x<\infty$, and set $\bar{f}=\int f(x)dx$, $\bar{g}=\int g(x)dx$. Theorem: if $\bar{f}\neq 0$ then, for every u, $\lim_{T\to\infty} Pr\left[(1/\bar{f}T^{1/2})\int_0^T f(X(t))dt\leq u\right]=G(u)$, where $G(u)=(2/\pi)^{1/2}\int_0^u e^{-y^2/2}dy$ for $u\geqq 0$, $=0$ for $u<0$, if $\bar{g}\neq 0$, $\lim_{T\to\infty}\int_0^T f(X(t))dt/\int_0^T g(X(t))dt=\bar{f}/\bar{g}$ in probability. The same theorem holds for two-dimensional Brownian motion $X(t)=(X_1(t), X_2(t))$ with independent components, with the factor $1/\bar{f}T^{1/2}$ replaced by $2\pi/\bar{f}\log T$ and with $G(u)=1-e^{-u}$ for $u\geqq 0$, $=0$ for $u<0$. (Received April 15, 1953.)

420. Eugene Lukacs: *On strongly continuous stochastic processes.*

A condition is derived which assures that the increments of a stochastic process are normally distributed. It is assumed that the increments over non-overlapping time intervals are completely independent and it is shown that the normality of the increments follows from a certain continuity property. A characterization of the Wiener process in terms of this continuity property is given. (Received March 9, 1953.)

TOPOLOGY

421. R. D. Anderson: *Continuous collections of continuous curves.*

By a modification of the argument used by the author in his paper *Monotone interior dimension-raising mappings* (Duke Math. J. vol. 19 (1952) pp. 359–366) it can be shown that there exists a continuous collection G of mutually exclusive compact nondegenerate continuous curves filling up a one-dimensional compact continuum such that G with respect to its elements as points is homeomorphic to the compact Hilbert cube H. As any compact metric closed point set can be imbedded in H, it follows immediately that for any compact metric closed point set K there exist a one-dimensional compact closed point set M_K and a monotone interior mapping f of M_K onto K such that for each point k of K, $f^{-1}(k)$ is locally connected. (Received March 23, 1953.)

422. J. D. Baum: *An equicontinuity condition for transformation groups.*

The following necessary and sufficient condition for equicontinuity in transformation groups (dynamical systems) is proved: Let (X, T) be a transformation group, X compact and T_2 and minimal under T, and T abelian. Then T is equicontinuous if and only if there exists $x\in X$ such that for every continuous mapping f of X into the reals, the function $f_x(t)=f(xt)$ $(t\in T)$ is almost periodic in the left uniform functional transformation group over T to R, the reals. This result is a generalization of an unpublished theorem of Kakutani's and is proved using methods due to Gottschalk and Hedlund. Particular use is made of the equivalence of almost periodicity and equicontinuity proved by Gottschalk (Bull. Amer. Math. Soc. vol. 52 (1946) p. 633). (Received February 5, 1953.)

423. Morton Brown: *A countable connected Hausdorff space.*

The points are the positive integers. Neighborhoods are sets of integers $\{a+bx\}$, where a and b are relatively prime to each other ($x=1, 2, 3, \cdots$). Let $\{a+bx\}$ and $\{c+dx\}$ be two neighborhoods. It is shown that bd is a limit point of both neighborhoods. Thus, the closures of any two neighborhoods have a nonvoid intersection. This is a sufficient condition that a space be connected. (Received March 12, 1953.)

424. Haskell Cohen: *Cocantorian manifolds.* Preliminary report.

A locally compact Hausdorff space is an n-cocantorian manifold if its codimension (Bull. Amer. Math. Soc. Abstract 58-1-70) is n and it cannot be disconnected by a closed subset of codimension less than $n-1$. After several preliminary lemmas it is shown that every locally compact Hausdorff space of codimension n has a compact n-cocantorian manifold. (Received March 11, 1953.)

425*t*. M. L. Curtis: *Monotone deformation-free mappings.*

If B is an open subset of the n-sphere S^n and B is uniformly locally connected in terms of Čech cycles with integer coefficients and also in terms of Čech cocyles with reals (mod 1) as coefficients in dimensions $0, \cdots, n$, then $A=S^n-B$ is uniformly locally connected in terms of Čech cycles with integer coefficients in the same dimensions. If M is a continuum separating S^n, and M is deformable into a domain A of S^n-M with a deformation $h: M\times I\rightarrow \bar{A}$ such that $h(M\times t)\cap M=0$ for $t>0$, then $B=S^n-A$ satisfies the hypothesis of the above result so that A is uniformly locally connected in terms of integral Čech cycles. If X and Y are connected and locally connected by arcs, Y is simply connected and $f: X\rightarrow Y$ is monotone and onto, then the induced map $f_*: \pi_1(X)\rightarrow\pi_1(Y)$ is onto. These three lemmas are combined to show that if the deformation of M is monotone and M is locally simply connected, then A is ULC^n (=uniformly locally connected in the homotopy sense in dimensions $0, \cdots, n$). (Received March 11, 1953.)

426. M. L. Curtis: *Universal spaces for fibre spaces.* Preliminary report.

A fibre space consists of spaces E, B and a map p of E onto B such that the covering homotopy theorem is satisfied for arbitrary topological spaces. Following a definition given by R. Thom, one says that two fibre spaces (E_1, B_1, p_1) and (E_2, B_2, p_2) are fibre homotopy equivalent if $B_1=B_2$ and there exist fibre mappings $\alpha: E_1\rightarrow E_2$ and $\beta: E_2\rightarrow E_1$ such that $\alpha\beta\simeq 1$ and $\beta\alpha\simeq 1$, all maps and homotopies inducing the identity on the base. Given a topological space B and a point x in B, a class $C(B, x)$ of fibre spaces is defined. (E, K, p) belongs to $C(B, x)$ if: (1) K is covered by a collection $\{V_\nu\}$ of open sets such that for each V_ν a definite fibre homotopy equivalence $\phi_\nu: p^{-1}(V_\nu)\rightarrow V_\nu\times\Omega$ is given, where Ω is the space of loops at x in B. (2) on $V_\nu\cap V_\mu$, ϕ_ν and ϕ_μ are fibre homotopic. If B is locally contractible and E_ω is the space of paths from x in B, then (E_ω, B, p) is a universal space for fibre spaces in $C(B, x)$ which have a finite complex as base space. The classification theorem also holds for this class of spaces. Principal fibre bundles with compact connected Lie group G as fibre belong to $C(B_G, x)$ where B_G is the base of the contractible universal bundle for G. This leads to the result that such principal bundles are equivalent if and only if they are fibre homotopy equivalent. (Received March 11, 1953.)

427. Robert Ellis: *Continuity and homeomorphism groups.*

Let X be a T_2-space, let Φ be a group of homeomorphisms of X onto X, let $\pi: X\times\Phi\rightarrow X$ be the map such that $(x, \phi)\pi = x\phi$ $(x\in X, \phi\in\Phi)$, and for each $x\in X$ let $\pi_x: \Phi\rightarrow X$ be the map such that $\phi\pi_x = (x, \phi)\pi = x\phi$. The group Φ is *rigid* provided that if $\mathcal{J}$ is a filter base on Φ and $x, y, z\in X$ are such that $x\mathcal{J}\rightarrow z$ and $y\mathcal{J}\rightarrow z$, then $x = y$. The following two theorems and related results are proved. (1) Let X be locally compact, let Φ be abelian and rigid, let $\mathcal{T}$ be a first countable topology on Φ which makes each $\pi_x (x\in X)$ continuous, and let $x_0\in X$ be such that the closure of $x_0\Phi$ is X. Then π is continuous at (x_0, ϕ) for all $\phi\in\Phi$. (2) Let X be a minimal orbit closure under Φ, let X be first countable compact, let Φ be abelian, let each $\pi_x (x\in X)$ be one-to-one, and let each $\pi_x^{-1}\pi_y$ $(x, y\in X)$ be homeomorphic. Then Φ is equicontinuous. (Received March 11, 1953.)

428. Mary E. Estill: *A biconnected set having no widely connected subset.*

In footnote number 8 of *The closure of types of connected sets* which appears in the Proceedings of the American Mathematical Society vol. 2 (1951) pp. 178–185, P. M. Swingle raises a question raised with him by E. W. Miller. The question he intended to raise was that of the existence of a biconnected set in the plane without a dispersion point such that no subset of it is widely connected. There is such a set and an example is given. (Received March 12, 1953.)

429*t*. F. A. Ficken: *Closed linear spaces of continuous functions orthogonal to closed subsets of a completely regular space.* Preliminary report.

Let X be a completely regular topology space. Let Y be a complex (or real) linear space of complex- (or real-) valued functions $y(x)$ continuous on X, let Y contain all bounded continuous $y(x)$, and let Y be so topologized that the operations of addition and multiplication by scalars are continuous. For each closed $F\subset X$ (or closed linear $L\subset Y$) define $F^\perp$ (or $L^\perp$) to consist of those $y\in Y$ (or $x\in X$) such that $y(x) = 0$ for all $x\in F$ (or $y(x) = 0$ for all $y\in L$). Let $\mathcal{J}$ (or $\mathcal{L}$) denote the complete lattice of closed subsets F of X (or closed linear subspaces L of Y). The correspondence $F\rightarrow F^\perp$ maps $\mathcal{J}$ into $\mathcal{L}$, and $F^{\perp\perp} = F$. The correspondence $L\rightarrow L^\perp$ maps $\mathcal{L}$ into $\mathcal{J}$, but L may be properly contained in $L^{\perp\perp}$. The correspondence $L\rightarrow L^{\perp\perp}$ is a closure operation on $\mathcal{L}$. Let $\mathcal{P}$ denote the complete lattice consisting of those $L\in\mathcal{L}$ for which $L^{\perp\perp} = L$. Then $F\rightarrow F^\perp$ maps $\mathcal{J}$ into $\mathcal{P}$. The principal result of this note is that $F\leftrightarrow F^\perp$ is a lattice anti-isomorphism between $\mathcal{J}$ and $\mathcal{P}$. (Received March 9, 1953.)

430*t*. E. E. Floyd: *A theorem on maps from an n-sphere to n-space.*

Let f be a continuous map from an n-sphere S to n-space E. If x and y are points of S, it is proved that there exists a rotation r of S with $f(rx) = f(ry)$. This result is an extension of a well known theorem of Borsuk concerning antipodal points. (Received March 12, 1953.)

431. E. E. Floyd: *Orbit spaces of finite transformation groups.* I.

Let X be a finite simplicial complex, and let G be a finite transformation group operating simplicially on X. Let Y denote the space of orbits of X. It is proved that if X is either homologically trivial over the integers or an absolute retract, then so also is Y. (Received March 12, 1953.)

432. S. I. Goldberg: *Representation theory of Lie algebras.* Preliminary report.

In the paper (E) *Extensions of Lie algebras and the third cohomology group* (to appear in the Canadian Journal of Mathematics) the author developed certain aspects of representation theory and showed that the extension problem under consideration was susceptible to treatment by the cohomology theory of Lie algebras. Certain other notions in (E) are rephrased in the language of cohomology thereby further indicating the close contact between cohomology and representation theory. The following result is proved: "Let L be a Lie algebra over an arbitrary field. An extension $U=(L, V, W; \beta)$ of the L-module W by the L-module V determines a unique 1-cohomology class of L in the L-module Hom (V, W). Conversely, every 1-cohomology class of L in Hom (V, W) determines an extension U up to L-isomorphism leaving W elementwise fixed." The factor set β is actually an element of Z^1 $(L,$ Hom $(V, W))$. A cup product is defined and it is then conjectured that H^r $(L,$ Hom $(V, W))$ $\cong H^{r+1}(L, W)$. (Received March 9, 1953.)

433*t*. J. S. Griffin, Jr.: *Fibre spaces with totally disconnected fibres.*

Let $\{X, B; \pi\}$ be a fibre space in the sense of Hu (Proc. Amer. Math. Soc. vol. 1 (1950) p. 756) such that if $b\in B$ then $\pi^{-1}(b)$ is totally disconnected. Then if Y and A are topological spaces and p: $Y\rightarrow A$ is continuous onto, for any continuous functions f: $Y\rightarrow X$ and g: $A\rightarrow B$ such that $\pi f=gp$, each homotopy N of g may be uniquely covered by a homotopy K of f, i.e., there is exactly one homotopy K of f such that $\pi K=Np$. Immediate consequences of this lemma are first and second covering homotopy theorems for $\{X, B, \pi\}$, with no hypotheses on the domains of the functions being deformed, together with the uniqueness of these homotopies. If C is a topological space which is simply connected and locally arcwise connected and f: $C\rightarrow B$ is continuous, then for each $c\in C$ and $x\in\pi^{-1}f(c)$ there is a continuous function g: $C\rightarrow X$ such that $g(c)=x$ and $\pi g=f$; if C is connected this function is unique. Thus if B is simply connected and locally arcwise connected, $\{X, B, \pi\}$ has a cross section. (Received February 10, 1953.)

434. O. G. Harrold, Jr.: *The enclosing of simple arcs by simple closed surfaces in three-space.*

The set K in 3-space is called locally polyhedral at p if some neighborhood of p meets K in the null set or a finite polyhedron. The arc J is said to have property $\mathfrak{P}$ provided that to each point x of J and positive ϵ there is a set K in 3-space such that (i) K is a topological 2-sphere, (ii) x lies in the interior of K, (iii) diameter of K is less than ϵ, (iv) cardinal $K\cap J=$ order (Menger-Urysohn) of x in J, and (v) K is locally polyhedral at each point of the complement of J. Theorem I. If the arc J has Property $\mathfrak{P}$ and ϵ is positive, there is a polyhedral 2-sphere K whose interior contains J and which lies in the ϵ-neighborhood of J. Theorem II. The complement of J is an open 3-cell. (Received March 10, 1953.)

435*t*. V. L. Klee, Jr.: *A remark on compact convex sets.*

By using a theorem of Keller [Math. Ann. vol. 105 (1931) pp. 748–758], the following result is established. Theorem: Suppose K is an infinite-dimensional compact convex subset of a linear topological space E. Then each of the following implies that K is homeomorphic with the Hilbert parallelotope: (a) there is a countable set F of continuous linear functionals on E such that for any two distinct points x and y

of K there is an $f\in F$ for which $f(x)\neq f(y)$; (b) E is a normed linear space; (c) K is separable and E is a normed linear space in its weak topology. The result (b) solves the problem proposed in the author's *Convex bodies and periodic homeomorphisms in Hilbert space* [Trans. Amer. Math. Soc. vol. 74 (1953) pp. 10–43]. (Received February 23, 1953.)

436. E. E. Moise: *Affine structures in 3-manifolds.* VIII. *Invariance of the classical knot-types.*

Let P and P' be polygons in 3-space, and let f be a homeomorphism of 3-space onto itself, preserving orientation, and throwing P onto P'. It is shown that the knot-types of P and P' must then be the same. The basic definitions of knot-theory can therefore be generalized as follows: A knot is a tamely imbedded simple closed curve J. The knot-type of such a J is the set of all curves J' for which there is an orientation-preserving homeomorphism throwing 3-space onto itself and throwing J onto J'. The above theorem shows that the classical knot-types are subsets of the classical knot-types. The methods of proof are extensions of those used in *Affine structures in 3-manifolds.* V. *The triangulation theorem and Hauptvermutung* (Ann. of Math. vol. 56 (1952) pp. 96–114). (Received March 30, 1953.)

437. J. H. Roberts: *A topological mapping of the rational points of Hilbert space into the plane.*

An affirmative answer is given to the following question, raised by P. Erdös: Is the set of rational points of Hilbert space (a 1-dimensional set) topologically equivalent to a subset of the Euclidean plane? The proof is dependent upon a particular homeomorphism of Hilbert space into a subset of the Hilbert cube, under which, for any point, the number of irrational coordinates is fixed or else increases by one. The following more general theorem is proved: If M_k is the set of points of Hilbert space each of which has at most k irrational coordinates (M_k is $(k+1)$-dimensional), then M_k is topologically equivalent to a subset of E_{2k+2}. (Received April 6, 1953.)

438. J. W. Siry: *Chromatic polynomials of large maps.*

Birkhoff and Lewis (Trans. Amer. Math. Soc. vol. 60 (1946) pp. 355–451) stated a strong form of the four-color conjecture in terms of conditions on chromatic polynomials and established, by determining the polynomials involved, that these conditions are satisfied for certain proper maps which are regular in the sense of Birkhoff (Amer. J. Math. vol. 35 (1912) pp. 115–128). In a program to extend this study to include the truncated icosahedron, Rudebock (Thesis, Maryland, 1948, and further unpublished results) and Hall (results as yet unpublished) obtained polynomials for numerous maps having 25 or fewer regions. Further progress has been greatly facilitated by the method of the present paper for determining chromatic polynomials of large maps. The method is based upon an analysis of the structure of sets of maps by means of a certain class of reflexive, symmetric binary relations and a related class of equivalence relations. When applied to a class, M, of maps each having a certain proper 9-ring and a homeomorph invariant under a transformation group isomorphic with the symmetric group of degree 3, the method yields an expression for each $P_n(\lambda)\in M$ in terms of a set of thirty-six $P_{n-k}(\lambda)$, where $k\geq 6$. This brings within range the determination of the chromatic polynomial of the truncated icosahedron itself. (Received March 14, 1953.)

439. R. L. Wilder and J. P. Roth (p): *On certain inequalities relating the Betti numbers of a manifold and its subsets.*

Let M be a closed subset of a compact space S. Then, with a field as coefficient group, the homology group $H^r(M)$ is isomorphic to the direct sum of groups $H^r_0(S)$ and $H^{r+1}_1(S, M)$, subgroups of $H^r(S)$ and $H^{r+1}(S, M)$, respectively. Thus $p^r(M) \leqq p^r(S) + p^{r+1}(S, M)$. In particular, if S is an orientable n-gcm, then $p^r(M) \leqq p^r(S) + p^{n-r-1}(S-M)$, a relation which for $r=n-1$ was recently obtained by P. A. White [Proc. Amer. Math. Soc. vol. 3 (1952) pp. 488–498, Theorem 4]. Or, if U is an open subset, with boundary M, of an orientable n-gcm, then $p^r(M) \leqq p^r(\overline{U}) + p^{n-r-1}(U)$; so that if U is ulc^{n-r-1}, then $p^r(M) \leqq p^r(\overline{U}) + p^{n-r-1}(\overline{U})$, which is a generalization of a relation due to M. Morse [Proc. Nat. Acad. Sci. U.S.A. vol. 38 (1952) pp. 247–258]. Another inequality of Morse (loc. cit.) generalizes to the form $p^{n-r-1}(S-M) \leqq p^{n-r-1}(S) + p^r(M)$, where again S is an orientable n-gcm, M a closed subset of S. A third relation (Morse, loc. cit.) will be treated in a subsequent paper. (Received March 10, 1953.)

440. Hidehiko Yamabe: *On the conjecture of Iwasawa and Gleason.*

Let G be a connected locally compact group, and U be a compact neighborhood of the identity element. Suppose that a sequence of compact sets $\{D_n\}$ satisfies the following properties: (i) $\{D_n\}$ converges to a compact subgroup contained in U. (ii) $D_n^n \subset U$, $D_n^{n+1} \not\subset U$. (iii) For all rational numbers between 0 and 1, $\lim D_n^{[sn]} = D(s)$ does exist. Moreover $D(1) - \bigcup_{\mu<1} D(\mu) \neq \emptyset$. Then we can find a sequence $\{\theta_n\}$ of functions in $L_2(G)$, and a sequence $\{x_n\}$ of elements with $x_n \in D_n$, such that $\{n(x_n\theta_n - \theta_n)\}$ converges weakly to a function $\tau \neq 0$ in $L_2(G)$. Here n has only to range over a suitable subsequence of natural numbers. By making full use of above statement we can prove the following theorem: a locally compact connected group with no small normal subgroup is a Lie group. This gives an affirmative answer to the conjecture of Iwasawa and Gleason. The above theorem is known to be equivalent to the theorem: every locally compact group is a generalized Lie group. (Received March 26, 1953.)

L. W. COHEN,
Associate Secretary

THE APRIL MEETING IN CHICAGO

The four hundred ninety-first meeting of the American Mathematical Society was held at the University of Chicago on Friday and Saturday, April 24–25, 1953. Approximately 230 persons attended including the following 191 members of the Society:

Brian Abrahamson, W. R. Allen, W. F. Ames, Louis Auslander, D. G. Austin, J. H. Bell, Gerald Berman, R. H. Bing, R. L. Blair, G. M. Bloom, R. P. Boas, W. M. Boothby, Raoul Bott, Joseph Bram, G. U. Brauer, C. C. Braunschweiger, R. H. Bruck, R. C. Buck, R. H. Cameron, Buchanan Cargal, K. H. Carlson, W. B. Caton, S. S. Chern, E. W. Chittenden, H. M. Clark, M. D. Clement, Harvey Cohn, C. W. Curtis, H. J. Curtis, John DeCicco, J. C. E. Dekker, R. F. Deniston, W. E. Deskins, R. J. DeVogelaere, V. E. Dietrich, J. A. Dieudonné, N. J. Divinsky, J. L. Doob, L. A. Dragonette, R. J. Driscoll, Benjamin Epstein, H. P. Evans, R. L. Evans, H. S. Everett, Walter Feit, Chester Feldman, Jacob Feldman, J. S. Frame, Evelyn Frank, W. G. Franzen, C. G. Fry, L. E. Fuller, Leonard Gillman, Seymour Ginsburg, Casper Goffman, H. E. Goheen, Michael Golomb, A. W. Goodman, S. H. Gould, L. M. Graves, R. L. Graves, Harold Greenspan, L. W. Griffiths, V. G. Grove, William Gustin, M. M. Gutterman, Franklin Haimo, Marshall Hall, Jr., P. R. Halmos, Camilla Hayden, R. G. Helsel, Melvin Henriksen, I. N. Herstein, Fritz Herzog, J. J. L. Hinrichsen, J. G. Hocking, R. E. von Holdt, S. S. Holland, T. C. Holyoke, B. E. Howard, S. P. Hughart, H. K. Hughes, Ralph Hull, W. E. Jenner, Meyer Jerison, H. T. Jones, K. E. Kain, Samuel Kaplan, Irving Kaplansky, N. D. Kazarinoff, J. L. Kelley, R. B. Kellogg, J. B. Kelly, L. M. Kelly, J. H. B. Kemperman, Erwin Kleinfeld, Fulton Koehler, Jacob Korevaar, W. C. Krathwohl, A. H. Kruse, M. Z. Krzywoblocki, R. G. Kuller, H. G. Landau, C. G. Latimer, G. F. Leger, Jr., G. R. Lehner, B. L. Lercher, W. J. LeVeque, W. D. Lindstrom, A. J. Lohwater, R. D. Lowe, Richard McKinney, Saunders MacLane, Arne Magnus, E. A. Michael, D. W. Miller, Marian A. Moore, M. D. Morley, S. T. C. Moy, S. B. Myers, Isaac Namioka, August Newlander, Jr., E. A. Nordhaus, R. Z. Norman, Rufus Oldenburger, C. C. Oursler, Gordan Pall, M. H. Payne, M. H. Pearl, Sam Perlis, G. B. Price, A. L. Putnam, C. R. Putnam, Gustave Rabson, O. W. Rechard, W. T. Reid, Haim Reingold, R. B. Reisel, R. W. Rempfer, Daniel Resch, R. K. Ritt, Alex Rosenberg, M. A. Rosenlicht, Arthur Rosenthal, E. H. Rothe, J. M. Sachs, D. E. Sanderson, R. G. Sanger, R. L. San Soucie, A. C. Schaeffer, O. F. G. Schilling, Morris Schreiber, W. R. Scott, W. T. Scott, I. E. Segal, Esther Seiden, M. A. Seybold, D. H. Shaftman, M. E. Shanks, K. S. Shih, Annette Sinclair, Abe Sklar, M. F. Smiley, K. T. Smith, Jerome Spanier, E. J. Specht, George Springer, H. E. Stelson, B. M. Stewart, Fred Supnick, T. T. Tanimoto, J. S. Thale, H. P. Thielman, E. A. Trabant, R. N. Van Norton, Andrew Van Tuyl, Bernard Vinograde, R. D. Wagner, G. L. Walker, L. M. Weiner, P. J. Wells, George Whaples, L. R. Wilcox, F. B. Wright, Jr., F. M. Wright, Oswald Wyler, L. C. Young, P. M. Young, J. W. T. Youngs, Daniel Zelinsky, Antoni Zygmund.

By invitation of the Committee to Select Hour Speakers for Western Sectional Meetings, Professor J. A. Dieudonné of the University of Michigan addressed the Society on the topic *Recent developments in the theory of topological vector spaces*. Professor P. R. Halmos presided at this session.

There was a total of seven sessions for the presentation of contributed papers; these were held on Friday and Saturday morning and on Friday afternoon. Presiding officers were: Professors R. C. Buck, S. S. Chern, G. B. Price, E. H. Rothe, A. C. Schaeffer, M. F. Smiley, and G. L. Walker.

The local department entertained the Society with a tea held in the Common Room of Eckhart Hall on Friday afternoon.

Abstracts of the papers read follow. Those having the letter "*t*" after their numbers were read by title. Paper number 445 was presented by Dr. Henriksen, number 459 by Professor Smith, number 462 by Mr. Fagen, number 467 by Dr. Rabson, and number 490 by Mr. Norman. Mr. Mostert was introduced by Professor M. E. Shanks and Mr. Brown by Professor P. C. Rosenbloom.

ALGEBRA AND THEORY OF NUMBERS

441*t*. C. G. Chehata: *Commutative extension of partial automorphisms of groups.*

Let ϵ be an isomorphism which maps a subgroup A of the group G onto a second subgroup B (not necessarily distinct from A) of G; then ϵ is called a "partial automorphism" of G. A partial (or total) automorphism ϵ^* "extends" or "continues" a partial automorphism ϵ if ϵ^* is defined for, at least, all those elements for which ϵ is defined and moreover ϵ^* coincides with ϵ where ϵ is defined. It is known (G. Higman, B. H. Neumann, and Hanna Neumann, J. London Math. Soc. vol. 24 (1949)) that any partial automorphism of a group can always be extended to a total automorphism of a supergroup, and even to an inner automorphism of a supergroup. Moreover any number of partial automorphisms can be simultaneously extended to inner automorphisms of one and the same supergroup. In this paper sufficient conditions are derived which ensure that two partial automorphisms can be extended to commutative automorphisms of a supergroup. Although these conditions are too restrictive to be necessary as well, they are sufficiently wide to give the following special case as a corollary: If ϵ maps $A\subseteq G$ isomorphically onto $B\subseteq G$ and if ν maps $C\subseteq G$ isomorphically onto $D\subseteq G$, and if $A\cap C=B\cap C=A\cap D=B\cap D=\{1\}$, then ϵ and ν can be extended to commutative automorphisms of a supergroup. (Received February 18, 1953.)

442*t*. C. G. Chehata: *Simultaneous extension of partial endomorphisms of groups.*

Let ϵ be a homomorphic mapping of some subgroup A of the group G onto a subgroup B (not necessarily distinct from A) of G; then we call ϵ a "partial endomorphism" of G. A partial (or total) endomorphism ϵ^* "extends" or "continues" a partial endomorphism ϵ if ϵ^* is defined for at least all those elements for which ϵ is defined and moreover ϵ^* coincides with ϵ where ϵ is defined. In a paper by B. H. Neumann and Hanna Neumann (Proc. London Math. Soc. (3) vol. 2 (1952)) necessary and sufficient conditions are given for the extension of a partial endomorphism of G to a total endomorphism of a supergroup of G. In this paper these conditions are generalized to apply first to the simultaneous extension of two partial endomorphisms of G to total endomorphisms of one and the same supergroup of G, and then, using transfinite induction, to any well-ordered set of partial endomorphisms. The conditions here obtained are again necessary and sufficient. A number of special consequences are then derived. (Received February 18, 1953.)

443. Harvey Cohn: *Approach to Markoff's minimal forms through modular functions*. Preliminary report.

The author considers the subgroup of the modular group generated by P: $-1/(s+3)$, Q: $(2s-1)/(-s+1)$. Its fundamental domain is of genus 1, uniformized by $J(s)=[\wp'(u)]^2-4\wp^3(u)+1$. Markoff's forms correspond to the fixed points of the transformations P, Q, PQ, P^2Q, etc., generated as follows: Start with $[P, Q]$, and obtain from it four new pairs $[PQ, P]$, $[PQ, Q]$, $[QP, P]$, $[QP, Q]$. The transformations in question are found by repeating the process indefinitely. (Research sponsored by the Army Office of Ordnance Research.) (Received March 9, 1953.)

444*t*. W. F. Eberlein: *The inverse Fourier transform*.

Let G be a locally compact Abelian group. The Fourier transformation $F(f)=g$, where $g(s)=\int\sigma(x, s)f(x)dx$, maps $L^1(G)$ into a dense subspace of $C(G^*)$, the continuous functions on the dual group G^* vanishing at infinity. General operator theoretic methods introduced by the author in an earlier paper [Ann. of Math. (2) vol. 47 (1946) pp. 688–703] are applied to a study of the inverse transformation F^{-1}. In particular, F^{-1} is lower semi-continuous, whence F admits a natural extension defined on a space of measures on G. A by-product of the analysis is a simple proof of Segal's theorem [Acta Sci. Math. Szeged vol. 12, Part B (1950) pp. 157–161] that $F[L'(G)]=C(G^*)$ if and only if G is finite. (Received March 11, 1953.)

445. Leonard Gillman and Melvin Henriksen: *On the prime ideals of rings of continuous functions*. I.

Let $C=C(X, R)$ denote the ring of all real-valued continuous functions on a completely regular space X, $Z(f)$ the set of zeros of $f\in C$; an ideal I of C is *fixed* if $\bigcap_{f\in I}Z(f)$ is nonempty, otherwise I is *free* (Hewitt, Trans. Amer. Math. Soc. vol. 64 (1948) pp. 45–99). (1) Every prime fixed ideal of C is contained in a unique maximal ideal (the corresponding statement for free ideals holds if X is normal). (2) The subring of all bounded $f\in C$ always has nonmaximal prime ideals, unless X is finite. Denote by M_p the maximal fixed ideal consisting of all $f\in C$ such that $f(p)=0$; if M_p contains no prime ideal other than itself, then p is called a *P-point* of X. (3) p is a P-point if and only if every continuous function vanishing at p vanishes on a neighborhood of p. (4) p is a P-point if and only if every countable intersection of neighborhoods of p contains a neighborhood of p. (5) There exist spaces X without isolated points in which both the set of all P-points, and its complementary set, are dense in X. (Received April 23, 1953.)

446*t*. Leonard Gillman and Melvin Henriksen: *On the prime ideals of rings of continuous functions*. II.

Terminology is as in preceding abstract. If every point of (the completely regular space) X is a P-point, then X is called a *P-space*. (1) The following statements are equivalent. (i) X is a P-space. (ii) Every $Z(f)$ is open and closed. (iii) $C(X, R)$ is a regular ring. (iv) Every prime ideal (free or fixed) of $C(X, R)$ is maximal. (v) Every countable intersection of open sets of X is open, i.e., X is ω_1-additive (Sikorski, Fund. Math. vol. 37 (1950) pp. 125–126). (vi) Every ideal of $C(X, R)$ is $\mathfrak{m}$-closed (for definition see Hewitt, loc. cit. p. 49). (vii) Every prime fixed ideal of $C(X, R)$ is $\mathfrak{m}$-closed. (2) Finite products of P-spaces are P-spaces, but countable products need not be. (3) Every locally countably compact, or perfectly normal, or pseudo-compact (Hewitt, loc. cit. p. 67) P-space is discrete, as is every P-space satisfying the first axiom of countability. (4) Every P-space of power $\leq\aleph_1$ is normal. (5) For every $\mathfrak{m}\geq\aleph_1$ there is a normal P-space of power $\mathfrak{m}$ without isolated points. (6) For every $\mathfrak{n}\geq\aleph_1$ there is a non-normal P-space of power $\mathfrak{n}$ without isolated points. (Received April 23, 1953.)

447. Franklin Haimo: *Dihedral-type groups.* Preliminary report.

A generalization of the dihedral groups is defined; and, in a special case, its relation to the holomorph of a group is developed. An inverse limit of the set of dihedral groups can be constructed, a limit which turns out to be a centerless, 2-step, metabelian infinite group. All endomorphisms of a given dihedral group are enumerated, and the automorphisms are represented faithfully as a group of 2 by 2 matrices of integers. (Received March 11, 1953.)

448t. Marshall Hall, Jr.: *On a theorem of Jordan.*

In 1872 Jordan showed that a finite quadruply transitive group in which only the identity fixes four letters must be one of the groups S_4, S_5, A_6, or the Mathieu group on 11 letters. This theorem is generalized in two ways in this paper. Quadruply transitive groups are considered on any number of letters, finite or infinite, and instead of assuming that the subgroup fixing four letters is the identity alone, it is assumed that it is a finite group of odd order. The conclusion is essentially the same as Jordan's, there being only one other group satisfying this hypothesis, the alternating group A_7. (Received March 13, 1953.)

449. T. C. Holyoke: *Non-existence of certain transitive extensions.*

It is proved that the only dihedral groups (including the "dihedral" group with an infinite cyclic subgroup) possessing transitive extensions, when represented as permutations of cosets of a reflection subgroup, are those of orders 4, 6, and 10. The proof is based on a theorem given by the author in a previous paper (Amer. J. Math. vol. 74 (1952) pp. 787–796). (Received March 9, 1953.)

450. G. F. Leger, Jr.: *A note on derivations of Lie algebras.*

If L is a Lie algebra and V is an ideal of L, we say that L splits over V if there exists a subalgebra T of L so that $L = T + V$ and $T \cap V = (0)$. As is well known, derivations of L form a Lie algebra $D(L)$ and the inner derivations form an ideal $I(L)$ in $D(L)$. Let R denote the radical of L, $D(R)$ the derivations of R, and $I(R)$ the inner derivations of R. Theorem: If $D(R)$ splits over $I(R)$, then $D(L)$ splits over $I(L)$. An example is given of a nilpotent Lie algebra L such that $D(L)$ does not split over $I(L)$. (Received March 12, 1953.)

451. R. L. San Soucie: *Right alternative division rings of characteristic 2.*

L. A. Skornyakov (Izvestiya Akad. Nauk SSSR. Ser. Mat. vol. 15 (1951) pp. 177–184) has shown that right alternative division rings of characteristic not two are alternative. Fundamental to his proof is the lemma that such rings satisfy the identity

$(*)x(yz\cdot y)=(xy\cdot z)y$, for all x, y, z. R. H. Bruck has an example (unpublished) of a right alternative division ring, R, of characteristic two, which is not alternative. R, however, does not satisfy (*). In this paper, it is shown that a right alternative division ring of characteristic two is alternative if and only if it satisfies (*). Moreover, it is proved that, if R is a right alternative division ring of characteristic two having the right inverse property, then R satisfies (*), and, hence, is alternative. (Received March 10, 1953.)

452. W. R. Scott: *Some theorems on infinite groups.*

If $i(H)$ is the index of a subgroup H of a group G, then $i(\bigcap H_\alpha)\leqq \pi i(H_\alpha)$, $\alpha\in S$. This generalizes Poincaré's theorem. Other theorems on indices are given. It is shown that the set $L(\infty, H)$ of elements of G of infinite order with respect to H has order 0 or $o(G)$. A generalization is also given. The subgroup K of an infinite group G, as defined in W. R. Scott (*Groups and cardinal numbers*, Amer. J. Math. vol. 74 (1952) pp. 187–197) is shown to be overcharacteristic, i.e. $G/K\cong G/H$ implies $K\subseteq H$. Characterizations are obtained for those Abelian groups G all of whose subgroups H (factor groups G/H) of order equal to $o(G)$ are isomorphic to G. Again the Abelian groups, all of whose endomorphisms σ such that $o(G\sigma)=o(G)$ are onto, are found. (Received March 12, 1953.)

453. B. M. Stewart: *Sodd numbers.*

Let $\alpha(m)$ indicate the number of integers which can be represented as a *sum of distinct* (positive) *divisors* of the integer m. Define an even number m to be *sodd* if $\alpha(m)=\sigma(m)$; and define an odd number m to be *sodd* if $\alpha(m)=\sigma(m)-2$. In this paper theorems are given for generating new sodd numbers from known sodd numbers. In particular, the product of the first k odd primes is a sodd number for $k\geqq 5$; and these sodd numbers are used to solve Starke's variation of the Egyptian fraction problem, #4512, Amer. Math. Monthly. The *sodd-function*, defined by $s(m)=\alpha(m)/\sigma(m)$, is shown to be everywhere dense on the interval from 0 to 1. (Received February 24, 1953.)

454t. W. R. Utz: *A note on the Scholz-Brauer problem in addition chains.*

An addition chain for the positive integer n is a sequence of integers $1=a_0<a_1<a_2\cdots<a_r=n$ such that $a_i=a_j+a_s$ for $0\leqq j\leqq s<i\leqq r$. Let $l(n)$ denote the minimal value of r for which there is an addition chain for n. A. Scholz (Jber. Deutschen Math. Verein. vol. 47 (1937) p. 41) suggested that for all integers $q>0$, $l(2^q-1)\leqq q-1+l(q)$. A. T. Brauer (Bull. Amer. Math. Soc. vol. 45 (1939) pp. 736–739) established this inequality provided the $l(q)$ on the right in the inequality is the minimal length of chains in which $a_i=a_{i-1}+a_s$, $0\leqq s<i-1$. However, the problem as originally stated seems to be unsolved. The principal result of the present paper is the proof of the Scholz inequality for all q of the form $2^\alpha+2^\beta$, α and β being positive integers. (Received March 9, 1953.)

455. Bernard Vinograde: *Invariants of matrices under row- and column-constant addition.*

Over a division ring K of zero characteristic, the total matrix ring K_r maps isomorphically into the subring K_n^0 (of K_n, $n>r$) whose rows and columns individually add to zero, and the mapping is onto when $n=r+1$. The elements of K_r furnish the invariants of an equivalence relation in K_{r+1} defined by: $A\sim B$ if and only if $p(A-B)q$

$=0$ for all vectors p, q whose components sum to zero (hence $A-B$ is the sum of row-constant and column-constant elements of K_{r+1}). Suitably modified statements apply to the module of rectangular matrices. In particular, invariantively associated with each class is a pair of linear spaces and a pair of sets of parallel linear varieties by means of which can be established the identity $XAY-UAV=(X-U)B(Y-V)$, where: B is any matrix equivalent to A, $U(V)$ is a fixed column (row)-constant nonzero idempotent, and $X(Y)$ is a variable column(row)-constant nonzero idempotent. Over the real numbers, this is the same as an identity of H. W. Mills. It associates a support function invariantively with each class, a fact that intimately relates it to the theory of finite games, as will be explained elsewhere. (Received March 12, 1953.)

456. L. M. Weiner: *Algebras based on linear functions.*

Starting from an algebra A with products xy and a linear function f on the vector space of A to the base field F, an algebra A' is constructed over the same vector space as A by defining products in A' to be $x\cdot y=f(x)y+f(y)x+xy$. This type of algebra arises in the study of Lie admissible algebras. When A is associative, a necessary and sufficient condition that A' be associative is $f(x)f(y)=-f(xy)$. Assuming A to be power associative, it is necessary either that A be anticommutative or that f satisfy $f(x)f(y)=-[f(xy)+f(yx)]/2$ in order that A' be power associative. Either of these conditions is sufficient for the power associativity of A'. A necessary and sufficient condition that flexibility carry over from A to A' is $f(xy)=f(yx)$. In the case of a semi-simple Lie algebra A, f may be defined so that the resulting algebra is simple. In general, A' will be simple if A is anticommutative and every proper ideal of A contains at least one element x such that $f(x)\neq 0$. (Received March 6, 1953.)

457. George Whaples: *Generalized local class field theory.* II.

Let k be a regular local field (for terminology see Duke Math. J. vol. 19 (1952) pp. 505–517), $\bar{k}$ its residue class field, p the characteristic of $\bar{k}$, and $k^{\cdot}_{(i)}$ the group of elements congruent 1 mod π^i where π is a prime element. If h is any subgroup of $k^{\cdot}$ define $\bar{h}_{(i)}$ to be the additive group of residue classes $\bar{a}$ of elements $\alpha\in o$ for which $1+\alpha\pi^i\in(h\cap k^{\cdot}_{(i)})k^{\cdot}_{(i+1)}$ and call the $\bar{h}_{(i)}$ (which depend on choice of π) the image groups of h. Problem: Which subgroups of $\bar{k}^+$ can be image groups of finite abelian extensions of exponent p? Answer: All subgroups of $\bar{k}^+$ which are open in the additive polynomial topology (see Bull. Amer. Math. Soc. Abstract 59-1-83), provided of course that i is prime to p and in case k has characteristic 0, that $k^{\cdot}_{(i)}\not\subset k^{\cdot p}$. The case $i=1$ gives a complete description of all subgroups of conductor π^2 which are norm groups, thus solving problems raised by a counterexample of M. Moriya (J. Jap. Math. Soc. vol. 3 (1951) pp. 195–203). But a further counterexample shows that these results are not enough to settle the case of conductor π^3. (Received March 12, 1953.)

ANALYSIS

458*t*. J. W. Armstrong: *Point systems for Lagrange interpolation.*

S. Bernstein [Bull. Acad. Sci. URSS (1931) pp. 1025–1050] has shown that the Tchebichef abscissas have a Lebesgue function of uniform order $\log n$ for $|x|\leq 1$ and he demonstrates that new point systems with a Lebesgue function of uniform order $\log n$ can be obtained by "distorting" the Tchebichef abscissas in a particular manner. A new class of point systems including the Tchebichef abscissas is now obtained in the following manner. Let $\{w_n(x)\}$ be a sequence of polynomials with real distinct zeroes in $(-1, 1)$ and require these polynomials to have asymptotic extreme values, i.e., the maximum of $|w_n(x)|$ in $[-1, 1]$ remains less than a constant K (independent of n) times the smallest of the relative maxima of $|w_n(x)|$ between successive zeros including also the end points. The point system composed of these zeros is shown to have a Lebesgue function of uniform order $\log n$ for $|x|\leq 1$ and, furthermore, Bernstein's "distortion" theorem holds for any such point system obtained in the above manner. (Received February 6, 1953.)

459. Nachman Aronszajn and K. T. Smith: *Invariant subspaces of completely continuous operators.*

It is shown that every completely continuous linear operator on an arbitrary Banach space has closed proper invariant subspaces. (Received March 27, 1953.)

460. D. G. Austin: *A Lipschitzian characteristic of approximately derivable functions.*

Let m_nE denote the Lebesgue measure of a bounded set E in n-dimensional euclidean space and let $f(x)$ be a finite, single-valued, measurable function with domain E. The author uses the Egoroff Theorem to establish that if $f(x)$ possesses finite approximate partial derivates on E, then for any η, $\epsilon>0$ there exist numbers M, $\delta>0$ and a closed set $H\subset E$ with $m_nH>m_nE-\eta$ such that $s\in H$ implies $m_1[x_i;|(f(s_1,\cdots,s_{i-1},x_i,s_{i+1},\cdots,s_n)-f(s))/(x_i-s_i)|\leq M, s_i<x_i<s_i+\alpha]/\alpha>1-\epsilon$, for all α with $0<\alpha\leq\delta$. This result together with an extension theorem for Lipschitz functions implies that $f(x)$ has finite approximate partial derivates on E if and only if $f(x)$ is essentially Lipschitz on E. Certain classical theorems of Lebesgue, Denjoy, and Stepanoff are easy corollaries of this result. (Received March 9, 1953.)

461. Leon Brown: *The zeros of analytic functions in Banach spaces.*

This paper generalizes the Weierstrass Preparation Theorem (see Bochner and Martin P183). Consider a function f with range in $X\times C_1$ and domain in C_1, where X is a complex Banach space and C_1 is the space of complex numbers, which is analytic and bounded for $\|X\|\leq 1$, $|w|\leq 1$. Assuming that $f(0, w)$ has an s-fold zero at $w=0$, then $f=(w^s-\sum_{\nu=1}^{s-1} H_\nu(x)w^\nu)\Omega(x, w)=P\Omega$ where $H_\nu(x)$ is an analytic function on X to C_1 and Ω is a nonzero function of $X\times C_1$ to C_1, in a neighborhood of $(0, 0)$. The size of this neighborhood is estimated and P and Ω are represented as integrals of the function f. If one then considers the functions f as elements in a suitable Banach space, it is proved that P and Ω are analytically dependent on f. In proving this theorem the author arrives at a generalization of the Euclidian Algorithm. Given two functions f and g with domain $X\times C_1$ and range in C_1 analytic for $\|X\|\leq 1$, $w\leq 1$, and for every X_0 such that $\|X_0\|\leq 1$, $f(X_0, w)$ has S zeros in $|w|<1$, then there exist uniquely a polynomial P in w of degree $<S$ with coefficients analytic functions on X to C_1, and a $Q(x, w)$ analytic in the unit sphere such that $g=Qf+P$ and Q and P can be represented as integrals involving f and g. (Received March 9, 1953.)

462. R. H. Cameron and R. E. Fagen: *Nonlinear transformations of Volterra type in Wiener space.*

Let C be the space of continuous functions $x(t)$ on the interval $[0, 1]$ which vanish at $t=0$, and let Γ be a Wiener measurable subset of C. Let T be the transformation $y(t)=x(t)+\Lambda(x|t)$, and assume the (generally nonlinear) functional Λ is such that T takes Γ in a 1-to-1 manner into a subset $T\Gamma$ of C. Cameron and Martin have shown that if Λ satisfies certain smoothness conditions, $T\Gamma$ is Wiener measurable, and they have given an explicit formula for transforming Wiener integrals by the transformation T (Trans. Amer. Math. Soc. vol. 66 (1949) pp. 253–283). The present paper extends these results by weakening the smoothness conditions required of Λ. In particular, the kernel of the variation of Λ is permitted to have a jump along the diagonal. (Received March 17, 1953.)

463. Buchanan Cargal: *Approach properties of functions.*

Let X be a Hausdorff space, Y be a regular separable Hausdorff space, and f be a function on X into Y. Let λ be a set property defined for subsets of X, and S be a subset of X. Then a point ξ in X is a *point of* λ *f-approach by* S if for every neighborhood $M(f(\xi))$ of $f(\xi)$ and every neighborhood $N(\xi)$ of ξ, the set $E[x:f(x)\in M(f(\xi))]N(\xi)S$ has property λ. A property λ is an *ascending set property* if B has property λ whenever A has property λ and $A\subset B$. Several theorems are proved which reduce to well known theorems when λ is specialized to yield continuity. The following is a typical theorem: if (i) λ is an ascending set property, (ii) $\{f_n(x)\}$ converges uniformly to $F(x)$ on X, (iii) S is a set in X, (iv) there is a ξ in X such that for every n, ξ is a point of λf_n-approach by S, then ξ is a point of λF-approach by S. The concept λf-approach was introduced by H. Blumberg (*New properties of all real functions*, Trans. Amer. Math. Soc. vol. 24 (1922)). (Received March 2, 1953.)

464. R. L. Evans: *Solution of linear ordinary differential equations containing a parameter*. II.

The system (1) $y^{(n)}-\sum_{j=1}^{n}\lambda^{pj}P_j(\lambda, x)y^{(n-j)}=0$, (2) $y^{j}(\lambda, 0)=(j!)\sum_{\alpha=1}^{\infty} b_{\alpha,j}\lambda^{-\alpha}$ $(j=0, 1, \cdots, n-1)$, and (3) $P_j(\lambda, x)=\sum_{\mu=-pj}^{\infty}\sum_{\nu=[(p-\mu-1)/p]}^{\infty} a_{j,\mu,\nu}\lambda^{-\mu}x^{\nu}$ was solved, for bounded $(\lambda^p x)$, in part I (Proc. Amer. Math. Soc. (1953)). For solution in the large, combine this result with n independent solutions of (1) and (3) at unbounded $(\lambda^p x)$ by W. B. Ford's method (University of Michigan 1936) and its extensions. Neglecting special subcases, each independent solution for unbounded $(\lambda^p x)$ is such that, if $t^m=\lambda^p x$ and $m\geq$ l.c.m. $(1, 2, \cdots, j_0)$ (where $j_0=$ largest number of spaces under a single segment of the Puiseux diagram for $t=\infty$), then (4) $y(\lambda, x)=\exp(Q(\lambda, t))\cdot v(\lambda, t)$, where $Q(\lambda, t)=\sum_{k=0}^{\infty} q_k(t)/\lambda^k$ and $v(\lambda, t)=\sum_{k=0}^{\infty} v_k(t)/\lambda^k$. The q_k's and v_k's are determined in pairs $(k=0, 1, \cdots$, etc.). If s_0 is the Puiseux slope corresponding to the desired independent solution, $q_k(t)$ is that polynomial of degree $(s_0+1+[km/p])$ which converts the terminal Puiseux slope (for eq. for v_k) to -1 and which bounds the $\rho_{k,\sigma}$ growth (with k) as required below. Also, (5) $v_k(t)=\sum_{\sigma=0}^{k-1} u_{k,\sigma}(\log t)^{\sigma}+v_0(t)\sum_{\sigma=1}^{k} C_{k,\sigma}(\log t)^{\sigma}$ $(k=1, 2, \cdots)$, and (6) $u_{k,\sigma}\sim\sum_{\beta=0}^{\infty} A_{k,\sigma,\beta}t^{\rho_{k,\sigma}}t^{-\beta}$ where $\rho_{k,\sigma}<\rho_{k,0}\leq\rho_{0,0}+[km/p]$ $(\sigma=1, 2, \cdots, k-1)$ with proper $Q(\lambda, t)$. The C's and A's are constants determined by substitution of (4)–(6) in (1) and (3). The asymptotic series in (6) may be replaced by convergent factorial series by J. Horn's methods (Math. Zeit. (1924)) and their extensions. (Received March 9, 1953.)

465. Seymour Ginsburg: *Further results on order types and decompositions of sets*. I.

Let a and b be two order types for which $a<b$ (see Sierpiński, *Sur les types d'ordre des ensembles linéaires*, Fund. Math. vol. 37 (1950) pp. 253–264). If there exists an order type c such that $a<c<b$, then problem P, as applied to a and b, is said to admit of a solution c. Now let $\{d_e|e\in E\}$, where the power of E is $2^{\aleph_0}$, be a set of order types, each d_e being of power $2^{\aleph_0}$ and less than λ. The following two results are shown: (1) Problem P, as applied to $a=d_e$ and $b=\lambda$, admits of a solution c_e, such that the c_e are pairwise incomparable order types. (2) Problem P, as applied to $a=0$ and $b=d_e$,

admits of a solution c_a such that the c_a are pairwise incomparable order types. (Received March 2, 1953.)

466t. Seymour Ginsburg: *Further results on order types and decompositions of sets.* II.

All sets considered are linear and of power $2^{\aleph_0}$. A set E is called exact if the only similarity transformation of E into E is the identity. E is said to have property A if no two disjoint subsets are similar. The problem studied is the decomposition of a linear set into a family of disjoint sets, the order type of the sets being required to satisfy some specified conditions. Some of the results obtained are the following: (1) For each set E and each ordinal number $a \leqq \omega$, E is the union of a family $\{E_i \mid i<a\}$ of pairwise disjoint sets, the order type of the E_i being pairwise incomparable. (2) Let E contain a family $\{E_a \mid a \in G\}$ of disjoint, similar subsets. Then E is the union of a family $\{A_a \mid a \in G\}$ of disjoint, similar, exact sets, each having property A. (3) If $2^{\aleph_0} = \aleph_1$, then each set is the union of a family $\{E_n \mid n<\omega\}$ of disjoint, exact sets, each having property A. In addition, the order type of the E_n are pairwise incomparable. (Received March 2, 1953.)

467. Meyer Jerison and Gustave Rabson: *Fourier series on compact zero-dimensional groups.*

Let G be a compact zero-dimensional group. G is the limit of an inverse system of finite groups $\{G_\alpha\}$. Let π_α be the natural mapping of G onto G_α with kernel K_α. Define $\pi_\alpha^+\phi_\alpha(g)$ to be $\phi_\alpha(\pi_\alpha(g))$ and $\pi_\alpha^-\phi(g_\alpha)$ to be $\int_{K_\alpha}\phi(\pi_\alpha^{-1}(g_\alpha)k)dk$ where $g\in G$, $g_\alpha\in G_\alpha$, $\phi\in L^1$ on G, ϕ_α is a function on G_α and we choose any representative of $\pi_\alpha^{-1}(g^\alpha)$. It is proved that $\pi_\alpha^+\pi_\alpha^-\phi$ is a partial sum of the Fourier series of ϕ and that $\pi_\alpha^+\pi_\alpha^-\phi$ converges to ϕ in the mean, pointwise at a point of continuity, and uniformly if ϕ is continuous. If G is the countable product of groups of order two, considered as dyadic rationals, the theorem gives a slight strengthening of a theorem of Walsh which asserts that the 2^n partial sums of the Walsh-Fourier series of a continuous function on the unit interval converges uniformly to the function. (Received April 20, 1953.)

468. N. D. Kazarinoff: *Asymptotic forms for the Whittaker confluent hypergeometric functions when the parameter m is large.* Preliminary report.

In this paper forms are derived for the solutions of the differential equation $d^2w/dx^2+[-1/4+k/x+(1/4-m^2)/x^2]w=0$, which are valid for complex x and large complex m. Upon performing the transformations $s=\log(x/2im)$ and $u=e^{-s/2}w/2i$ of the independent and dependent variables, the differential equation becomes one of the class treated by R. E. Langer [Trans. Amer. Math. Soc. vol. 67 (1949) pp. 461–490]. Application to the theory in Langer's paper, generalized to include complex s, gives forms for the solutions of Whittaker's equation, customarily denoted by $M_{k,m}(x)$ and $W_{k,m}(x)$, which are asymptotic as to m and relative to sectors of the x-plane. The complex quantities x, k, and m are subject only to the restrictions that k be bounded and that $|m|$ be sufficiently large. (Received March 10, 1953.)

469. J. L. Kelley: *Averaging operators.*

Let $C_\infty(X)$ be the algebra of all those continuous real-valued functions on a locally compact Hausdorff space X which vanish at ∞, and let T be a bounded linear operator on $C_\infty(X)$. Break X into equivalence classes by agreeing x equiv. y if $T(f)(x) = T(f)(y)$ for all f. T is called averaging if for each equivalence class D, $T(f)$ is on D the average of the values of f on D (i.e. for some signed Baire measure m, for all $x \in D$ and all f, $T(f)(x) = \int f\,dm$ and $m(A) = 0$ if A is disjoint from D). Extending results of G. Birkhoff [Colloq. Int. Rech. Sci., no. 24, 1950, pp. 143–153] it is found that: 1. T is averaging if and only if $T(fT(g)) = T(f)T(g)$ for all f and g. 2. If T is idempotent and positive ($T(f) \geqq 0$ for $f \geqq 0$) and the range of T is a subalgebra, then T is averaging. 3. If X is a topological group, T commutes with right translation if and only if T is convolution on the left by a signed Baire measure m. If T commutes with right translation and is averaging, the measure m is $\pm$Haar measure on a compact subgroup of X. A theorem of Y. Kawada and P. R. Halmos' formulation of a result of Dieudonné are also derived. (Received March 11, 1953.)

470. J. H. B. Kemperman: *An asymptotic expression of $\sum_n g(\alpha n+\beta)s^n$ in terms of $\sum_n g(\rho n+\sigma)s^n$.*

For $u \geqq -k$, let $g(w) = g(u+iv)$ be analytic (no poles), $|g(u+iv)| \leqq K \exp(l\pi|u| + m\pi|v|)(1+v^2)^{-p}$, where K, l, m, and $p > 1/2$ are non-negative constants. Let $\alpha = |\alpha|e^{i\theta}$, $\rho = |\rho|e^{i\eta}$, β and σ be complex constants with $|\theta| < \pi/2$, $|\eta| < \pi/2$, $m|\rho| \leqq \cos\eta$. Put $F(s) = \sum g(\alpha n+\beta)s^{\alpha n}$ and $G(s) = (\rho/\alpha)s^{\sigma-\beta}\sum g(\rho n+\sigma)s^{\rho n}$, summing over the integers n for which Re $(\alpha n+\beta) > -k$ and Re $(\rho n+\sigma) > -k$, respectively. Let $\gamma = \exp(2\pi i/\alpha)$, $\Delta = \arg\gamma = 2\pi\cos\theta/|\alpha|$, $2m\pi = \nu\Delta + c$ (ν = integer $\geqq 0$; $0 \leqq c < \Delta$) and let k be an arbitrary integer. Restricting s to the sector $0 \leqq \arg s + m\pi \leqq c$ (or $c \leqq \arg s + m\pi \leqq \Delta$), the assertion is that the analytic function which, for small $|s|$, is defined by $H(s) = F(s\gamma^k) - \sum_{\mu=0}^{\nu} G(s\gamma^\mu)$ (or $H(s) = F(s\gamma^k) - \sum_{\mu=0}^{\nu-1} G(s\gamma^\mu)$, respectively), can be extended to a single-valued and analytic function throughout the entire sector in such a way that $H(s) = O(s-k-\beta)$ for large $|s|$. Especially, if $m|\alpha| < \cos\eta$ (hence, $\nu = 0$), $F(s\gamma^k) = O(s^{-k-\beta})$ for $c \leqq \arg s + m\pi \leqq \Delta$. E. M. Wright, Roy. Soc. London Philos. Trans. (A) vol. 238 (1940) p. 434 (generalizing a theorem of G. N. Watson), obtained the above result for the special case that $g(w) = O(\Gamma(w+b)^{-1})$ for $u \geqq -k$, $\rho = 1$, $\sigma = \beta = 0$. The above result can be extended to the case that $g(w)$ has for $u \geqq -k$ at most a finite number of poles. (Received March 16, 1953.)

471. Fulton Koehler: *Bounds for the moduli of the zeros of a polynomial.*

Let $f(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$, $a_0 \neq 0$, $a_n \neq 0$; and let the zeros of $f(z)$ be z_ν with $|z_1| \leqq |z_2| \leqq \cdots \leqq |z_n|$. Let the Newton diagram for $f(z)$, determined by the points $(\nu, -\log|a_\nu|)$, have slopes $\log R_1$, $\log R_2$, $\cdots$, $\log R_n$. In a paper by Ostrowski (*Recherches sur la méthode de Graeffe et les zéros des polynomes et des séries de Laurent*, Acta Math. vol. 72 (1940) pp. 99–257) it is shown that $|z_k|/R_k > 1-(1/2)^{1/k}$. The best bounds for $|z_k|/R_k$ as functions of n, are given only for $k=1$, $k=n$; and for $k=2$ when $n=3$. In this paper it is shown that the minimum of $|z_k|/R_k$ for all polynomials of nth degree with $a_k \neq 0$ is equal to the positive root of the equation $Q_k^n(x) = 1$, where $Q_k^n(x)$ is the partial sum of the series for $(1-x)^{-k}-1$ through the term in x^{n-k+1}. (Received March 5, 1953.)

472. Jacob Korevaar: *Best approximation in L_1 and the remainder in Littlewood's theorem.*

$f(x)$ is said to be of class $J_m(a, b)$ if (i) $f(x)$, $f'(x)$, $\cdots$, $f^{(m-1)}(x)$ exist and are continuous on $a \leqq x \leqq b$, (ii) $f^{(m)}(x)$ exists and is continuous except at a finite number of points, where it has a jump discontinuity, (iii) $|f^{(m)}(x_1)-f^{(m)}(x_2)|\leq A|x_1-x_2|$ on each of the subintervals of (a, b) on which $f^{(m)}(x)$ is continuous. Functions of class $J_m(a, b)$ are approximated by trigonometric and ordinary polynomials. The following result is typical. For any $f(x)$ of class $J_m(a, b)$, there exist constants H_1 and H_2 such that for every positive integer n one can find a polynomial $p_n(x)=\sum c_{nk}x^k$ of degree $n-1$ such that $\int_a^b|f(x)-p_n(x)|dx<H_1/n^{m+1}$, $|c_{nk}|<H_2^n$ $(k=0, 1, \cdots, n-1)$. This result is used to estimate the remainder in Littlewood's Tauberian theorem for power series. Let $|na_n|<K_1$, and let $\sum a_nx^n\rightarrow s$ as $x\uparrow 1$. To take a simple case, assume that $|\sum a_nx^n-s|<K_2(1-x)$, $0<x<1$. Then the above result with $m=0$ shows that $|s-s_n|<M_0/\log(n+2)$, $M_0=M_0(K_1, K_2)$; the above theorem with $m=1$ shows that $|s-(s_0+s_1+\cdots+s_n)/(n+1)|<M_1/\{\log (n+2)\}^2$, etc. These estimates for s_n etc. are best possible and improve the author's previous results (Duke Math. J. vol. 18 (1951) pp. 723–734). (Received March 11, 1953.)

473. M. Z. Krzywoblocki: *On the generalisation of fundamental existence theorems of W. M. Whyburn to partial differential systems.*

W. M. Whyburn proved some fundamental existence theorems for systems of first order differential equations where both the equations and the initial conditions differ from those that occur in the ordinary theory. The equations are different in that the quantities that appear on the right-hand side of the equations do not, as a whole, satisfy a Lipschitz condition. The boundary conditions differ in that they apply to more than one point of the interval. The author treats partial differential systems of the first order in two independent variables, which systems by a not one-to-one transformation are transformed onto an ordinary differential system with a parameter. For such systems, the author proves some existence theorems. The work consists of two parts. In Part I, the author generalizes many theorems on real-valued sets and functions to sets, and functions with a parameter. In that, he follows closely McShane's *Integration*. In Part II, he generalizes Whyburn's theorems to ordinary differential systems with a parameter. (Received January 30, 1953.)

474. Arne Magnus: *Volume-preserving maps in several complex variables.*

Let $u=u(z_1, z_2)$ and $v=v(z_1, z_2)$ be analytic in the complex variables z_1 and z_2. Let $(z_1, z_2)\rightarrow(u, v)$ be a volume-preserving map, that is, a map which carries any domain in the (z_1, z_2)-space into a domain in the (u, v)-space of the same 4-dimensional volume. Then $\partial(u, v)/\partial(z_1, z_2)=1$. Families of such maps are exhibited. A formula is derived which expresses u in terms of v when u and v are polynomials. All such polynomial pairs of degrees less than or equal to five are exhibited. All maps (polynomials or not) encountered are proved to be univalent. Some families of univalent but not volume-preserving maps are constructed by using the addition theorems for e^z, tan z, and sin z. Each map (volume-preserving or not) studied is shown to possess invariant 2-dimensional surfaces, that is, each map is assigned a 2-parameter family of 2-dimensional surfaces each of which is mapped into itself by $(z_1, z_2)\rightarrow(u, v)$. Extensions are made to several complex variables. (Received April 17, 1953.)

475*t*. R. K. Ritt: *A condition that* $\lim_{n\rightarrow\infty} n^{-1}T^n=0$.

Let T be a bounded linear transformation on a Banach space; let the spectrum of T be interior to the unit circle, with the possible exception $z=1$; further, suppose that there is an $M>0$ and an $\eta>0$ such that if z is in the resolvent set for T, $|z|\geq 1$, and $|s-1| \leq \eta$, then $\|(s-1)(s-T)^{-1}\| \leq M$. Then $\lim_{n\to\infty} n^{-1}T^n = 0$. The proof is obtained by computing an estimate of the integral representation for T^n. (Received March 13, 1953.)

476. R. K. Ritt: *Rings of functions.*

A B_*-algebra is a complete normed ring of functions on a connected compact Hausdorff space, containing, with any function, its complex conjugate. They are realizations of B^*-algebras [Gelfand, Neumark]. In a B_*-algebra there exist two types of maximal ideals: the first type—all functions zero at some point; the second type—a set of functions whose absolute value has its lower limit zero at some point. There are in general several ideals of second type associated with a point. In a B_*-algebra containing at least one nontrivial characteristic function, an arbitrary function can be uniformly approximated by a linear combination of disjoint characteristic functions if and only if no two distinct maximal ideals contain the same set of characteristic functions. Baire classes are defined, using uniformly bounded sequences of functions; they are shown to be B_*-algebras. In the Baire classes, if two maximal ideals contain the same set of characteristic functions they are both of the second type and are associated with the same point. Applications to spectral theory are given. (Received March 13, 1953.)

477t. R. K. Ritt: *Very weak convergence and the spectral resolution of bounded linear operators on a reflexive Banach space.*

If K is an arbitrary normed ring, with $\mathfrak{M}$ the space of maximal ideals, a uniformly bounded sequence, $\{x_n\}$, $x_n \in K$, converges very weakly to zero if $x_n(M) = 0$, all $M \in \mathfrak{M}$. Very weak convergence to zero is equivalent to weak convergence to zero if and only if every bounded linear functional on K has an integral representation of the Kakutani type [Ann of Math. vol. 42 (1941)]. A normed ring, $K(1, a)$, generated by the identity and a single element a is said to be normal if the uniform convergence to zero of a sequence of polynomials $\{p_n(\lambda)\}$ on the spectrum of a implies $\|p_n(a)\| \to 0$. If a is a bounded linear operator on a reflexive Banach space, and the spectrum of a is the unit interval, a will have an integral representation $\int_0^1 \lambda dE_\lambda$ if and only if $K(1, a)$ is normal. The proof employs the notion of very weak convergence and an argument of Eberlein [Bull. Amer. Math. Soc. vol. 52 (1946)]. (Received March 13, 1953.)

478t. L. B. Robinson: *Construction of systems of differential equations which functions given in advance satisfy.*

Given three functions I_i, write their infinitesimal transformations $\bar{I}_1 = (1+\psi_{22}\delta t + \psi_{33}\delta t)I_1 - \psi_{12}\delta t I_2 - \psi_{13}\delta t I_3$, $\bar{I}_2 = -\psi_{21}\delta t I_1 + (1+\psi_{11}\delta t + \psi_{33}\delta t)I_2 - \psi_{23}\delta t I_3$, $\bar{I}_3 = -\psi_{31}\delta t I_1 - \psi_{32}\delta t I_2 + (1+\psi_{11}\delta t + \psi_{22}\delta t)I_3$. It is easy to write down a system of partial differential equations which the I_i satisfy. Next write the symbolic products $\bar{I}_i \cdot \bar{I}_j = \bar{I}_{ij}$. An example would be $\bar{I}_1 \cdot \bar{I}_1 = \bar{I}_{11} = (1+2\psi_{22}\delta t + 2\psi_{33}\delta t)I_{11} - 2\psi_{12}\delta t I_{12} - 2\psi_{13}\delta t I_{13}$. It is easy to write down a system of partial-differential equations which the I_{ij} satisfy. And we can form the products $\bar{I}_i \cdot \bar{I}_j \cdot \bar{I}_k = \bar{I}_{ijk}$, and so on. The above functions are semitensors. Knowing a system of semitensors it is possible by symbolic multiplication to form an infinite chain of semitensors and also to write an infinite chain of differential equations which these semitensors satisfy. (Received April 6, 1953.)

APPLIED MATHEMATICS

479t. Nathaniel Coburn: *Stream lines for incompressible and compressible fluid flows (plane irrotational and rotational cases).*

The purpose of this paper is to use the intrinsic form of the equations of fluid dynamics (*Some intrinsic relations satisfied by the velocity and vorticity vectors in fluid flow theory*, N. Coburn, Michigan Mathematical Journal, vol. 2, 1952–53) to classify stream line patterns for incompressible and compressible fluids in the plane irrotational and rotational cases. In this method of attack, the equations of fluid dynamics become first order equations in functions of the magnitude of the velocity and the metric coefficients of the net determined by the stream lines and their orthogonal trajectories. The essential nonlinearity of the equations of fluid dynamics is thrown into the geometry of the stream lines (the vanishing of the Riemann tensor). By specifying the curvature of the stream lines or their orthogonal trajectories, this nonlinear equation may be discussed. In the incompressible irrotational case, it is shown that, if the curvature of a stream line (say, an obstacle) and if the magnitude of the velocity vector is known along this curve, then the determination of the flow depends upon the solution of a Neumann problem for an upper-half plane. For the rotational and compressible cases, the cases of straight stream lines are discussed. (Received January 14, 1953.)

480. R. J. De Vogelaere: *On a dynamical system with two degrees of freedom.*

At present only two dynamical systems are known with some details, the cosmic rays problem and the restricted three bodies problem. Attention is drawn to the plane problem corresponding to the isosceles triangle solutions of the general three bodies problem with the interesting feature of the singularities of the double and triple collisions. As known since the work of Fransen, the masses m at the vertices of the equal angles are to be equal. In the case with axial symmetry, the variables were chosen proportional, r proportional to the distances of the equal masses and s proportional to the distance of the other to the center of gravity. The first family of periodic orbits was discovered having as one end the collinear solution of Euler and as other end a solution with two symmetric double collisions. The results were put elegantly in terms of Fourier sums for the mass ratios 1, 1, 1; 1, 1, 4, and 1, 1, ϵ, the independent variable being analogous to the mean anomaly for s and to the eccentric anomaly for r. (Received March 10, 1953.)

GEOMETRY

481*t*. John DeCicco: *Physical systems of curves in Riemannian space.*

A system S_k of ∞^{2n-1} curves in a given field of force of a Riemannian space V_n consists of curves along which a constrained motion is possible such that the osculating geodesic surface at each point touches the force vector X, and the pressure P, along the principal normal to the curve, is proportional to the normal component N of X. Thus $P=kN$ where $k(\neq -1)$ is the constant factor of proportionality. For a system S_k, the author obtains extensions of the Lagrangian equations and the Hamiltonian equations of a dynamical system. A generalization is found of the Lie contact transformation of mechanics. A conservative system S_k can be defined as the extremals of the variation problem: $\int v^{1+k}ds=\min$. There is a similar integral for a conservative velocity system S_∞. Finally the theorem of Kasner which states that the ratio of the curvature of the rest trajectory to that of the line of force is $\rho=(1+k)/(3+k)$, at the initial point, is established for a system S_k in V_n. (Received March 10, 1953.)

482*t*. John DeCicco: *Transformation theory of physical systems of curves in Riemannian space.*

The following theorems are established concerning the transformation theory of physical systems of curves in Riemannian space. In any nonhomothetic conformal cartogram between two Riemannian manifolds V_n and $\bar{V}_n$, there is one and only one whole conservative system S_L, which is converted into a whole conservative system S_L. By a transformation T between V_n and $\bar{V}_n$, every physical system S_k, not a dynamical system S_0, in V_n becomes a physical system in V_n if and only if T is a homothetic transformation. A transformation T between V_n and $\bar{V}_n$ converts every dynamical system S_0 in V_n into a dynamical system if and only if T is a projective transformation such that $\partial(\bar{\Gamma}^i_{ii}-\Gamma^i_{ii})/\partial x^j=\partial(\bar{\Gamma}^j_{jj}-\Gamma_{jj})/\partial x^i$, where the Γ^i_{jk} denote the Christoffel symbols. The transformation T is affine if and only if the time is unchanged. Finally if both V_n and $\bar{V}_n$ are Euclidean spaces, then Appel's transformation for dynamical systems in n dimensions is found. (Received March 10, 1953.)

483. Marshall Hall, Jr.: *Uniqueness of the projective plane with 57 points.*

Using the result of W. A. Pierce that a plane with 57 points cannot contain a Fano configuration, it is shown that such a plane must be Desarguesian. A study of linear equations associated with incidences shows that Moufang's Desarguesian configuration D_9 must be valid in the plane and hence the plane is Desarguesian. (Received February 17, 1953.)

484*t*. P. C. Hammer: *Constant breadth surfaces.*

A quasi-pencil of lines in n-dimensional space is here defined to be a continuous family of lines simply covering some sphere and its exterior, including infinite points, and which has all points on each line in the family closest to any other line in the family interior to the sphere. A simple analytical formula is derived giving surfaces of constant breadth relative to an outwardly simple line family. When this line family is a quasi-pencil and other conditions are satisfied, the surface is a closed convex surface of constant breadth. The formula is shown to represent all such surfaces. (Received February 23, 1953.)

485*t*. P. C. Hammer: *Tangential similarity of surfaces.*

Weak tangential similarity and tangential similarity are defined for surfaces in E_4 parametrically representable over a domain in a real normed linear space. In certain cases of perspective surfaces weak tangential similarity is shown to imply similarity. Existence and uniqueness of curves satisfying certain differential conditions is shown when there is no differential equation and no Lipschitz type condition available. Cf. *Class-equivalence of functions and curves*, Bull. Amer. Math. Soc. Abstract 59-2-234. (Received March 12, 1953.)

STATISTICS AND PROBABILITY

486. Shu-Teh C. Moy: *Measure extensions and the martingale convergence theorem.*

In Doob's discussion of the relation between his martingale convergence theorem and a limit theorem on derivatives by Andersen and Jessen [Doob, *Stochastic processes*, New York, Wiley, pp. 630–632] the following two conditions concerning a martingale

$\{x_n, \mathcal{F}_n, n \geq 1\}$ are studied. 1. There is a countably additive set function ϕ, defined on the smallest Borel field $\mathcal{F}_\infty$ containing all the $\mathcal{F}_n$'s, of which the contraction ϕ_n to $\mathcal{F}_n$ is absolutely continuous with respect to the contraction P_n of the probability measure to $\mathcal{F}_n$ and for which x_n is the derivative of ϕ_n relative to P_n for every n. 2. Sup $\{E[|x_n|]: n \geq 1\} < \infty$. In this paper it is proved that 1 and 2 are equivalent if the basic space Ω on which the random variables x_n are defined is the space of real sequences $\xi = \{\xi_n\}$ and $\mathcal{F}_n$ is the smallest Borel field containing the sets of the form $[\{\xi_n\}: \xi_1 \leq \alpha_1, \cdots, \xi_n \leq \alpha_n]$ with $\alpha_1, \cdots, \alpha_n$ being any n real numbers. This result implies that the martingale convergence theorem can be deduced from the limit theorem of Andersen and Jessen. (Received April 22, 1953.)

TOPOLOGY

487. Raoul Bott: *On closed geodesics.* Preliminary report.

A closed geodesic g is a map of the reals R into a manifold M, which satisfies the geodesic differential equations, and is in addition of period 2π. $g^{(n)}$, the nth iterate of g, is defined by $g^n(x) = g(nx)$, $x \in R$, $n = 1, 2, \cdots$. Poincaré assigns to every g a system of numbers $\pm\alpha_1, \cdots, \pm\alpha_k$; $k \leq \dim(M)$, modulo 2π (see *Oeuvres*, vol. 7, p. 338). Morse (see Amer. Math. Soc. Colloquium Publications, 1934) assigns to every g two integers $N(g)$ and $\Lambda(g)$, the nullity and index of g respectively. It is proved that these numbers are related in the following fashion: $N\{g^{(n)}\} = \sum_{k=0}^{n-1} \nu(e^{2\pi ki/n})$, $\Lambda(g^{(n)}) = \sum_{k=0}^{n-1} \lambda(e^{2\pi ki/n})$ where ν and λ are non-negative integer-valued functions on the unit circle. Further, λ and ν are constant except at the points $e^{\pm i\alpha_k}$ and the jump of λ at $e^{\pm i\alpha_i}$ is bounded by the value of ν at $e^{\pm i\alpha_i}$. It follows that $\lim_{n\to\infty} (\Lambda g^{(n)}/n) = \sigma$ exists and $\sigma = \sum a_k\alpha_k$ mod 2π with the a_i integers. These formulae and their consequences generalized the results of Hedlund (Trans. Amer. Math. Soc. vol. 34 (1932) pp. 75–97) and certain of the announced results of M. Morse and E. Pitcher (Proc. Nat. Acad. Sci. U. S. A. (1934) pp. 282–287). (Received March 11, 1953.)

488t. E. E. Floyd and M. K. Fort, Jr.: *A characterization theorem for monotone mappings.*

The main result of the paper is the following theorem. A mapping f of the two-sphere S onto S is monotone if and only if there exists a continuous extension g of f such that g maps $S \cup Q$, Q the interior of S, onto itself with $g | Q$ a homeomorphism of Q onto Q. A key lemma states that the group of homeomorphisms of S onto S is uniformly locally connected. (Received March 12, 1953.)

489. William Gustin: *On the order of traversing a path.*

A continuous mapping f from an oriented closed linear interval Π onto a (necessarily Peano) compactum P is said to be a path with carrier P. A finite sequence $p_1 \cdots p_n$ of points in P is called an f-chain provided there exist parameters $\pi_1 \leq \cdots \leq \pi_n$ in Π such that $f(\pi_k) = p_k$ for $k = 1, \cdots, n$. The path order of f is defined as the set $\mathfrak{c}(f)$ of all f-chains. Seven properties of path order are established; and a set $\mathfrak{c}$ of finite sequences, termed abstract chains, of points in a compactum P satisfying these seven properties or axioms is said to be an abstract path order with carrier P. It is shown that every abstract path order $\mathfrak{c}$ can be concretely realized as the path order $\mathfrak{c}(f)$ engendered by some path f, and that the class of all such paths f with $\mathfrak{c}(f) = \mathfrak{c}$ is a Fréchet equivalence class of paths. Thus abstract path order intrinsically characterizes an oriented Fréchet curve, abstract chains together with their carrier space constituting a complete set of curve invariants. (Received March 9, 1953.)

490. Frank Harary and R. Z. Norman: *The dissimilarity characteristic of linear graphs.*

Otter's notion of dissimilarity characteristic for trees, previously extended by the authors to Husimi trees (to appear in Ann. of Math.) is here generalized to arbitrary linear graphs. Let G be a connected graph. Let p, k be the number of dissimilar points, lines of G respectively, and let k_s be the number of dissimilar symmetry lines of G. Consider an arbitrary cycle basis of G. From this basis we extract a set B of cycles forming a basis modulo similarity. A certain well-defined subset of the set of nonorientable cycles of B will be called the set of exceptional cycles of B. Let c, c_e be the number of cycles, exceptional cycles of B respectively. It is shown that $c-c_e$ is an invariant of G. The dissimilarity characteristic equation of G is $p-(k-k_s)+(c-c_e)=1$. This formula is potentially applicable to the solution of various counting problems, among them some of the combinatorial questions discussed by Uhlenbeck in his Gibbs Lecture. (Received March 11, 1953.)

491. P. S. Mostert: *On fibre spaces with totally disconnected fibres.*

Let $\mathcal{F}=\{X, B, \pi, \phi_U, \Omega\}$ be a fibre space in the sense of Hu (Proc. Amer. Math. Soc. vol. 1 (1950) pp. 756–762) where the base space B is arcwise connected and $\phi_U: U\times\pi^{-1}(b)\to\pi^{-1}(U)$ is a homeomorphism onto. Then it is proved that there is a collection $\mathcal{B}=\{X, B, Y, \pi, \psi_b, V_b, H_b, G\}$ which is an E-F bundle (Steenrod, *The topology of fibre bundles*, Princeton University Press, 1950, p. 18) where Y is homeomorphic with $\pi^{-1}(b)$, $b\in B$, ψ_b arises naturally out of ϕ_U, and G is isomorphic (algebraically) with a factor group of the fundamental group of B. The result is applied to fibre bundles to show that a fibre bundle $\mathcal{B}=\{X, B, Y, \pi, \phi_i, U_i, G\}$, where the fibre Y is totally disconnected, and the base space B is arcwise connected, is equivalent in G to a bundle with group $G'\subset G$ which is the closure in G of the image of the fundamental group of B under a homomorphism into. Hence, in particular, if B is also simply connected, it is G-equivalent to the product bundle. This result is then applied to locally compact groups of finite dimension. (Received March 9, 1953.)

492t. E. S. Northam: *Topology in lattices.*

Various results concerning the order topology and interval topology of a lattice are obtained. It can be shown that a necessary condition for the interval topology to be Hausdorff is that each interval of the lattice have a finite separating set in the following sense: A set $\{a_i\}$ separates the interval $[a, b]$ if $a<a_i<b$ for all a_i and each member of $[a, b]$ is comparable with some a_i. This result can be applied to show that the interval topology of a Boolean algebra is Hausdorff if and only if every element is over an atom, and also to solve Problem 104 of Birkhoff's list. A necessary and sufficient condition for an element to be isolated in the interval topology is obtained. Also an example can be found which shows that the order topology, L-topology of Rennie, and transfinite sequential topologies of a lattice need not be Hausdorff. (Received March 16, 1953.)

J. W. T. YOUNGS,
Associate Secretary

THE MAY MEETING IN PALO ALTO

The four hundred ninety-second meeting of the American Mathematical Society was held at Stanford University, Palo Alto, California, on May 2, 1953. Attendance was approximately 120, including the following 88 members of the Society:

H. L. Alder, C. B. Allendoerfer, T. M. Apostol, H. M. Bacon, E. M. Beesley, Donald C. Benson, Stefan Bergman, Kurt Bing, W. W. Bledsoe, R. N. Bradt, J. L. Brenner, Paul Brock, F. H. Brownell, Herman Chernoff, Randolph Church, P. A. Clement, K. L. Cooke, E. L. Crow, P. H. Daus, A. C. Davis, E. A. Davis, C. R. DePrima, Douglas Derry, Roy Dubisch, Arthur Erdélyi, Paul Erdös, F. D. Faulkner, Solomon Feferman, J. M. G. Fell, W. J. Firey, Harley Flanders, G. E. Forsythe, A. L. Foster, Joel Franklin, K. S. Ghent, M. A. Girshick, J. W. Green, R. M. Hayes, J. G. Herriot, M. R. Hestenes, Edwin Hewitt, Jan Kalicki, William Karush, R. S. Lehman, Joseph Lehner, R. B. Leipnik, Hans Lewy, Charles Loewner, A. T. Lonseth, A. V. Martin, R. B. Merkel, A. B. Mewborn, F. R. Morris, A. P. Morse, T. S. Motzkin, A. F. Moursund, R. G. Needels, Ivan Niven, C. D. Olds, T. K. Pan, C. L. Perry, Jr., George Pólya, F. M. Pulliam, Joseph Putter, C. H. Rawlins, Jr., R. M. Redheffer, Edgar Reich, H. J. Reiter, J. B. Robinson, R. M. Robinson, E. B. Roessler, H. L. Royden, Herman Rubin, M. M. Schiffer, Abraham Seidenberg, M. A. Shader, W. H. Simons, Ernst Snapper, J. M. Stark, E. G. Straus, M. V. Sunseri, Irving Sussman, Gabor Szegö, Alfred Tarski, J. W. Weihe, Robert Weinstock, G. T. Whyburn, A. R. Williams.

The general session at ten o'clock was presided over by Professor Charles Loewner. At eleven, Professor Ernst Snapper of the University of Southern California gave the invited address, *Equivalence relations in algebraic geometry*. There were two sessions in the afternoon for contributed papers, at which Professors Roy Dubisch and E. M. Beesley presided.

The abstracts of papers presented at the meeting follow. Those whose numbers are followed by "t" were presented by title. Professor Fraïssé was introduced by Professor Tarski and Professor Karlin by the Associate Secretary. In the absence of Mr. Vaught, paper 522 was presented by Mr. Solomon Feferman. Paper 508 was presented by Professor Straus, paper 519 by Dr. Motzkin, and paper 523 by Professor Hewitt.

ALGEBRA AND THEORY OF NUMBERS

493. J. L. Brenner: *Orthogonal matrices of modular polynomials.*

The result of bordering a 2×2 orthogonal matrix is called a plane rotation, as usual. Theorem. Every orthogonal matrix, the elements of which are polynomials in t with coefficients in the field of two elements, is a product of constant matrices and plane rotations. This theorem is false for matrices over $C_p[t]$, $p>2$, p prime [Amer. Math. Monthly vol. 58 (1951) pp. 327–329]. A whole class of nonconstant orthogonal

$n\times n$ matrices over $C_p[t]$ is constructed ($n\geq 3$). (L. Carlitz, in an article to appear in Amer. Math. Monthly, had constructed $p\times p$ matrices of this type.) When $p>2$, it is not determined whether the construction yields all matrices A such that det $(E+A)=0$. (Received March 5, 1953.)

494. Roland Fraïssé: *On a decomposition of relations which generalizes the sum of ordering relations.*

Let $A(x_1, \cdots, x_n)$ be an n-ary relation with a base E (a function defined on E^n, but not on the diagonal elements, assuming two values). A *decomposition* of A is defined by a partition of E into E_i's (with some E_i containing at least two elements) and by a *main* relation P with the base $\{E_i\}$, such that if x_m $(1\leq m\leq n)$ are not all in the same E_i, then $A(x_1, \cdots, x_n)=P(E_{i_1}, \cdots, E_{i_n})$ where $x_m\in E_{i_m}$. A relation is *primitive* if no such decomposition exists. The following are proved: (1) Given A with a finite base E, if the main relation is primitive, then it is unique up to isomorphism. If we apply this process to each restriction of A to E_i, we obtain, by iteration, a finite set Γ of primitive relations. Any two Γ's obtained by different sequences of decompositions can be put into 1-1 correspondance such that the corresponding relations are isomorphic. (2) Let Σ be the class of n-ary relations with a denumerable base, such that $A\in\Sigma$ iff every n-ary relation with a denumerable base is isomorphic to a restriction of A. (a) Σ contains some primitive relations. (b) If $A\in\Sigma$, either $P\in\Sigma$ or at least one of the restrictions of A to an E_i is in Σ. (Received March 11, 1953.)

495t. J. R. Jackson: *On the existence problem of linear programming. I.*

This paper is concerned with the following linear programming problem: Given $m\times n$ matrix A and $m\times 1$ vector b, does there exist $n\times 1$ vector $x\geq 0$ such that $Ax=b$? It is first shown that *the following three conditions are equivalent*: (i) *there exists* $m\times 1$ *vector* w *such that* $w'A>0$ (i.e., each component of $w'A$ is positive); (ii) *if* $x\geq 0$ *and* $Ax=0$, *then* $x=0$; (iii) *for any* $m\times 1$ *vector* b, *the set* $[x\mid Ax=b,\ x\geq 0]$ *is bounded.* Obviously, if $w'A>0$, then a necessary condition for the existence of $x\geq 0$ with $Ax=b$ is that $w'b>0$. If $w'A>0$ and $w'b>0$; then w can be modified slightly to get v with $v>0$, $v'A>0$, and $v'b>0$; and then a simple transformation of A yields matrix B such that each component of $v'B$ equals $v'b$; while the existence of $x\geq 0$ with $Ax=b$ is equivalent to the existence of $y\geq 0$ with $By=b$ (and, in fact, the sets of vectors x and y satisfying these two conditions are very simply related). Having formed matrix H by subtracting b from each column of B, the following is proved: *The zero-sum two-person game with matrix* H *has non-negative value; its value is zero if and only if there exists* $x\geq 0$ *with* $Ax=b$; *moreover, if the value is zero, then the optimal mixed strategies for the second player in this game are precisely the vectors* $y\geq 0$ *with* $By=b$. (Received March 4, 1953.)

496t. J. R. Jackson: *On the existence problem of linear programming. II.*

Under the assumption that there exists $m\times 1$ vector w such that $w'A>0$ and $w'b>0$, it is possible to construct another game with the properties of the game H of the previous abstract. Let B^* be a matrix whose columns form a maximal linearly independent subset of the columns of B (considered as vectors). Then there exists matrix D such that $B=B^*D$. Obviously, a necessary condition for the existence of $x\geq 0$ with $Bx=b$ is that there exist vector s such that $b=B^*s$. In this case, it is not difficult to see that $Bx=b$ if and only if $Dx=e$. It is easily seen that if I is the vector of suitable dimension each of whose components is 1, it follows that $I'D>0$ and $I'e>0$. Thus, the theorem of the previous abstract can be applied to the problem of whether there exists $x\geqq 0$ such that $Dx=e$, with $v=I$. This result is of potential interest from a computational point of view, since B^*, D, and e are easily obtained in many applications. (Received March 4, 1953.)

497. R. S. Lehman: *Approximation of improper integrals by sums over multiples of irrational numbers.*

Let α be a positive irrational number. Let p_i/q_i ($i=0, 1, 2, \cdots$) be the convergents in the regular continued fraction expansion of α. Suppose $f(x)$ is a periodic function with period 1, improperly Riemann integrable on the interval (0, 1), bounded in every closed sub-interval which excludes the points 0 and 1, and either bounded or monotone near 0 and near 1 in (0, 1). While Weyl's result for properly integrable $f(x)$ that $N^{-1}\sum_{n=1}^{N} f(n\alpha)\rightarrow\int_0^1 f(x)dx$ as $N\rightarrow\infty$ does not apply in general, it is found that the sum does tend to the integral provided certain terms are struck out of the sum. Which terms are struck out does not depend on the particular function $f(x)$ but only on the arithmetic character of α. More precisely, let $q_i\leqq N<q_{i+1}$ so that $N=bq_i+r$, $q_i>r\geqq 0$; then the omitted terms are those for which $n=kq_i$ ($k=1, 2, \cdots, b$). It is shown that cancelling terms from the sum in the above approximation formula becomes necessary only for a set of α having measure zero; in particular it is unnecessary for α having continued fractions with bounded quotients. (Received March 11, 1953.)

498t. R. D. Stalley: *A new number density.*

The density of a sequence of positive integers $n_1<n_2<\cdots$ is defined to be glb k/n_k. Two theorems concerning the densities α, β, and γ of sequences A, B, and $C=A+B$ are proved. First, if $\alpha+\beta>1$, then $\gamma=1$. Second, if $1, 2, \cdots, k\in A$, then $\gamma\geqq k(k+1)^{-1}\alpha+\beta$ or $\gamma=1$, where this is a best possible result. Since this density is a modified Schnirelmann density, it is unexpected that these results are identical to known results for asymptotic density. (Received March 9, 1953.)

499t. J. D. Swift: *Some remarks on the numbers of Chernick and Carmichael.*

A Chernick number is defined as a product $n_1n_2n_3=N$ such that $N=1$ mod (n_i-1), $i=1, 2, 3$. In Bull. Amer. Math. Soc. vol. 45, pp. 269–273, J. Chernick showed that a sufficient condition that N be an absolute pseudo-prime or Carmichael number is that n_1, n_2, and n_3 be prime. This condition is not necessary. On the other hand, Carmichael numbers exist which are not Chernick numbers. For example, a Carmichael number with four prime factors p, q, r, s is a Chernick number if and only if at least one of the relations of the type $pq-1$ divides $rs-1$ holds. If the quotient of the last named numbers is k, a Chernick number $(pq)\cdot r\cdot s$ is an absolute pseudo-prime if $p-1$ divides $(kq+1)(q-1)$ and the similar relation holds with p and q interchanged. The tests may be extended to more complicated cases and applied to the construction of Carmichael numbers. Similar considerations apply to the pseudo-prime character relative to particular bases such as 2. (Received March 12, 1953.)

500. Alfred Tarski: *Universal arithmetical classes of mathematical systems.* Preliminary report.

For notation see Tarski, Proceedings of the International Congress of Mathematicians, 1950, vol. 1, p. 705 ff. Consider systems $\mathfrak{A}$, $\mathfrak{B}$, $\cdots$ formed by nonempty sets A, B, $\cdots$ and binary relations $R\subseteq A\times A$, $S\subseteq B\times B$, $\cdots$. $\mathfrak{A}\cong\mathfrak{B}$ means that $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic; $\mathfrak{A}\subseteq\mathfrak{B}$ means that $\mathfrak{A}$ is a $\mathfrak{B}$-subsystem, i.e., $A\subseteq B$ and $R=S\cap(A\times A)$. $\mathfrak{S}$ being a set of systems, the meaning of formulas $\mathfrak{S}\in AC$, $\mathfrak{S}\in AC_\delta$ is known. $\mathfrak{S}\in UC$ means that $\mathfrak{S}$ is a *universal arithmetical class*, i.e., $\mathfrak{S}$ consists of all systems satisfying a first-order sentence of the form $\prod_x\cdots\phi(x,\cdots)$ where $\prod$ is the universal quantifier and $\phi(x,\cdots)$ a formula without quantifiers. $\mathfrak{S}\in UC_\delta$ means that $\mathfrak{S}$ consists of all systems satisfying a certain set of such sentences. Theorems: I. If $\mathfrak{A}$ is a finite and $\mathfrak{S}$ consists of all $\mathfrak{B}$'s such that $\mathfrak{A}\cong\mathfrak{A}'$ for no $\mathfrak{A}'\subseteq\mathfrak{B}$, then $\mathfrak{S}\in UC$. II. $\mathfrak{S}\in UC_\delta$ if and only if $\mathfrak{B}\in\mathfrak{S}$ whenever (i) $\mathfrak{B}\cong\mathfrak{A}$ or $\mathfrak{B}\subseteq\mathfrak{A}$ for some $\mathfrak{A}\in\mathfrak{S}$, or (ii) $\mathfrak{S}$ contains all finite $\mathfrak{B}$-subsystems. III. $\mathfrak{S}\in UC$ (or $\mathfrak{S}\in UC_\delta$) if and only if $\mathfrak{S}\in AC$ (or $\mathfrak{S}\in AC_\delta$, respectively), and $\mathfrak{B}\in\mathfrak{S}$ whenever $\mathfrak{B}\subseteq\mathfrak{A}$ for some $\mathfrak{A}\in\mathfrak{S}$. I–III extend to arbitrary systems with finitely many relations and operations of finite ranks. (Received March 11, 1953.)

501t. R. L. Vaught: *Remarks on universal arithmetical classes*. Preliminary report.

For notation see the preceding abstract of Tarski. The following theorem analogous to Theorem II of that abstract holds: $\mathfrak{S}\in UC$ if and only if (i) $\mathfrak{B}\in\mathfrak{S}$ whenever $\mathfrak{B}\cong\mathfrak{A}$ or $\mathfrak{B}\subseteq\mathfrak{A}$ for some $\mathfrak{A}\in\mathfrak{S}$, and (ii) there is a natural number n such that $\mathfrak{B}\in\mathfrak{S}$ whenever $\mathfrak{S}$ contains all $\mathfrak{B}$-subsystems with at most n elements. Neither this theorem nor Theorems I and II of the preceding abstract apply in general to systems with infinitely many relations and operations of finite ranks. (However, as was remarked by Tarski, his Theorem III applies to arbitrary such systems. In this case the formula $\mathfrak{S}\in AC_\delta$ is understood to mean that $\mathfrak{S}$ is an intersection of arbitrarily many sets $\mathfrak{T}\in AC$, and similarly for $\mathfrak{S}\in UC_\delta$.) (Received March 11, 1953.)

ANALYSIS

502. F. H. Brownell: *Flows and noncommuting projections on Hilbert space.*

Let $\{E(A)\}$ over Borel subsets A of the real line be a resolution of the identity for the Hilbert space X and consider the flow $u_t=U_tu_0=\int_{-\infty}^{\infty}e^{it\lambda}dE(\lambda)u_0$ for a fixed $u_0\in X$ and t real, the integral existing as the X norm limit of the Lebesgue-Stieltjes sums. Let P be a projection in X, not commuting with $E(A)$ in the nontrivial case. The problem is to study the asymptotic behavior of $\|Pu_t\|^2=(Pu_t,u_t)$ as $t\to+\infty$. If P is compact (i.e. finite-dimensional), it is possible to shift the integration so that $\|Pu_t\|^2=\int_{-\infty}^{\infty}e^{its}d\psi(s)$, where $\psi(A)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\chi_A(\lambda-\beta)d_\lambda d_\beta(PE(\lambda)u_0,\ E(\beta)u_0)$ suitably defined is σ-additive and of bounded variation over the Borel sets of $(-\infty,\ \infty)$. Thus it is known that $\|Pu_t\|^2$ converges densely to a constant C if and only if $\psi(\{s\})=0$ for all real $s\neq0$, and in this case $C=\psi(\{0\})$. This condition on P and the extension to noncompact P are studied. (Received March 13, 1953.)

503t. H. E. Chrestenson: *A class of generalised Walsh functions.*

For each integer $\alpha\geqq2$ the system $\Psi_\alpha=\{\psi_j(x)\}$ $(j=0,1,\cdots)$ of Walsh functions of order α is defined. Each Ψ_α is an orthonormal system, complete in $L(0,1)$. Let $f(x)$ be integrable and periodic, and let the Fourier series for $f(x)$ be $\sum_{n=0}^{\infty}c_n\psi_n(x)$. Write $s_n(x)=\sum_{r=0}^{n-1}c_r\psi_r(x)$. The following results are proved: (i) As $n\to\infty$, $s_{\alpha^n}(x)\to f(x)$ a.e. (ii) If $f(x)$ has total variation $V<\infty$, then $s_n(x)\to f(x)$ a.e., and $|c_n|\leqq Vk^{-1}\csc\pi/\alpha$.

(iii) If $[f(t)-c][t-x_0]^{-1}\in L(x_0-\delta, x_0+\delta)$ for some $\delta>0$, then $s_n(x_0)\to c$. (iv) If the modulus of continuity of $f(x)$ satisfies $\omega(\delta;f)=o((\log \delta^{-1})^{-1})$ as $\delta\to 0$, then $s_n(x)\to f(x)$ uniformly. (v) If $f(x)$ is continuous, then $\sum_{n=0}^{\infty} c_n\psi_n(x)$ is uniformly $(C, 1)$ summable to $f(x)$. These results include some of those of N. J. Fine (Trans. Amer. Math. Soc. vol. 65 (1949) pp. 372–414) and J. L. Walsh (Amer. J. Math. vol. 55 (1923) pp. 5–24). (Received March 9, 1953.)

504*t*. H. E. Chrestenson: *Some groups of orthonormal functions.*

Let $\Lambda=\{\lambda_j(x)\}$ $(j=0, 1, \cdots)$ be a set of complex-valued orthonormal functions on (0, 1). If for each j and k, $\lambda_j(x)\bar{\lambda}_k(x)\in\Lambda$, then Λ is a multiplicative group. For each integer $\alpha\geqq 2$, a system $\Psi_\alpha=\{\psi_j(x)\}$ is defined so that each $\psi_j(x)$ has order dividing α. Ψ_2 is the system of Walsh functions. Suppose that the group Λ is isomorphic to Ψ_α so that $\lambda_j(x)$ and $\psi_j(x)$ correspond. Necessary and sufficient conditions are given for the existence of a measurable transformation Tx having the properties (i) for any measurable set $E\subset[0, 1)$, $\mu(T^{-1}E)=\mu(E)$, and (ii) for all $n\geqq 0$, $\lambda_n(x)=\psi_n(Tx)$ a.e. These results include those of Paul Civin (Pacific Journal of Mathematics vol. 2 (1952) pp. 291–295) for $\alpha=2$ and $\lambda_n(x)$ real-valued. (Received March 9, 1953.)

505*t*. Paul Civin: *Orthonormal cyclic groups.*

Let $\{A_n(x)\}$ $(n=0, \pm 1, \cdots; 0\leqq x\leqq 1)$ be a set of complex measurable functions which is a multiplicative cyclic group. A necessary and sufficient condition that $\{A_n(x)\}$ be an orthonormal system over $0\leqq x\leqq 1$ is that the generator of the group admit a representation exp $(2\pi ic(x))$ almost everywhere, with $c(x)$ equimeasurable with x. A similar result relating real functions such that $A_n^2(x)=1$ and the Walsh functions is contained in a note of the authors in the Pacific Journal of Mathematics vol. 2 (1952) pp. 291–296. (Received March 9, 1953.)

506. P. A. Clement: *Generalized convexity and inequalities im grossen of functions on surfaces of negative curvature.*

Certain explicit functions involving the circumference $l(r)$ and area $a(r)$ of a geodesic circle on an analytic surface whose Gaussian curvature K satisfies $K\leqq K_0<0$ ($K_0=$constant) are shown to satisfy a *Condition* C: they are subfunctions of the two-parameter family of solutions of the second order differential equation $(D^2+K_0)u=0$, zero at the pole, monotonic nondecreasing, and identically zero if and only if $K=K_0$. A function $f(r)$ is shown to be a subfunction of the family by verifying that $f(r)$ satisfies the defining functional inequality, or, if $f(r)$ is of class C^2, that it satisfies a differential inequality [see Peixoto, *Generalized convex functions and second order differential inequalities*, Bull. Amer. Math. Soc. vol. 55 (1949) pp. 563–572]. For example, the isoperimetric function, $l^2(r)+K_0a^2(r)-2\pi a(r)$, satisfies Condition C, as do also a number of functions obtained from it by a "modification" process. The inequality implied for a function satisfying Condition C characterizes the surface for some functions. Other configurations than geodesic circles are considered and some of the functions retain characteristic properties on generalization; for example, some functions which are generalizations of Steiner's formulas are shown to satisfy a generalized Condition C. (Received March 10, 1953.)

507. K. L. Cooke: *The rate of increase of real continuous solutions of algebraic differential-difference equations of the first order.*

In this paper, bounds are established on the possible rate of growth, as $t\to+\infty$, of real continuous solutions of algebraic differential-difference equations of the form $P(t, u(t), u'(t), u(t+1), u'(t+1)) = 0$, where t is a real variable and where $P(t, u, v, w, x)$ is a polynomial, with real coefficients, in the variables t, u, v, w, x. The paper extends previous work by Borel (Ann. École Norm. vol. 16 (1899) pp. 9–136) and Shah (Bull. Amer. Math. Soc. vol. 53 (1947) pp. 548–558), who discussed the analogous problem for algebraic differential equations and for algebraic difference equations, respectively. The methods used here are similar to those of Borel and Shah. A typical theorem is the following: To each polynomial $P(t, u, v)$ there corresponds a positive number A with the property that if $u(t)$ is a real solution of $P(t, u(t), u'(t+1)) = 0$ which exists and has a continuous first derivative for $t \geqq t_0$, then there is no number $T>0$ for which $|u(t)| \geqq e_3(At)$ for all $t \geqq T$. Stronger theorems can be proved if, and only if, the differential-difference equations are of more restricted type. (Received March 9, 1953.)

508. Paul Erdös and E. G. Straus: *Linear independence of sequences in a Banach space.*

The following theorem is proved: Let $x_1, x_2, \cdots$ be a sequence of unit vectors in a Banach space such that every finite subset is linearly independent, and let $M_1, M_2, \cdots$ be a sequence of non-negative numbers. Then there exists a subsequence $\{x_{i_n}\}$ with the property that $\lim_{m\to\infty} \sum_n c_{mn}x_{i_n} = 0$ and $|c_{mn}| \leqq M_n$ implies $\lim_{m\to\infty} c_{mn} = 0$ for all n. (Received March 11, 1953.)

509. G. E. Forsythe: *Arbitrarily close lower bounds for the fundamental frequency of certain vibrating membranes.*

Divide the plane into equal squares S_h of side h by a network of parallels to the axes. A "half-square" is an isosceles right triangle which is half an S_h. Let the fixed convex open region R with boundary C be a finite union of squares and half-squares, for all h under consideration. Let λ be the least number such that (*) $u_{xx}+u_{yy}+\lambda u = 0$ in R and $u = 0$ on C. (Subscripts denote differentiation.) Let λ_h be the least number such that $v(x+h, y)+v(x-h, y)+v(x, y+h)+v(x, y-h)-4v(x, y)+\lambda_h h^2 v(x, y) = 0$ at the network nodes (x, y) in R, and $v = 0$ at the nodes on C. Theorem: For all h under consideration, as $h \to 0$ and $a = a(R) = \iint_R (u_{xx}^2+u_{yy}^2)dxdy / \iint_R (u_x^2+u_y^2)dxdy$, where u solves (*), $\lambda_h/\lambda \leqq 1 - ah^2/12 + o(h^2)$. Application: for all sufficiently small h the ordinary finite-difference approximation λ_h is a rigorous lower bound for λ. Since $\lambda_h \to \lambda$ as $h \to 0$, for all sufficiently small h these λ_h can be teamed with upper bounds from other methods [e.g., Rayleigh-Ritz; or Pólya, C. R. Acad. Sci. Paris vol. 235 (1952) pp. 995–997] to bound λ arbitrarily closely. In proving the theorem one sets the u solving (*) into the Rayleigh quotient of quadratic forms, $N(v)/D(v)$, whose minimum is λ_h, and compares $N(u)/D(u)$ with $-\iint_R u(u_{xx}+u_{yy})dxdy / \iint_R u^2 dxdy = \lambda$. (Received March 9, 1953.)

510. Samuel Karlin: *On the renewal equation.*

A unified approach to the renewal equation $u(x) = \int_{-\infty}^{\infty} u(x-t)dF(t)+G(x)$ allowing negative arguments is presented. Only special examples of this general case have been treated thus far. This includes both the discrete and continuous equations. All the asymptotic relations are reduced to the general Wiener Tauberian theorem. Various rates of convergence of $\lim_{x\to\infty} u(x)$ and $\lim_{x\to-\infty} u(x)$ are obtained by assuming existence of higher moments of F and G. This presentation includes as special cases the recent results of papers of Chung and Wolfowitz (Ann. of Math. (1952)) and Chung and Pollard (Proc. Amer. Math. Soc. (1952)). Finally, limiting theorems for the renewal equations in higher dimensions are obtained and the treatment is generalized to include renewal type theorems involving groups and semi-groups. (Received April 13, 1953.)

511. Joseph Lehner: *Note on the Schwarz triangle functions.*

The triangle functions referred to are those that map a certain curvilinear triangle on a half-plane; they are automorphic on the group $\Gamma(\lambda)$ generated by $S . z \rightarrow z+\lambda$, $T: z \rightarrow -1/z$,where λ takes one of the values $\lambda = 2 \cos \pi/q$, $q = 3, 4, 5, \cdots$. If $\phi_\lambda(z)$ is a triangle function, it will have a simple pole at $z = i\infty$ and be otherwise regular. We normalize $\phi_\lambda(z) = x^{-1} + \sum_{n=0}^{\infty} c_n(\lambda) x^n$, where $x = \exp 2\pi i z/\lambda$. We prove: I. *For every* λ *and every* $n > 0$, $c_n(\lambda)$ *is a rational number*. II. $c_n(\lambda) \sim (2\lambda)^{-1/2} \cdot n^{-3/4} \cdot \exp 4\pi n^{1/2}/\lambda$, *as* $n \rightarrow \infty$. I is proved by considering the inverse of ϕ, which is known to be the quotient of two solutions of the hypergeometric differential equation (with parameters depending on λ). II follows from the Hardy circle method, where only the largest arc in the dissection of the circle is treated. (Received February 27, 1953.)

512. R. B. Leipnik: *Heaviside calculus of abstract power series.*

The notation of a powered set is defined. The group algebra of unconditionally convergent power series over a powered set is shown to be a powered set by validating some formal manipulations due to Heaviside. Applications to Heaviside calculus are obtained. (Received March 12, 1953.)

513*t*. B. L. McAllister and C. J. Thorne: *Reverse derivatives and reverse differential equations.*

A formula for the nth derivative of y with respect to x is given as a function of derivatives of x with respect to y. If, in a solvable differential equation of order 2 or larger in y, the derivatives are replaced by reverse derivatives, the new differential equation in x has the same solution, in general. The new differential equation is usually nonlinear. The use of this method on the nonlinear differential equations presently in the literature usually leads to a form of Abel's equation. Occasionally this leads to solutions; more often, it shows that the solution to the nonlinear differential equation has movable singularities. That is, the singularities are functions of the initial conditions. (The results presented in this paper were obtained in carrying out Contract No. N7-ONR-45107 between the office of Naval Research and the University of Utah.) (Received March 2, 1953.)

514*t*. T. S. Motzkin and J. L. Walsh: *Least pth power polynomials on a complex finite point set.*

Let E be the set of m distinct complex points $\{z_1, z_2, \cdots, z_m\}$, and for variable ζ_ν let the polynomial $T_n(z) = \prod_1^n (z-\zeta_\nu)$ minimize $\sum_1^m \mu_\nu |T_n(z_\nu)|^p$, $\mu_\nu > 0$, $0 < n < m$. Let any polynomial with no zero on E be called *proper*. Let $S_n(z) = z^n + \cdots$ denote a ("stationary") polynomial such that $\sum_1^m \mu_\nu |S_n(z_\nu)|^{p-2} S_n(z_\nu) \bar{u}(z_\nu) = 0$ for every polynomial $u(z)$ of degree $n-1$. For each $p > 0$ and arbitrary μ_ν the set of proper stationary polynomials is precisely the set of proper extremal polynomials in the sense of Fekete (1951) ($p = \infty$). For $p > 1$ and given μ_ν the polynomials $S_n(z)$ and $T_n(z)$ are unique and coincide. For $p \leqq 1$ and $\mu_1 > \sum_2^m \mu_\nu$, every $T_n(z)$ has a zero in z_1. For $p = 1$ and given μ_ν the $T_n(z)$ form in the $2n$-space of coefficients a convex polyhedron P of dimension $s \leqq n$. For $s = n$ and noncollinear z_ν, P is a simplex. If a proper $T_n(z)$ (or equivalently $S_n(z)$) exists, then $s \geqq 2n - m + 1$. For $n = m - 1$, set $\omega(z) = \prod_1^m (z - z_\nu)$

and suppose $\mu_1|\omega'(z_1)| = \mu_2|\omega'(z_2)| = \cdots = \mu_r|\omega'(z_r)| < \mu_{r+1}|\omega'(z_{r+1})| \leq \cdots \leq \mu_m|\omega'(z_m)|$; then P is precisely the simplex $\omega(z)\sum_1^r \lambda_i/(z-z_i)$, $\sum\lambda_i = 1$, $\lambda_i \geq 0$. (Received December 9, 1952.)

515. R. M. Redheffer: *Even canonical products.*

Let $F(z) = \prod(1-z^2/\lambda_n^2)$, $\lambda_n > 0$, $\Lambda(u) =$ number of λ's less than u. Then the following statements are equivalent: (a) $\limsup \log |F(x)|/x = 0$ and $\limsup \Lambda(u)/u = D < \infty$; (b) $\limsup \log |F(re^{i\theta})|/r = \pi D'|\sin\theta|$ for some $D' < \infty$ and all θ. Also the following statements are equivalent: (a) $\lim \Lambda(u)/u = D$; (b) Given $\epsilon > 0$ there is a constant $C = C(\epsilon)$ such that for $\pi - \epsilon \geq |\theta| \geq \epsilon$ one has $\pi Dr|\sin\theta| - \epsilon r \leq \log |F(re^{i\theta})| \leq \pi Dr|\sin\theta| + \epsilon r$ whenever $r > C$. The latter generalizes to complex λ_n, to give the following: Let $F(z) = \prod(1-z^2/\lambda_n^2)$, $\lim n/\lambda_n = D$, let $G(z) = \prod(1-z^2/\mu_n^2)$, $\lim n/\mu_n = D' \neq D$. Then the roots $\pm z_n$ of $F(z) = G(z)$ satisfy $\lim n/z_n = \max(D, D')$. (Received February 12, 1953.)

516. Edgar Reich: *An inequality for subordinate analytic functions.*

Let $f(z)$ and $F(z)$ be nonconstant analytic functions, regular in the unit circle, $|z| < 1$, and suppose that f is subordinate to F (J. E. Littlewood, *Lectures on the theory of functions*, London, 1944, pp. 163–171). Let $a(r)$ and $A(r)$ be the areas of the regions on the Riemann surfaces onto which the disk $|z| < r$ is mapped by f and F respectively. It is shown that $a(r)/A(r) \leq \max_{k=1,2,\cdots}(kr^{2k-2})$ $(0 < r < 1)$, where the inequality is the best possible one in the sense that for any fixed r in the range $(0, 1)$ there exist functions $f(z)$ and $F(z)$ satisfying the hypotheses, such that the inequality becomes an equality. The above result supplements one of G. M. Goluzin [Mat. Sbornik N.S. vol. 29 (1951) pp. 209–224]. (Received March 20, 1953.)

APPLIED MATHEMATICS

517t. K. S. Miller and R. J. Schwarz: *On the interference of pulse trains.*

In various electronic applications one frequently has to deal with trains of periodic pulses. This paper considers the problem of determining explicit formulas for the coincidence time of two or more pulse trains of arbitrary period and pulse width. Formulas for the fraction of time two pulse trains (with arbitrary initial phase) are coincident for a duration greater than a prescribed length of time are also deduced. Multiple pulse trains are briefly considered. Number theoretic methods are employed. (Received March 23, 1953.)

GEOMETRY

518. Douglas Derry: *On differentiable closed curves of order n in n-space.*

Let C_n be a closed curve in real projective n-space for which osculating k-spaces (k, s), $0 \leq k \leq n-1$, are defined for every curve point s of C_n. Those $n-2$-spaces L which contain no points within n distinct $(n-1, s)$ are shown to be exactly those $n-2$-spaces L for which every hyperplane through L cuts C_n in n points. A characterization of such $n-2$-spaces is given. The result is applied to obtain characterizations of those points of the projective space through which less than n $(n-1, s)$ pass. (Received March 9, 1953.)

519. T. S. Motzkin and C. B. Tompkins: *Boundedness of sequential projections.* Preliminary report.

A convex set S is *polyhedral at infinity* if S is the union of a bounded convex set and of the intersection of a finite number of half-spaces. It is shown that for finitely many closed convex sets S_i in an inner-product space that are polyhedral at infinity there exists a nonvoid bounded closed convex set C such that for every point P of C also the projection P_i of P on S_i (that is, the nearest point of S_i to P) belongs to C for each i. It follows that the set of all points obtained from a given point P by projection on some S_{i_1}, from the resulting point on S_{i_2}, and so on, is bounded. The same is true if, in all former statements, P_i is replaced by $P+\lambda(P_i-P)$, $0\leqq\lambda<2$. Applications are given to the solutions of systems of linear equations and inequalities. The theorems subsist for general strictly convex Banach spaces of dimension 2, but not of higher dimension. (Received May 1, 1953.)

520. T. K. Pan: *Relative first curvature and relative parallelism in a subspace of a Riemannian space.*

Let V_n be a subspace of a Riemannian space V_m. Associate to each point of V_n a set of $m-n$ arbitrary independent unit vectors in V_m, which are functions of the point and the direction at the point of V_n and which are not in V_n except in the asymptotic directions of V_n. With respect to these vectors, relative first curvature, relative associate curvature, pseudogeodesics and relative parallelism are defined in V_n in V_m to generalize the concepts and results obtained by the author in a hypersurface (Bull. Amer. Math. Soc. Abstract 58-6-699). (Received March 16, 1953.)

LOGIC AND FOUNDATIONS

521. Kurt Bing: *Definability in the theory of integers.* Preliminary report.

The following theorems are proved: Addition and multiplication of integers are definable in terms of lower predicate calculus without identity and a single symmetric dyadic relation. Addition and multiplication of integers are not each definable in terms of lower predicate calculus (or even second order predicate calculus) and monadic relations. For each sequence of relations of integers there exists a single symmetric dyadic relation H of integers such that each relation of the sequence is definable in terms of lower predicate calculus without identity and H. Addition and multiplication of integers are definable in terms of a single operator and a single symmetric dyadic relation. These results are extensions, to the theory of integers, of theorems proved by Church and Quine (J. Symbolic Logic vol. 17 (1952) p. 186), Craig and Quine (ibid. p. 188), and Myhill (J. Symbolic Logic vol. 15 (1950) p. 130) for the theory of natural numbers. (Received March 11, 1953.)

522. R. L. Vaught: *Applications of the generalized Skolem-Löwenheim theorem to problems of completeness and decidability.*

For terminology see Tarski, Proceedings of the International Congress of Mathematicians, 1950, vol. 1, pp. 705 ff. The following theorem easily follows from the generalized Skolem-Löwenheim theorem (stated in op. cit., Theorem 23): If K is a non-empty arithmetically closed class of algebras (with at most denumerably many relations and operations) such that for some infinite power, all members of K with that power are isomorphic, then all members of K are arithmetically equivalent. A metamathematical consequence of this result is: If T is a consistent theory formalized within first order logic such that every model of T is infinite, and for some infinite power, all models of that power are isomorphic, then T is complete; if, moreover, T has a general recursive axiom system, then T is decidable. This immediately implies the completeness and decidability of various theories for which these results were originally established by other methods. Examples: the theories of densely ordered sets without end points, of atomless Boolean algebras, of infinitely divisible torsion-free Abelian groups, of Abelian groups with all elements of a fixed prime order, and of algebraically closed fields. The ordinary Skolem-Löwenheim theorem (asserting the existence of countable models) was previously used in completeness proofs by A. Robinson and Tarski. (Received March 11, 1953.)

STATISTICS AND PROBABILITY

523. Edwin Hewitt and L. J. Savage: *Symmetric measures on Cartesian products.*

Two theorems of B. de Finetti (Ann. Inst. H. Poincaré vol. 7 (1937) pp. 1–68) are generalized as follows: Let X be a compact Hausdorff space, and let Q be the Cartesian product of $\aleph_0$ replicas of X. For ϕ a countably additive probability measure on X, let π_ϕ denote the product measure on Q obtained by taking each factor-measure equal to ϕ. Let ψ be an arbitrary countably additive probability measure on Q invariant under all permutations of a finite number of co-ordinates. Then there exists a Borel measure Ψ on the space Π of all measures π_ϕ (which is made into a compact Hausdorff space in a certain way) such that $\psi=\int_\Pi \phi_\pi d\Psi(\phi_\pi)$, the integral being of Gel'fand-Pettis type. Uniqueness is obtained under certain additional assumptions. Related theorems are proved for locally compact X and for general measurable spaces X, with finitely additive measures. (Received March 13, 1953.)

TOPOLOGY

524t. C. E. Burgess: *Collections and sequences of continua in the plane.*

The following theorems are proved. (1) If, in the plane, W is a collection consisting of n mutually exclusive bounded continua and G is a collection consisting of n^2-n+1 mutually exclusive continua each intersecting every set of W, then some continuum of G contains a bounded continuum which intersects every continuum of W. (2) If $\{M_n\}$ is a sequence of mutually exclusive nondegenerate continua in the plane converging to a continuum M, then there is a sequence $\{T_n\}$ of bounded continua converging to M such that for each n, T_n is a subset of M_n and is irreducible between some two points. This generalizes a result previously obtained by the author [*Continua and their complementary domains in the plane*, Duke Math. J. vol. 18 (1951) pp. 901–917, Theorem 5]. (3) Let K be a continuum having the property that for some positive integer n there exist n domains intersecting K such that no bounded subcontinuum of K intersects each of these domains. The plane does not contain uncountably many mutually exclusive continua such as K. (Received March 6, 1953.)

525. A. V. Martin: *Decompositions and quasi-compact mappings.*

A decomposition of a topological space M is any collection G of nonempty, pairwise-disjoint sets (not necessarily closed) of M whose union is M. The hyperspace determined by G is defined in the natural way. It is easy to prove that a topological space X is the image of M under a quasi-compact mapping (for definition see Whyburn, *Open and closed mappings*, Duke Math. J. vol. 17 (1950) pp. 69–74) if and only if X is homeomorphic to the hyperspace of some decomposition of M. Let M be a separable metric space and let f be a quasi-compact mapping of M onto a topological space X. The following results are proved: If M is locally compact and X is metric, then X is separable and locally compact. If f is monotone and M is locally compact and X is Hausdorff, then X is separable, metric, and locally compact. If f is closed, and if for every x in $X, f^{-1}(x)$ is compact, then X is separable and metric. (Received March 16, 1953.)

J. W. GREEN,
Associate Secretary

EDWARD VERMILYE HUNTINGTON
1874–1952

Edward Vermilye Huntington was born in Clinton, New York, on April 26, 1874. He was graduated from Harvard University in the class of 1895, a class which numbered among its members two other mathematicians, J. L. Coolidge and J. K. Whittemore. Two years later he received the A.M. degree from Harvard, serving also as Instructor in Mathematics during those two years. Two years followed at Williams College as instructor in mathematics. 1899 to 1901 were spent in study in Europe, culminating in the Ph.D. degree from Strassburg, on a topic dealing with the foundations of the number systems, thus foreshadowing the field of his later major interest. He returned to Harvard, where he remained until his retirement in 1941, passing through the then conventional stages of advancement until his appointment in 1919 as Professor of Mechanics under the Faculty of Arts and Sciences, an appointment probably unique under that Faculty. From 1926 on he served also as Tutor in the Department of Mathematics. During the first world war he served in uniform in Washington, as Major on statistical duty for the General Staff. During the first half of 1925–26 he held what was known as the Exchange Professorship to the Western Colleges, and in the second half of the same year he attended, as delegate of the American Mathematical Society, a scientific congress in Lima, Peru, where he received the honorary Sc.D. degree from the University of San Marcos. After his retirement he was able, to his great gratification, to render valuable services during the second world war as Consultant in Research to the National Defense Research Committee on various confidential problems. In the last years of his life he waged with fortitude a foredoomed battle against the encroachments of paralysis agitans, supported by the devoted ministrations of his wife. He finally succumbed on November 25, 1952.

(This article is a re-publication of an article which appeared in the Harvard University *Gazette*. It was prepared by a committee consisting of Professors P. W. Bridgman, Chairman, W. V. Quine, J. H. Van Vleck, D. V. Widder, and was spread upon the records of the Harvard faculty in recognition of Professor Huntington.)

BOOK REVIEWS

Linear algebra and projective geometry. By R. Baer. New York, Academic Press, 1952. 8+318 pp. $6.50.

A half-century or so has elapsed since the great treatises on foundations of geometry appeared. In these works it was taken for granted that the reader could stand any amount of intricate geometrical reasoning; but the idea that geometry could be done over an arbitrary division ring was quite a novelty and was accordingly treated with respect and caution. In the present book we see how much the mathematical climate has changed. The necessity of grappling with an arbitrary division ring should be the least of the reader's worries. Nor will a lack of geometric intuition seriously impede him (it is interesting to note that there are only nineteen figures in the book, of which two are non-geometrical and the last seven are concerned with the introduction of coordinates). But a good backlog of experience with the trickery of modern algebra is recommended to any prospective reader.

Let F be a division ring, A a vector space over F, both more or less arbitrary. While it is true that the case of characteristic two often gets "cavalier treatment," possible non-commutativity of F is allowed every scope for its operations. Infinite dimensionality of A is permitted, but plays a subdued role, coming to the reader's attention mainly when considerations of duality make it impossible. With a minimum of delay the object of central interest makes its appearance: the lattice L of subspaces of A. If L_1 is a second such lattice, call a lattice isomorphism between L and L_1 a projectivity. The first fundamental theorem asserts that any projectivity is induced by a semi-linear transformation, provided the dimension of A is at least three.

A lattice isomorphism of L upon itself is an auto-projectivity. In the group of auto-projectivities we can pick out the subgroup of collineations, consisting of those auto-projectivities which can be induced by linear transformations. A pitfall awaits us here, for the same projectivity may be induced by both a linear and a semi-linear transformation, provided the automorphism for the latter is inner. This is the first of many occasions on which inner automorphisms have to be accorded special consideration. It is incidentally gratifying that perspectivities, which play such an inflated part in classical accounts, are here cut down to their proper role of being possible building blocks of collineations. The discussion moves on to the second fundamental theorem (fixing a collineation by its effect on a simplex), Pappus's theorem, and cross ratio.

A lattice anti-isomorphism (order-inverting map) of L upon L_1 is a duality, and a duality of L upon itself is an auto-duality. Chapter IV (the longest in the book) is devoted to their study. The theory of auto-dualities turns out to be almost identical with that of semi-bilinear forms; we say "almost" because the forms $f(x, y)$ and $f(x, y)c$ induce the same dualities. Of major interest are polarities, which are auto-dualities of period two, the corresponding algebraic gadgets being alternate and Hermitian forms. The structure of these forms and of groups over the forms is discussed in considerable detail.

In Chapter V the ring of linear transformations is taken up. This is perhaps somewhat less novel than the remaining chapters, for in the last decade algebraists have been operating successfully in somewhat more general rings. Next the groups of non-singular linear and semi-linear transformations are studied, and their isomorphisms completely determined. The method, originated by Mackey, is to use the elements of order two to reconstruct the vector space.

The final chapter introduces coordinates in a projective geometry, assumed to be Desarguesian if it is a plane. The method is to prove directly that the given geometry can be embedded as a hyperplane H in a larger geometry K, a tour de force which may leave the reader slightly stunned. Then the desired vector space is obtained by manipulating the group of collineations of K leaving H pointwise fixed, an idea stemming from Dehn and Artin. Happily, the author concludes with some indications of how one can detour around the embedding theorem.

The author has done much more than collecting and proving elegantly theorems already in the literature. Here is just a sample of important results difficult or impossible to locate elsewhere. (1) If A is a c-dimensional vector space over a division ring with d elements, then the dimension of the full dual of A is d^c. (2) The collineation group is generated by the projectivities which have a line of fixed points. (3) Even without commutativity, cross ratio is definable up to an inner automorphism. The formula is

$$(\boldsymbol{h}-\boldsymbol{n})(\boldsymbol{k}-\boldsymbol{n})^{-1}(\boldsymbol{k}-\boldsymbol{m})(\boldsymbol{h}-\boldsymbol{m})^{-1}.$$

(4) The generalized theorem of von Staudt-Ancochea: if a mapping on a line preserves one central cross ratio, other than 0 or 1, it is induced by an automorphism or anti-automorphism. (5) An auto-duality is of period two if and only if the corresponding semi-bilinear form has the property that orthogonality is symmetric; and the form may then be normalized to be alternate or Hermitian. (6) Sylvester's theorem of inertia, generalized to any division ring with

suitable order properties. (7) Pascal's theorem holds only for the usual symmetric forms and in particular requires commutativity (conic sections make a fleeting appearance at this point in the book). (8) Determination of the two-sided ideals in the ring of all linear transformations, there being one for each cardinal number.

There are two generalizations which the author calls to the reader's attention. The first is to replace the vector space by a suitable kind of module over a ring; in a series of papers the author himself has carried this program nearly to completion. The second is to replace the vector space (implicity paired to its full dual) by an arbitrary pair of dual vector spaces; here there has been substantial work by Mackey and Rickart. Further in the distance lies the project of uniting these two generalizations by studying dual modules. In yet a different direction lie the still largely mysterious rings and lattices without minimal elements, typified by von Neumann's continuous geometry. So there is much to be done; and the coming generations of young algebraists, with this book happily tucked under their arms, will find the path well laid out.

I. KAPLANSKY

Calculus of variations with applications to physics and engineering. By Robert Weinstock. New York, McGraw-Hill, 1952. 10+326 pp. $6.50.

This book, which appears in the International Series in Pure and Applied Mathematics, has been written to fill the need for an elementary introduction to the calculus of variations, followed by extensive applications to physics and theoretical engineering. By far the greater emphasis is placed on the applications, and the list of chapter headings will show the scope: 1. Introduction; 2. Background preliminaries; 3. Introductory problems; 4. Isoperimetric problems; 5. Geometrical optics: Fermat's principle; 6. Dynamics of particles; 7. Two independent variables: the vibrating string; 8. The Sturm-Liouville eigenvalue-eigenfunction problem; 9. Several independent variables: the vibrating membrane; 10. Theory of elasticity; 11. Quantum mechanics; 12. Electrostatics.

A book with this scope should have a wide appeal at the present time, particularly among those physicists and engineers who find variational methods tricky and evasive. For the author's aim is clarity of exposition. He goes slowly at the beginning, where slowness is essential, and he provides, at the ends of the chapters, sets of exercises which should prove very useful. He is writing for those who know the concepts and techniques of a first year calculus course, including a smattering of ordinary differential equations, and who are also familiar with many of the matters encountered in a short course on advanced calculus. To assist the reader, the second chapter contains a list of essential formulae, with derivations partially indicated.

I regard this as a very useful book which I shall refer to frequently in the future. It contains a wealth of material, and the inclusion of elasticity and quantum mechanics in one volume marks a breaking down of barriers, greatly to be desired.

In tempering this praise with some adverse criticism, I would like as far as possible to distinguish between criticism stemming from personal prejudice and criticism called forth by actual errors in the book. Errors are easy to deal with. They creep in somehow or other, hide when the author reads the proofs, and then shame him on the printed page. The court is indulgent, merely sentencing the author to correct them in the next edition. Here are a few things that should be otherwise:

p. 9. The wronskian is shown as a determinant with $(n+1)$ rows and n columns, and it is stated wrongly (cf. E. L. Ince, *Ordinary differential equations*, London, 1927, p. 116) that the vanishing of the wronskian is a sufficient condition for the linear dependence of n solutions of a linear homogeneous differential equation of order n.

p. 44. For Schwartz read Schwarz.

p. 89. This extension of Hamilton's principle to a system without a potential function is a muddle. For a correct treatment, see E. T. Whittaker, *Analytical dynamics*, Cambridge, 1927, p. 248.

p. 149. Some clumsy work here in the replacement of a line integral by the double integral of a divergence over the contained area, confusing since ds/dx becomes infinite twice at least on going round a closed curve. A couple of pages here could with advantage be compressed into a clearer treatment contained in a few lines.

p. 164. The non-degeneracy of eigenvalues for a rectangular membrane of sides a and b depends on the irrationality of (a^2/b^2), not (a/b).

Let me now turn to criticism based on personal prejudice. I like simple English—the simpler the better. Therefore I criticize a sentence such as the following (p. 7):

"Quite often involved in the integrand of a line integral taken about a simple closed curve C in the xy plane is the *normal derivative* of a function $w(x, y)$."

But, happily, this is not a typical sentence. On the whole the style is clear enough, although loaded a bit too much with long words where short ones would have done the job better.

But that is not important. In the category of prejudiced criticism I have two important and interesting matters to discuss. These issues are: (i) the δ method in the calculus of variations, (ii) mathematics by authority, or "we state without proof." The second arises in connection with expansions in terms of eigenfunctions, and plays a fundamental part in Chapters 7 and 9.

The author abjures the "vague mechanical δ method," and uses instead what may be called the ϵ method. The difference is this: in the former the variation of y is δy, and in the latter it is $\epsilon\eta$. What do we gain by the use of the ϵ method? A great deal, I readily admit, if whenever we see the symbol δ we mutter to ourselves archaically "An infinitely small quantity." But we need not do so. We can rationalize the δ of the calculus of variations in the same way as we rationalize the d of calculus, and so enjoy a notation which is precise, suggestive, and economical. It is true that in this process of rationalization we would bring in ϵ, and we would keep it permanently in the backs of our minds. But we would not have to drag it into situations already complicated enough.

Mathematical notations exist to save thought by substituting for it mechanical processes which can be carried out automatically. But we do need, at any given moment, to be able to snap out of our trance and know what we are really doing. So, in the calculus of variations the mathematical physicist needs two wives, δ to relax with and ϵ to tell him where he gets off. He will give up the latter as soon as he understands the former.

He will never give up δ, and I very much fear that the author's harsh rejection of δ will limit the appeal of his book among those who might profit most from it. With no δ, we cannot express Hamilton's principle as $\delta\int L dt = 0$; we have to make a sentence out of it. Indeed, the whole of Chapter 6, devoted to Hamiltonian theory, cries aloud for δ, not only for economy in writing but to express the basic structure of the theory. The author seems to lose sight of the simplicity obtained by taking

$$\delta \int \left(\sum p_i dq_i - H dt\right) = 0$$

as the central law, with free variations of p_i and q_i except for $\delta q_i = 0$ at the ends; and the whole of p. 87 may with advantage be compressed into two lines if we recognize the principle of least action as

$$\delta \int \sum p_i dq_i = 0, \quad H(p, q) = E.$$

Only time will tell. My bet is that two centuries hence δ and d will still be going strong, unless nature happens to find out about second variations; but she has not yet, to any serious extent, and so the author very wisely excludes them from his book.

And now to the other matter—mathematics by authority. On p. 101 we read:

"We state without proof the following theorem concerning the expansion of an arbitrary function in terms of the known set of eigenfunctions:

"If the arbitrary function $g(x)$ is piecewise continuous and piecewise differentiable in $0 \leqq x \leqq L$, the series

$$\sum_{n=1}^{\infty} c_n\phi_n(x), \quad \text{with} \quad c_n = \int_0^L \sigma\phi_n g\,dx,$$

converges uniformly to $g(x)$ in every subinterval of $0 \leqq x \leqq L$ in which $g(x)$ is continuous. We may therefore write

$$g(x) = \sum_{n=1}^{\infty} c_n\phi_n(x) \qquad \left(c_n = \int_0^L \sigma\phi_n g\,dx\right). \tag{29}$$

Moreover, in any subinterval in which $g'(x)$ is continuous, we may differentiate (29) term by term to obtain

$$g'(x) = \sum_{n=1}^{\infty} c_n\phi_n'(x), \tag{30}$$

and the convergence is uniform. (Possible exceptions at $x=0$ and $x=L$ are mentioned below.)"

Here is a situation not uncommon. An author wants to use a technique the validity of which he cannot establish in the text; it would take him too far afield. So he says: "We state without proof . . . " Thus he commits one sin (imposing mathematics by authority, "teacher knows best") to avoid another (incomplete proof or proof by analogy). For my part, I would choose the latter every time. A statement is set down before me, and it is I and no one else who must put *true* or *false* opposite it in the recesses of my own mind. If the statement is too wide for me to handle in all its generality, then I shall test it in particular instances which lie within my scope.

That is precisely what I did for the statement quoted above. I took $g(x)=x$ in the range $(0, \pi)$ and expanded it in a series of sines. Then I differentiated term by term, and got a series which did not converge for any value of x in the range. Thus, by quite elementary methods, well within the scope of the potential reader of the book, I was able to satisfy myself that either (a) the statement is false, or (b) I do not understand what it means.

I believe that there is a rather bad error here, which the author would do well to correct if he gets the chance in a later edition—maybe he will have to do quite a bit of rewriting. But even if the error is corrected, that does not answer the question: Shall we impose mathematics by authority? My own view is that we should not. Authoritative statements cannot be completely avoided, but they should be supported by plausibility-arguments and by the working out of special cases within the scope of the reader. This takes space, but it is space well used if one thereby establishes confidence and a sense of reality. In the last analysis, it is the special case that establishes confidence, not the general theorem, and this holds for everyone, high and low.

J. L. SYNGE

Les nombres inaccessibles. By É. Borel. Paris, Gauthier-Villars, 1952. 10+141 pp. $3.72.

The author prefaces the work under review as follows:

"This little book is the result of half a century of reflections on the principles of mathematical analysis and, in particular, on the definition of numbers. Some of these reflections have already been sketched here and there in the works of this Collection, but it seemed to me that it would be useful to coordinate them in a connected account.

"The profound transformations of physics in the twentieth century, and especially the theories of relativity, quanta, and wave mechanics, have been inspired by the fundamental idea that phenomena must be observed *en eux-mêmes*, without taking account of *a priori* conceptions such as time, space, matter, or energy—conceptions with which one has associated absolute and immutable entities.

"It seems to me that mathematicians as well, while maintaining the full right to work out abstract theories deduced from arbitrary noncontradictory axioms, have an interest in distinguishing, among the objects of thought which are the substance of their science, those which are truly accessible, that is to say, have an individuality, a personality, which characterizes them without ambiguity. One is thus led to define in a precise manner a science of the accessible and of the real, beyond which it remains possible to develop a science of the imaginary and of the imagined, these two sciences being able, in certain cases, to lend each other mutual support.

"Such is the spirit in which I have written this book, which I submit to the reflections of the young mathematicians whose efforts will contribute to the orientation towards perpetually new developments, of the science which is the starting point of all the others, and which will undoubtedly continue for a long time to be the vital source whence the progress of our knowledge and the perfection of our techniques will spring."

Borel's thesis is that the overwhelming majority of numbers will always remain inaccessible to the human race as we know it, in the sense that it will never be possible to define these numbers effectively in such a manner that any two mathematicians will be certain that they are speaking about one and the same entity. Indeed, some integers of one hundred digits, even if written down, will never be "known," because of the human impossibility of discovering any but trivial consequences of their definition. The boundary between the accessible and the inaccessible, however, is itself inaccessible. The properties of a set whose elements are inaccessible can be studied—even though it is impossible to study the elements individually—but only by means of the calculus of probabilities.

The author is thus naturally led to discuss systems for representing numbers, methods of defining numbers, enumerable sets and the continuum in relation to measure and probability, and the axiom of choice.

Many parts of the book, e.g. the chapter on systems of numeration, are delightful because of interesting remarks which lend a surprisingly new aspect to simple, familiar mathematical objects. The somewhat novelistic style of writing makes superficial reading easy, penetrating reading more difficult. A definition, for instance, given merely by analogy with some previously described examples is not an aid to understanding an inherently difficult subject, and may actually serve to cast suspicion on the clarity of the motivating idea.

Readers will find some of Borel's assertions provocative, and this is probably one of the chief merits of the book: it contains material for thought and discussion. Several statements in the above-quoted preface alone, e.g., the comment about physics, would serve (and, in fact, have served the author himself) more appropriately as subjects for another book rather than for brief evaluation in a book review. We shall therefore leave aside for the most part the broader philosophical aspects of the volume. The ensuing remarks are devoted mainly to pointing out what appear to be some specific inconsistencies.

Borel infers from the homogeneity of the continuum that two equal subintervals of the unit interval are indiscernible, and that therefore the probability that a point chosen at random in the unit interval fall in one subinterval is equal to the probability that it fall in the other, and, further, is equal to the length of the subinterval. It is difficult to see how he can consider two equal intervals indiscernible if, say, the left end point of one is accessible and that of the other is inaccessible. Nor does he give adequate reason for assigning the same probability to two equal subintervals if, e.g., one is near the origin and the other is near the center of the unit interval, especially in view of the fact that he interprets choice and probability in the naïve sense.

If an enumerable set, such as that of the natural numbers, is considered instead of the unit interval, then the assignment of equal probabilities to the elements of this set reduces to zero the global probability of any number of accessible integers, which, according to Borel, is absurd because it precludes the possibility of ever getting one of these numbers, so that every choice leads to an inaccessible number. This interpretation of probability zero as an expression of impossibility is, however, not justified, and the author conveniently avoids it when admitting that the accessible points of the unit interval have probability zero.

Borel proposes to assign a *positive* probability to *each* element of an enumerable set in such a manner that these probabilities form a convergent series whose sum is unity. He admits that no simple and natural method for doing this presents itself, but argues that there are many arbitrary conventions for making such an assignment (e.g., one under which probability 1/2 is assigned to the set of accessible integers as well as to the set of inaccessible integers) and that, in many problems, which particular assignment is selected does not influence the conclusion drawn. Now in many applications of Zermelo's axiom of choice it is also true that the conclusion drawn does not depend on the particular choice function, but Borel ignores this fact: He interprets Zermelo's axiom as affirming that it is possible to choose (in the literal sense) a definite number from, e.g., the set of inaccessible integers, so that if this number is denoted by a, then a designates a well-determined number, the same for all mathematicians. He then regards this as a meaningless operation because a is not, and never will be, distinguished from the other inaccessible numbers. It is not clear why he does not regard the assignment of a positive probability to every inaccessible a, or of probability 1/2 to the set of inaccessible integers, as an equally meaningless operation, inasmuch as the set of inaccessible integers is no more well-determined than the a referred to above in connection with Zermelo's axiom.

Borel discusses the familiar decomposition of the circumference of a circle into an enumerable number of mutually exclusive, congruent sets of points. He asserts that it is not possible to attribute equal probabilities to these sets without running into contradictions, and that it is therefore necessary to attribute unequal probabilities to them. "But then we contradict the Euclidean principle of equality, according to which two superposable figures are equal." As the construction of these sets "requires the use of Zermelo's axiom, our conclusion is that it is necessary to choose between Zermelo's axiom and Euclid's axiom according to which two superposable figures are equal, that is to say, identical from all points of view, and that, in particular, equal probabilities correspond to them. The simultaneous application of the two axioms leads, in fact, to a contradiction." (Borel, needless to say, chooses "Euclid's axiom.") This argument is open to objections. First, it is not inconceivable that a nonmeasurable set can be constructed without the intervention of Zermelo's axiom. Secondly, there is another way in which two superposable figures may be "identical from all points of view," without having equal probabilities correspond to them, and that is, by having *no* probabilities correspond to them. Euclid's axiom cannot be interpreted as stating that congruent figures—if the "figures" in question are not the elementary Euclidean ones—*have probabilities* and that these probabilities are equal. The author seems to be taking account here of just such an *a priori* conception—probability—as he implies opposition to in his preface.

It is undoubtedly valuable to have in one volume the ideas and opinions of so famous a mathematician on such an important and controversial subject as the foundations of analysis. The history of mathematics shows, however, that what is considered real or imaginary is not an absolute concept, but is relative to the development of our knowledge, and furthermore, that "abstract theories" often throw sufficient light on objects of thought to transfer them from the realm of the imaginary to that of the real. Complex numbers were once considered meaningless, whereas today some mathematicians consider Zermelo's axiom and its consequences meaningless. Borel's notion of accessibility, although of heuristic significance, seems too subjective, temporal, and, by precluding intrinsically the possibility of delimiting the realm of the accessible, vague, according to his own standards, to "define in a precise manner a science of the accessible and of the real."

F. BAGEMIHL

General topology. By W. Sierpiński. Trans. by C. C. Krieger. (Mathematical Expositions, No. 7.) University of Toronto Press, 1952. 12+290 pp. $6.00.

This book retains many of the characteristics that brought popularity to the author's earlier book, *Introduction to general topology*. However, it has a different axiomatic treatment and is much enlarged so that it could be classified as more than a revision. Readers who have objected to the corruption of the vocabulary and notation of topology by those who have undertaken to rewrite it may find it refreshing to find the way in which such expressions as compact, bicompact, basis, +, $\sum, \prod$ are used.

Chapter I treats Fréchet (V) spaces, that is, spaces in which each point is associated with one or more subsets called neighborhoods but these neighborhoods need not satisfy any additional conditions. Topological spaces are studied in Chapter II when it is supposed that the neighborhoods satisfy additional conditions. It is of interest that in the cartesian product of topological spaces, a neighborhood is defined to be the cartesian product of neighborhoods. Topological spaces with a countable basis are considered in Chapter III, while Chapter IV treats Hausdorff spaces with such a basis. Chapter V deals with the properties normality and regularity.

An extended treatment of metric spaces is given in Chapters VI and VII. A large part of the study of topology is devoted to the important metric spaces and Sierpiński devotes over one-half of his book to them. In the study of complete metric spaces in the last chapter, particular attention is given to various kinds of their subsets, such as those that are F_σ, G_δ, Borel, or analytic sets. The treatment is more extensive than that in the corresponding chapter of the *Introduction*. The Appendix is much the same as in the *Introduction*.

It is logically satisfying to give a sequence of axioms and definitions and prove theorems on the basis of these without appealing to the intuition. However, such a sequence of axioms may omit some of the spaces that are of interest to topologists. Nevertheless, this book deals with many of the spaces usually studied. A feature that makes it especially suitable as a text is the collection of problems and theorems left as exercises in each chapter.

R. H. BING

Anfangswertprobleme bei partiellen Differentialgleichungen. By R. Sauer. Berlin, Göttingen, Heidelberg, Springer, 1952. 14+229 pp. 26 DM; bound, 29 DM.

This book is concerned mainly with the initial value problems for partial differential equations or systems of equations of hyperbolic type. Many examples from compressible flow and supersonic flow are given as applications of the mathematical theory. Numerical and graphical methods for obtaining solutions are discussed. These include differencing procedures and lattice constructions. The author observes that he did not find it possible to include the recent work of L. Schwartz. He does include an account of Hadamard's theory but not of the results of M. Riesz.

An introductory chapter discusses the classification of linear second-order partial differential equations, the Cauchy-Kowalewski existence theorem, simple examples of the wave equation and its properties as well as difference equations, and applications to gas dynamics and acoustics. The second chapter is devoted to the equation of the first order, which is treated fully, and concludes with the Hamilton-Jacobi equation and applications to mechanics.

The third chapter treats systems of quasilinear differential equations of the first order and the general second-order equation for the case of two independent variables. Many applications are made. The chapter concludes with Riemann's method of integration for the linear second-order equation. In the last chapter the restriction to two independent variables is removed. Most of the chapter is on the linear case. Huygens' principle and the Hadamard theory are given. A number of applications are made.

N. Levinson

Inequalities. By H. G. Hardy, J. E. Littlewood and G. Pólya. 2d ed. Cambridge University Press, 1952. 12+324 pp. $4.75.

The second edition of this book differs from the first (published in 1934) by the inclusion of three appendices amplifying a few points of the text. The first gives an elementary proof of the Hilbert-Artin theorem concerning the representation of a strictly positive homogeneous polynomial in several variables as the ratio of two sums of squares. The second gives a proof of the Riesz-Thorin theorem about the convexity of the maxima of bilinear forms. The last proves Hilbert's familiar inequality by the elementary method of maxima and minima.

Unfortunately the first edition of the book was not reviewed in this Bulletin. Clearly, there is not much point in writing a detailed review now when the book has been available to a whole generation of analysts, and a few words of comment may suffice.

In retrospect, one sees that "Hardy, Littlewood and Pólya" has been one of the most important books in Analysis during the last few decades. It had an impact on the trend of research and is still influencing it. In looking through the book now one realizes how little one would like to change the existing text, though, not unnaturally, one would like to see the book expanded by the inclusion of new material. Much of this material is already in the book in the form "Miscellaneous Theorems and Examples" at the ends of the chapters, where many results are stated without proofs, or merely with indications of proofs. Today, when the main results of the theory are comparatively familiar, due, to a great extent, to the book itself, a "promotion" of a part of the small type material to a more prominent place and an elaboration of this material would seem desirable. Also, the inclusion of results pertaining to the theory of linear operations would be useful, if only in connection with the work of M. Riesz and Thorin. We now realize their importance, and today, due to the simplifications of the proofs and the further development of the theory of linear operations, such results are much more within the reach of the beginner than they were twenty years ago.

A. ZYGMUND

Theory of elasticity and plasticity. By H. M. Westergaard. (Harvard Monographs in Applied Science, no. 3.) New York, Wiley, 1952. 14+176 pp. $5.00.

It was the intention of the author to write a textbook on elasticity and plasticity, containing, in particular, a unified account of his own researches and of the aspects of the subject to which they pertain. The introduction indicates that the present work corresponds to the first half of the project, and there are passages which probably would have been revised if even this part had not been completed under the pressure of time and illness. The author died in 1950.

The work as it stands refers almost exclusively to elasticity and may be divided roughly into three parts: scope and history, fundamental concepts, inverse and semi-inverse solutions obtained by stress functions and strain functions.

The author's historical remarks, both in his second chapter and in the numerous careful annotations throughout the rest of the work, are drawn almost entirely from his own experience in the literature. They constitute a valuable supplement to what is generally known, particularly since they refer to work whose date, or at least whose main interest, is subsequent to the definitive form of Love's treatise (1906). Many of the authors whose work is discussed are still active (Prandtl, v. Kármán, v. Mises, Mindlin, Nadai, and younger writers).

The last two chapters contain the only systematic exposition in English of the use of stress and strain potentials in elasticity theory. A surprising omission is the work of Neuber (Zeitschrift für angewandte Mathematik und Mechanik vol. 14 (1934) pp. 203–212); there is no reference to the recent Russian inundation. The virtue of Westergaard's presentation lies in his showing the reader the value of general solutions in terms of arbitrary functions and in his ability to adjust these to yield special cases of real interest.

There are numerous interesting exercises, stated mostly without answer. Some will give the student a surprise if he works them (e.g. Problem 6 on p. 114).

The reader accustomed to other engineering texts will find this one rather different. It is free from "approximations," yet many of the pages contain compact, individual, and interesting remarks on practical applications (e.g. pp. 43–45, 98–99). An example of the author's simple and personal manner of relating theory to experience, even in old material, is as follows (p. 80): "The constant μ is called Poisson's ratio; Poisson, in his extensive paper of 1829, presented arguments, later found untenable, that its value should be $\frac{1}{4}$. Good approximate values are: for steel, 0.3; for concrete, 0.2; for cork, close to zero, which is important in the operation of pressing a cork into a bottle; and for rubber, slightly less than 0.5, which makes it desirable to insert a rubber stopper into a bottle by turning it rather than by pressing."

To one accustomed to working in the mathematical theory for its own sake, this expression of a distinguished research engineer's belief in the mathematical method and quiet confidence in the exact solutions of the theory of elasticity will be welcome reassurance, especially at a time when traditional mechanics is besieged on opposite sides by computing machines and existence theorems. Every student of elasticity can read this book easily and with both profit and pleasure.

C. TRUESDELL

Séries adhérentes. Régularisation des suites. Applications. By S. Mandelbrojt. Paris, Gauthier-Villars, 1952. 14+279 pp. 4000 fr.

This book is an account of researches, mainly due to the author, on various problems which may seem unconnected at first sight. Their collection in one book is justified by the fact that all these questions can be treated by two main tools, the regularisation of sequences and Mandelbrojt's "fundamental inequality." This inequality gives a bound for the coefficients d_k of the formal Dirichlet series $\sum d_k e^{-\lambda_k s}$, if it is known that this series représents asymptotically the regular function $F(s)$ with a certain degree of accuracy in a domain Δ of the s-plane (hence the title Séries adhérentes).

The main contents chapter by chapter are: Chapter 1, Regularisation of sequences. Chapter 2, Watson's problem and generalisations. Chapter 3, The fundamental inequality. Chapter 4, Carleman's theorem on quasi-analytic functions, proved by Bang's real variable method as well as by the standard complex variable technique. Generalisations of Carleman's theorem are given in which the conclusion $f(x) \equiv 0$ follows from restrictions on the size of $f(x)$ and its derivatives on an infinite interval together with the hypothesis that $f(0) = f^{(\lambda_n)}(0) = 0$, where $\{\lambda_n\}$ is a subset of the positive integers. Examples show that the results are very close to best possible. Chapter 5, Results on the closure of $\{x^{\lambda_n}/F(x)\}$ in the uniform topology on an infinite interval. Transition to the conjugate space immediately gives theorems on the uniqueness of the generalized moment problem $\int x^{\lambda_n} d\psi(x) = m_n$. Chapter 6, The Cartan-Mandelbrojt solution of the equivalence problem of classes of differentiable functions. Chapter 7, Location of singularities of functions defined by Dirichlet series. Generalizations of Picard's Theorem for such functions.

Most of the results have appeared in previous publications, but the book contains many improvements and completions of earlier work.

The presentation is clear and assumes no specialist knowledge of analysis. It is to be hoped that the book will encourage some mathematicians, who may have been deterred from a study of the original papers by the massive, rather tedious technicalities, to get acquainted with the fine results of the author. The technicalities now take a proportionately much smaller space, even though some tiresome detail might still have been omitted with the loss of some generality, but with a gain of perspicuity. This would have been especially justified by the fact that many of the theorems even at their most complicated just miss being best possible.

Perhaps one might regret that this presentation of the fundamental inequality and its applications has appeared at a time when we seem close to best possible results, but have not quite reached them. On the other hand it may be good that something is left for the readers of this book to do. They can be grateful to the author for giving them this careful account of the full facts of the case.

There are several misprints; most of them very minor.

W. H. J. FUCHS

Deuxième Colloque de Géométrie Algébrique. Centre Belge de Recherches Mathématiques. Liège, Thone, and Paris, Masson, 1952. 243 pp. 2625 fr.

This book gives accounts of the talks presented at the conference held at Liège in June 1952. The contents are: *Courbes de diramation des plans multiples et tresses algébriques*, by O. Chisini; *Quelques travaux récents concernant la classification des courbes algébriques*, by L. Gauthier; *Transformations ponctuelles et transformations crémoniennes*, by M. Villa; *Sur la théorie des corps algébriques*, by E. Kähler; *Formes différentielles méromorphes sur les variétés kählériennes compactes*, by P. Dolbeault; *Problèmes résolus et non résolus de la théorie des fonctions abéliennes dans ses rapports avec la géométrie algébrique*, by F. Conforto; *Les problèmes de classification dans la théorie des surfaces algébriques irrégulières*, by A. Andreotti; *La théorie de la base pour les diviseurs sur les variétés algébriques*, by A. Néron; *La théorie des idéaux et la géométrie algébrique*, by W. Gröbner; *Quelques progrès récents dans la classification des variétés algébriques d'un espace projectif*, by F. Gaeta; *Modèles de surfaces canoniques normales de* S_3 *et de genre linéaire* $11 \leqq p^{(1)} \leqq 17$, by P. Burniat; *Introduction des courbes quasi irréductibles d'une surface algébrique. Application à la régularité de certains systèmes linéaires*, by L. Nollet; and *Les singularités des points de diramation isolés des surfaces multiples*, by L. Godeaux.

Most of the articles can be used to obtain a good general idea of the directions in which their respective authors have been working in the past few years. Some treat specific problems in more or less full detail while others are retrospective or provide outlines of work to appear elsewhere. A few words about some of the more interesting papers: Dolbeault's is essentially a statement of certain existence results on higher degree differential forms of second and third kind. Conforto summarizes the classical theory and some of his own work on theta functions, abelian varieties and their subvarieties, and fundamental domains for equivalence classes of Riemann matrices. Andreotti outlines his work on the classification of irregular surfaces via their natural mappings into their Albanese varieties and ideals of theta functions. Néron provides a readable outline of the main part of his important recent paper (Bull. Soc. Math. France, 1952) in which he proves Severi's theorem of the base.

M. ROSENLICHT

BRIEF MENTION

The theory and applications of harmonic integrals. By W. V. D. Hodge. 2d ed. Cambridge University Press, 1952. 10+282 pp. $5.50.

The first edition (1941) was reviewed in this Bulletin, vol. 50, p. 43. The second edition incorporates some corrections; §26.3, dealing with the relation between products of forms and intersections of cycles, has been replaced by a brief account of de Rham's version of the theory. There is a bibliography of 21 papers on harmonic integrals which have appeared since 1941.

Opere. By L. Bianchi. Ed. by the Unione Matematica Italiana, with the assistance of the Consiglio Nazionale delle Ricerche. Vol. 1, part 1. Rome, Cremonese, 1952. 615 pp. 5000 lire.

Bianchi's works are to appear in 10 volumes. This one contains a bibliography, and papers on finite groups, algebraic equations, theory of numbers, infinite discontinuous groups, and arithmetic forms.

NOTES

The International Congress of Mathematicians 1954 will be held in Amsterdam from September 2 to September 9 under the auspices of Het Wiskundig Genootschap (The Mathematical Society of the Netherlands). Those who wish to attend the Congress are requested to communicate their name (with degrees, qualifications etc.) and full address to the secretariat, 2d Boerhaavestraat 49, Amsterdam, The Netherlands, as soon as possible.

The Board of Directors of the Heineman Foundation for Research, Educational, Charitable and Scientific Purposes, Inc. announce the establishment of a prize to be known as the "Dannie Heineman Prize," in the sum of $5,000, to be awarded every three years to the author of an outstanding book or manuscript in the mathematical or physical sciences. Submissions for the next award shall be made not later than December 31, 1955. Further information may be obtained from the Secretary of the Foundation, 50 Broadway, New York 4, New York.

The American Mathematical Society has received a grant from the National Science Foundation for research in experimental methods for reducing costs of publication and distribution of mathematical papers.

National Science Foundation Postdoctoral Fellowships have been awarded to the following: Assistant Professor N. J. Fine, University of Pennsylvania; Visiting Associate Professor J. L. Kelley, University of Kansas; Dr. J. C. Moore, Princeton University; Associate Professor Ernst Snapper, University of Southern California.

The National Science Foundation has announced grants in mathematics to the following institutions, to support studies by the professor indicated: University of Michigan, Wilfred Kaplan; Princeton University, N. E. Steenrod; University of Virginia, E. J. McShane; Washington University, H. Margaret Elliott.

Associate Professor Arthur Bernhart of the University of Oklahoma has received a five hundred dollar award "for extraordinary excellence in student counseling and in the teaching of freshman and sophomore students." A grant of $25,000 to the University will make it possible to present ten similar awards each year during the next five years.

Assistant Professor Louise H. Chin of the University of Arizona has received a Ford Foundation Fellowship and will study at Harvard University and the University of California.

Professor Arthur Erdélyi of the California Institute of Technology has been elected a Foreign Member of the Academy of Sciences of Turin.

Professor W. G. Madow of the University of Illinois has received a grant from the Fund for the Advancement of Education and will spend the year at Princeton University.

Professor K. O. May of Carleton College has been awarded a Ford Foundation Fellowship.

Associate Professor H. M. Schaerf of Washington University has been awarded a Ford Foundation Fellowship.

Professor C. F. Stephens of Morgan State College has been awarded a Ford Foundation Fellowship and will study at the Institute for Advanced Study.

Dr. L. G. Fourès of the Institute for Advanced Study has been appointed a lecturer at the University of Marseille.

Professor Laurent Schwartz of the University of Nancy has been appointed a lecturer at the Institut Henri Poincaré, University of Paris.

Dr. Eleazer Bromberg of the Office of Naval Research has accepted a position as assistant chief of the Division of Computing Services, Institute of Mathematical Sciences, New York University.

Mr. R. G. Brown of the Willow Run Research Center, University of Michigan, has accepted a position as member of the Operations Research Group, Arthur D. Little, Inc., Cambridge, Massachusetts.

Mr. D. E. Coffey of Saline High School, Saline, Michigan, has been appointed to an associate professorship at Lawrence Institute of Technology.

Dr. Robert Davies of the Rand Corporation has accepted a position as senior research engineer with the General Motors Corporation, Detroit, Michigan.

Dr. A. H. Diamond of the Office of Ordnance Research, Durham, North Carolina, has been appointed to a professorship at Stevens Institute of Technology.

Dr. H. A. Dye of the California Institute of Technology is on leave of absence at the Institute for Advanced Study.

Dr. G. W. Evans of the Argonne National Laboratory has accepted a position as applied mathematician with the University of California Radiation Laboratory, Livermore, California.

Professor Franz Hohn of the University of Illinois is on leave of absence and has accepted a position with the Bell Telephone Laboratories.

Associate Professor Emeritus K. W. Lamson of Lehigh University has been appointed to a visiting professorship at the University of Puerto Rico.

Dr. R. A. Leibler of the Sandia Corporation has accepted a position as mathematician with the Defense Department, Washington, D. C.

Dr. Harold Luxenberg of the Hughes Aircraft Company has accepted a position as consulting engineer with Remington Rand, Inc., Philadelphia, Pennsylvania.

Dr. Harry Polachek of the Naval Ordnance Laboratory, White Oak, Silver Spring, Maryland, has accepted a position as head of the applied Mathematics Laboratory, David Taylor Model Basin, Washington, D. C.

Mr. B. A. Rattray of McGill University has been appointed a lecturer at the University of New Brunswick.

Dr. Joan E. Robinson of Bryn Mawr College has accepted a position as mathematician with the National Security Agency, Washington, D. C.

Assistant Professor R. D. Schafer of the University of Pennsylvania has been appointed to a professorship at the University of Connecticut.

Associate Professor Henry Scheffé of Columbia University has been appointed to a professorship at the University of California, Berkeley.

Associate Professor R. M. Whitmore of Southwestern University is on leave of absence and has been appointed a staff member at the Sandia Corporation.

Assistant Professor Fumio Yagi of the University of Washington has accepted a position as mathematician with the Ballistic Research Laboratories, Aberdeen Proving Ground, Maryland.

The following promotions are announced:

Iacopo Barsotti, University of Pittsburgh, to a professorship.

E. J. Gumbel, Columbia University, to an adjunct professorship.

G. W. Morgan, Brown University, to an associate professorship.

W. J. Thron, Washington University, to an associate professorship.

The following appointments to instructorships are announced: California Institute of Technology: Dr. J. M. G. Fell, Dr. F. L. Spitzer; University of Washington: Dr. A. E. Livingston.

Dr. D. F. Campbell of Chicago, Illinois, died on February 5, 1953, at the age of eighty-six years. He had been a member of the Society for fifty-three years.

Professor Emeritus D. R. Curtiss of Northwestern University died on April 28, 1953, at the age of seventy-five years. He had been a member of the Society for fifty-one years.

Mr. T. B. Ripton of Mattapan, Massachusetts, died in March, 1953, at the age of forty-two years.

Mr. L. G. Simon of New York, New York, died on February 18, 1953, at the age of sixty-three years. He had been a member of the Society for thirty-three years.

NEW PUBLICATIONS

AMBARCUMYAN, V. A., MUSTEL', É. R., SEVERNYĬ, A. B., and SOBOLEV, V. V. *Teoretičeskaya astrofizika.* Moscow, Gosudarstvennoe Izdatel'stvo Tehniko-Teoretičeskoi Literatury, 1952. 635 pp., 1 plate. 15 rubles.

APOSTLE, H. G. *Aristotle's philosophy of mathematics.* University of Chicago Press, 1952. 10+228 pp. $6.00.

BALLIEU, R. See GILLIS, P. P.

BELLMAN, R. See HARRIS, T. E.

BERICOVICI, I. S. *Calcule aproximative.* Bucarest, Editura Tehnică, 1952. 440+2 pp. 21.46 Lei.

BLUMENTHAL, L. M. *Theory and applications of distance geometry.* Oxford, Clarendon, 1953. 11+347 pp. $10.00.

BOREL, É. *Les nombres premiers.* Paris, Presses Universitaires de France, 1953. 135 pp.

BOUCKAERT, L. See GILLIS, P. P.

BRAITHWAITE, R. B. *Scientific explanation. A study of the function of theory, probability and law in science.* Cambridge University Press, 1953. 12+376 pp. $8.00.

BRILLOUIN, L. *Wave propagation in periodic structures. Electric filters and crystal lattices.* 2d ed. New York, Dover, 1953. 12+255 pp. Cloth bound, $3.50; paper bound, $1.75.

BUCHHOLZ, H. *Die konfluente hypergeometrische Funktion mit besonderer Berücksichtigung ihrer Anwendungen.* (Ergebnisse der angewandten Mathematik, vol. 2.) Berlin, Springer, 1953. 16+234 pp. 36.00 DM.

CAMPUS, F. See GILLIS, P. P.

CARSON, J. R. *Electric circuit theory and the operational calculus.* 2d ed. New York, Chelsea, 1953. 10+197 pp. $3.95.

CHANDRASEKHARAN, K., and MINAKSHISUNDARAM, S. *Typical means.* Oxford University Press, 1952. 10+139 pp. $6.50.

CLAGETT, M. See *Medieval science of weights.*

COCHRAN, W. G. *Sampling techniques.* New York, Wiley, 1953. 14+330 pp. $6.50.

COUTREZ, R. See GILLIS, P. P.

VAN DANTZIG, D. See GILLIS, P. P.

DOMIZLAFF, H. See TORNIER, E.

DOOB, J. L. *Stochastic processes.* New York, Wiley, 1953. 8+654 pp. $10.00.

EINSTEIN, A. *The meaning of relativity.* 4th ed. Princeton University Press, 1953. 4+168 pp. $3.50.

FAUVILLE, A. See GILLIS, P. P.

FINIKOV, S. P. *Kurs differencial'noĭ geometrii.* Moscow, Gosudarstvennoe Izdatel'stvo Tehniko-Teoretičeskoi Literatury, 1952. 343 pp. 8.6 rubles.

FLÜGGE, S., and MARSCHALL, H. *Rechenmethoden der Quantentheorie dargestellt in Aufgaben und Lösungen.* Part 1. *Elementare Quantenmechanik.* 2d ed. (Die Grundlehren der Mathematischen Wissenschaften in Einzeldarstellungen mit besonderer Berücksichtigung der Anwendungsgebiete, vol. 53.) Berlin, Springer, 1952. 8+272 pp. 29.80 DM; bound, 32.80 DM.

FRAENKEL, A. A. *Abstract set theory.* Amsterdam, North-Holland, 1953. 12+479 pp. 38.00 florins.

FRÉCHET, M. *Pages choisies d'analyse générale.* Paris, Gauthier-Villars, 1953. 213 pp. $5.94.

FRÉCHET, M. See GILLIS, P. P.

Gillis, P. P., von Mises, R., Ballieu, R., van Dantzig, D., Coutrez, R., Bouckaert, L., Prigogine, I., Campus, F., Fauville, A., Fréchet, M., and Hirsch, G. *Théorie des probabilités. Exposés sur ses fondements et ses applications.* Louvain, Nauwelaerts, 1952. 195 pp. 200 Belgian fr.

Goffman, C. *Real functions.* New York, Rinehart, 1953. 12+263 pp. $6.00.

Gould, S. H. See Ore, O.

Graf, U., and Henning, H. J. *Formeln und Tabellen der mathematischen Statistik.* Berlin, Springer, 1953. 6+102 pp. 9 DM.

Harris, T. E., Bellman, R., and Shapiro, H. N. *Studies in functional equations occurring in decision processes.* (The Rand Corporation Report P-382.) Santa Monica, 1953. 62 pp.

Hasse, H. *Mathematik als Wissenschaft, Kunst und Macht.* Wiesbaden, Verlag für Angewandte Wissenschaften, 1952. 34 pp.

Haupt, O. *Einfuhrung in die Algebra.* Part 1. 2d ed. Leipzig, Geest and Portig, 1952. 18+370 pp.

Henning, H. J. See Graf, U.

Herschel, R. See Petrovskij, I. G.

Hirsch, G. See Gillis, P. P.

Itô, K. *Kakuritsu-ron.* [*Theory of probability.*] Tokyo, Iwanami Shôten, 1953. 6+405 pp. 650 Yen.

Jordan, P. *Schwerkraft und Weltall. Grundlagen der theoretischen Kosmologie.* Braunschweig, Vieweg, 1952. 8+207 pp. 15.80 DM.

Kraitchik, M. *Mathematical recreations.* 2d ed. New York, Dover, 1953. 330 pp. Clothbound, $3.00; paperbound, $1.60.

Kuratowski, C. *Topologie.* Vol. I, 3d ed. (Monografie Matematyczne, vol. 20.) Warsaw, Polskie Towarzystwo Matematyczne, 1952. 12+450 pp. $7.50.

Leja, F. *Funkcje analityczne i harmoniczne.* Vol. I. [*Analytic and harmonic functions.* Vol. I.] (Monografie Matematyczne, vol. 29.) Warsaw, Polskie Towarzystwo Matematyczne, 1952. 4+174 pp. 15.00 Zl.

McNaughton, R. See Wang, H.

Mann, H. B. *Introduction to the theory of stochastic processes depending on a continuous parameter.* (National Bureau of Standards Applied Mathematics Series, no. 24.) Washington, Government Printing Office. 6+45 pp. $.30.

Marschall, H. See Flügge, S.

The medieval science of weights. (Scientia de ponderibus.) Treatises ascribed to Euclid, Archimedes, Thabit ibn Quarra, Jordanus de Nemore, and Blasius of Parma. Ed. by E. A. Moody and M. Clagett. Madison, University of Wisconsin Press, 1952. 10+438 pp. $5.00.

Minakshisundaram, S. See Chandrasekharan, K.

von Mises, R. See Gillis, P. P.

Moody, E. A. See *Medieval science of weights.*

Mustel', È. R. See Ambarcumyan, V. A.

Ollendorff, F. *Berechnung magnetischer Felder.* Vienna, Springer, 1952. 10+432 pp. $15.70.

Ore, O. *Cardano, the gambling scholar.* With a translation from the Latin of Cardano's "Book on games of chance" by S. H. Gould. Princeton University Press, 1953. 14+249 pp. $4.00.

Petrovskij, I. G. *Vorlesungen über die Theorie der Integralgleichungen.* Trans. by R. Herschel. Würzburg, Physica-Verlag, 1953. 100 pp. 7.80 DM.

PIAZOLLA BELOCH, M. *Geometria descrittiva*. 2d ed. Istituto di Geometria dell'Università di Ferrara, 1953. 15+172+172+118 pp.

PRIGOGINE, I. See GILLIS, P. P.

RIEMANN, B. *Gesammelte mathematische Werke und wissenschaftlicher Nachlass*. New York, Dover, 1953. 19+558+8+116 pp. Clothbound, $4.95; paperbound, $2.25.

SCHLÄFLI, L. *Gesammelte mathematische Abhandlungen*. Vol. II. Basel, Birkhäuser, 1953. 381 pp. 56.15 Swiss fr.

SERRUYS, M. *Étude générale de l'écoulement d'un gaz à travers une tuyère quelconque et du passage par la vitesse du son (en régime permanent ou quelconque avec apport de chaleur et réaction chimique éventuels)*. (Publ. Sci. Tech. Ministère de l'Air, no. 272.) Paris, 1952. 8+62 pp. 750 fr.

SEVERNYĬ, A. B. See AMBARCUMYAN, V. A.

SHAPIRO, H. N. See HARRIS, T. E.

SIBAGAKI, W. *Theory and applications of the gamma function with a table of the gamma function for complex arguments significant to the sixth decimal place*. (In Japanese.) Tokyo, Iwanami Syoten, 1952. 202 pp. 680 Yen.

ŠILOV, G. E. *Vvedenie v teoriyu lineĭnyh prostranstv*. Moscow, Gosudarstvennoe Izdatel'stvo Tehniko-Teoretičeskoi Literatury, 1952. 384 pp. 7.30 rubles.

SOBOLEV, V. V. See AMBARCUMYAN, V. A.

THRON, W. J. *Introduction to the theory of functions of a complex variable*. New York, Wiley, 1953. 10+230 pp. $6.50.

TORNIER, E., and DOMIZLAFF, H. *Theorie der Versuchsvorschriften der Wahrscheinlichkeitsrechnung*. Stuttgart, Kohlhammer, 1952. 108 pp. 12.00 DM.

VINOGRADOV, I. M. *Izbrannye trudy*. Moscow, Izdatel'stvo Akademii Nauk SSSR, 1952. 436 pp., 1 plate. 21.70 rubles.

WALLOT, J. *Grössengleichungen, Einheiten und Dimensionen*. Leipzig, Barth, 1953. 8+216 pp. 14.70 DM; bound, 16.35 DM.

WANG, H., and MCNAUGHTON, R. *Les systèmes axiomatiques de la théorie des ensembles*. Paris, Gauthier-Villars, 1953. 55 pp. 750 fr.

YANO, T. *Kokumin Sûhyô*. [*People's Tables*.] Tokyo, Kokuseisha, 1952. 4+146 pp., 1 plate. 230 Yen.

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THE ORIGIN AND GROWTH OF MATHEMATICAL CONCEPTS

R. L. WILDER

1. Introduction. According to A. N. Whitehead [1, p. 20], "The science of Pure Mathematics, in its modern developments, may claim to be the most original creation of the human spirit"; a statement with which probably few mathematicians would quarrel. A layman, however, might and probably would take this to mean that modern mathematics is something which has already been created; a creation, that is, which has already been accomplished and is now safely embalmed with remains on view in any good library.

Of course, as mathematicians we know that mathematics is in no such static shape; that it is, on the contrary, a dynamic affair, changing even from day to day. Our late colleagues who were in the forefront of mathematical creation about the turn of the century, 1900, would be amazed to see what mathematics is like today; indeed, many of them, I'd wager, would not like some of the modern developments, probably taking the same attitude toward them that some mathematicians of the late 19th century took toward Cantor's innovations regarding the infinite. Moreover, those of us who are active today would probably feel the same way toward the mathematics of the year 2000, if we were in some way able to view it. We ourselves probably have not sufficient perception of the changes going on in mathematics at the present time; it is well known that the participants in great social changes are usually unaware of them. And there is some evidence that our awareness of the process of mathematical change, although not so at variance with the facts as that of the layman, is still so defective in some ways as to lead to unfortunate but avoidable situations. This is partly due, I suppose, to our being so busy creating new mathematics that we have little time or patience to view our behavior from the outside and study its characteristics. And even when we do so, we seem prone to take such a specialized angle from which to make our observations that we get only a partial perspective.

2. Motives. Poincaré remarked, in one of his numerous essays [2, p. 376], that "mathematical science must reflect on itself," and

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A retiring address, as Vice President and Chairman of Section A, American Association for the Advancement of Science, delivered before a joint meeting of Section A, the American Mathematical Society, and the Mathematical Association of America, at St. Louis, Missouri, December 29, 1952; received by the editors March 13, 1953.

supported this dictum with several studies of the psychological aspects of mathematical creation. These studies inspired others of a similar nature, among which might be noted especially the little book [3] by Hadamard on the *Psychology of invention in the mathematical field*, published 7 years ago.

However, the angle from which these studies were made, that of psychology, can furnish only part of the general picture. Studies of a psychological nature are significant in that they analyze the mental processes by which the individual mathematician uses the materials and tools present in his culture to make new constructs. But they fail to make proper connection with the cultural stream in which they are imbedded. The mathematician is not an isolated entity grinding out new ideas, with everything coming out and nothing going in. Rather he is making new syntheses out of the concepts that are going in. Young men who have just received the doctorate usually seem to have at least an intuitive recognition of this, for how often do we hear them express their fears of having to accept positions where there are few or no mathematicians who have interests similar to their own; or, if they find it necessary to accept such positions, they worry over whether the libraries, by which they mean the means of keeping contact with their fellow workers through the medium of journals, are good or not. The popularity of such centers as Göttingen in the twenties, and now the Institute for Advanced Study, furnishes further evidence of the gregariousness of the mathematician. And of course there are our societies and associations, through whose meetings and journals we may exchange ideas.

I believe, therefore, that even if it only leads to a rational understanding and appreciation of these matters, a study of these *cooperative* features of mathematical creation, or what we might call a study of the "group dynamics" of mathematical creation, is warranted. Because of its high level of abstraction, we are inclined to look upon mathematics as strictly an *individual* activity. In his address opening the recent International Congress of Mathematicians in Cambridge [4, p. 125], Oswald Veblen comments that "Mathematics is terribly individual. Any mathematical act, whether of creation or apprehension, takes place in the deepest recesses of the individual mind." However, he goes on to say, "Mathematical thoughts must nevertheless be communicated to other individuals and assimilated into the body of general knowledge. Otherwise they can hardly be said to exist." Note that last sentence, by the way— "Otherwise they can hardly be said to exist." Obviously Veblen is aware that while, in one of its aspects, mathematical creation is, as he says, "terribly individual," in its other aspects it is not an individual affair at all. Later, in the same address, we find an affirmation of the necessity for mathematicians forming such associations as we have today: "The resultant organizations of various kinds have accomplished many important things known to us all. Of these accomplishments I am sure that the most important is the maintenance of a set of standards and traditions which enable us to preserve *that coherent and growing something which we call Mathematics.*" [Italics mine.]

To summarize my motives, then, I wish to inquire, above the individual level, into the manner in which mathematical concepts originate, and to study those factors that encourage their formation and influence their growth. I think that much benefit might be derived from such an inquiry. For example, if the individual working mathematician understands that when a concept is about to make its appearance, it is most likely to do so through the medium of more than one creative mathematician; and if, furthermore, he knows the reasons for this phenomenon, then we can expect less indulgence in bad feelings and suspicion of plagiarism to ensue than we find in notable past instances. Mathematical history contains numerous cases of arguments over priority, with nothing settled after the smoke of battle has cleared away except that when you come right down to it practically the same thing was thought of by someone else several years previously, only he didn't quite realize the full significance of what he had, or did not have the good luck to possess the tools wherewith to exploit it. Coolidge, in his *A history of geometrical methods*, remarks [5, p. 122], "It is a curious fact in the history of mathematics that discoveries of the greatest importance were made simultaneously by different men of genius." This is quite true, except that there is nothing "curious" about it, nor is it confined to mathematics. And it is exactly what one should expect if he is acquainted with the manner in which concepts evolve.

I shall have to be brief and incomplete in my presentation here, giving only a general description of my study. In particular, I shall pay little attention to questions such as "What is a concept?", or to the different type levels of concepts, important though these are. I shall take it for granted that we know what we mean by such terms as "the group concept," "the concept of limit," and the like—these are well defined today. Other things which we label "concepts" are not at all well defined. And of course some concepts embrace other concepts, such as that of the calculus embracing the concept of limit of a set of real numbers.

Furthermore, in the case histories which I shall presently mention, you will probably be impressed with the fact that in tracing the roots of a concept under discussion, we find invariably that some of them go back to Greece. It should not be inferred from this that they *began* in Greece. The fact is, that these roots have many smaller tendrils that reach out into pre-Greek or contemporary cultures, and I am not enough of a historian to have their identities, if such exist, at my disposal. This will not, however, affect my main conclusions in the least. I do want to warn, however, against the impression that these roots necessarily start in Greece.

In reviewing such case histories, I am of course going over much material which is well known. What we get out of such material, however, is strongly influenced by our orientation, and in repeating these historical details it is with the purpose of bringing out aspects that are not ordinarily emphasized and frequently not observed.

3. **Case studies.** Now how do mathematical concepts originate; where do they come from? I suppose we might, without thinking very seriously about our answer, reply, "By taking thought;" or "by shutting one's self in his study, getting out a pencil or piece of chalk, and going to work." This kind of reply is reminiscent of the answer which someone[1] made to the question, "How does one write novels?", viz., "By pressing the seat of one's pants to a chair for three hours at a time and writing."

Like most pat answers, this is not, on further reflection, quite acceptable. To be sure, concepts don't get born without someone taking thought. We must not forget, however, that gregarious aspect of mathematical activity which I mentioned above. Not only does this have its influence on the formation of concepts, but it must be conceded an equal partner, at least, in the process.

A concept doesn't just pop up full grown "like Venus from the waves," although it may seem to, to the individual mathematician who does the conceiving. Usually its elements are lying in what, if the term were not already in technical mathematical use, one might call the *mathematical continuum*, but which we might better call the *mathematical culture stream*. As the eminent historian of science, George Sarton, comments [6, p. 36]: "—creations absolutely *de novo* are very rare, if they occur at all; most novelties are only novel combinations of old elements, and the degree of novelty is thus a matter of interpretation, which may vary considerably according to the historian's experience, standpoint, or prejudices. —the determination of an event as the 'first' is not a special affirmation relative to that event, but a general negative proposition relative to an undetermined number of unknown events."

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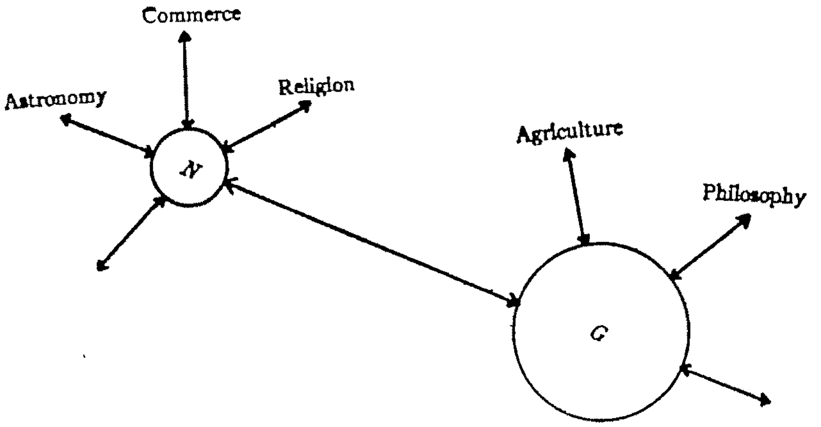
[1] I think it was the writer Sinclair Lewis, but I have been unable to confirm this.

3a. *Number and geometry concepts.* The absurdity of always striving for a point of origin is nowhere better exemplified than in the origins of mathematics itself. We are unable here to find any "names" by which to label innovations even if we would. And this, I am certain, is not due simply to absence of adequate written record. The accompanying figure is supposed to represent the origin and growth of the earliest mathematical concepts, those of *number* and *geometry*.



Just when these began to merit being called *concepts* we don't know, but enough evidence exists to warrant the conclusion that this occurred at no particular date, since each gradually emerged from the cultures of the Middle East and Greece; and of course they may have emerged in other cultures, as the number concept did in the Mayan culture. The short arrows represent such influences as those of a Babylonian priesthood facing the necessity of assessing taxes and keeping records of a host of tax-gatherers, the desire to compute the quantity of seeds necessary to sow a given piece of land, and other social needs. From such matters to the concept of a number—"four" for example, as contrasted with "four men," "four dogs"—must have been a long evolution. Certainly no one has suggested a prehistoric Archimedes leaping out of his bath or primitive swimming pool and running through the forest primeval shouting, "I've got it—the number four, the number four!" Similarly, geometry did not begin with Euclid.

3b. *Analytic geometry.* Or to take a very elementary, but historically important example, in histories of mathematics one finds serious discussion of whether Fermat or Descartes should be credited with the "invention" of analytic geometry, sometimes accompanied by a choice between the two and an extended argument supporting the choice.

In such cases, I am inclined to feel that the historian is unconsciously influenced by that principle of popular folklore; "everything has a beginning." And conceding this, no historian worth his salt can fail to produce one. In some cases, particularly in that of a very specialized concept, such as that of the "theta-Fuchsian series" whose genesis in the mind of Poincaré is so interestingly described in his article on *Mathematical creation* [2, p. 387], to speak of a "beginning" is perhaps justified—although even in such instances one could without doubt, given sufficient evidence, trace the inception of the notion from prior concepts. But in the case of broader concepts, such as "analytic geometry," we seem prone to error due to the vagueness of the concept.

It is interesting to note the way in which Coolidge handles the question in his *A history of geometrical methods*. To quote [5, p. 117]: "The opinion is currently held among mathematicians that analytic geometry sprang full-armed from the head of Descartes as did Athene from that of Zeus. . . . There is much to be said in favor of this thesis, but . . . another opinion is certainly possible. The fact is that in inquiring into the origin of analytic geometry we run into a difficulty that lies at the bottom of a good proportion of our disputes in this Vale of Tears. What do we mean by the words 'analytic geometry'? Till that is settled, it is futile to inquire as to who discovered it." That the essence of the subject, however we define it, was known to the Greeks, such as Apollonius, seems beyond dispute; but as a coordinate geometry with a workable algebra like that which we associate with the notion, it evolved through the works of Fermat and Descartes as well as various predecessors. Had the Greeks possessed a good algebra, the story might have been different.

3c. *Calculus*. Everyone is familiar with the way the basic postulate of popular folklore that everything has a beginning has operated in the case of the Calculus. The dispute over its so-called "authorship" between Newton and Leibniz and their followers is, I suppose, the classical example of useless disputation attributable to the impulse to conform to the postulate. In a recent article [7], A. Rosenthal makes clear the indebtedness of both Newton and Leibniz to their forerunners as well as their contemporaries; and that no matter how you define it, the calculus is a product of a slow evolution that has been recorded as far back as the Greeks. And although I suspect that the author privately feels that it makes little sense to speak of the invention or the discovery of the calculus as an actual temporal event, he tries to justify the thesis that Newton and Leibniz originated the concept, by affording a definition.

After remarking that "there certainly was an extensive development of the theory of integration and differentiation in the period immediately before Newton and Leibniz," he asks [7, p. 83], "What then was missing at that time," —i.e., at the time when Newton and Leibniz entered the picture. He points out that one "very important point still missing was the general fact that differentiation and integration are inverse processes, that is, the so called fundamental theorem of integral calculus." And after some discussion as to whether various mathematicians, who evidently came very close to it, really recognized it or not, he discloses that it was known in full generality by Isaac Barrow, the teacher of Newton at Cambridge. So he repeats, "What more remained to be done?" replying, "What had to be created was just the *Calculus*, a general symbolic and systematic method of analytic operations, to be performed by strictly formal rules, independent of the geometric meaning. . . . it is just this Calculus which was established by Newton and Leibniz, independent of each other and using different types of symbolism." Although this is not really a *definition* in the true sense, at least it is an attempt to fix the general nature of just what it was that Newton and Leibniz did do. Whether it is a good thing to try to assign an originator to as broad a concept as the calculus, even when it is impossible to make the originator *unique*, is another question.

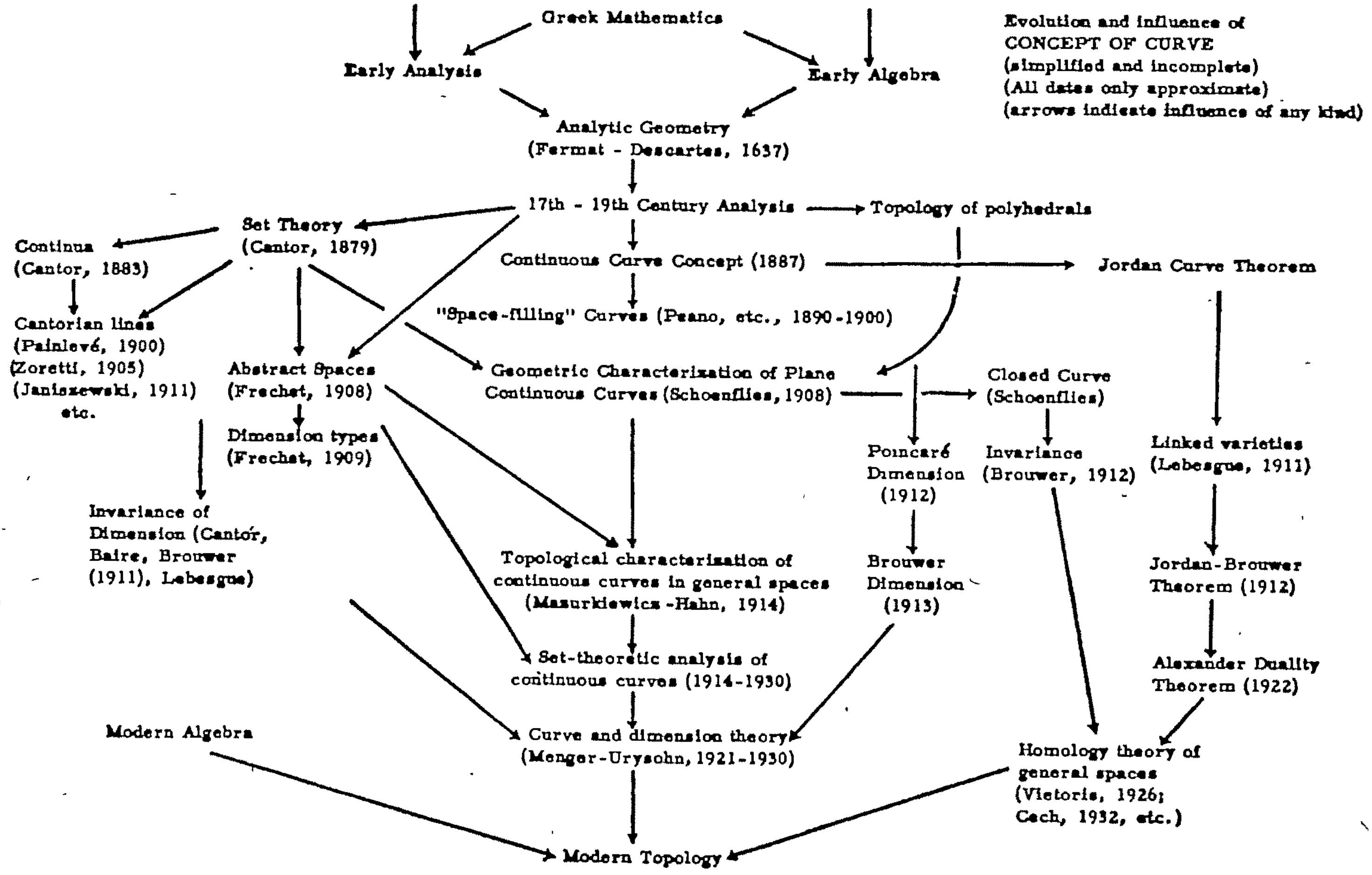
It should be noted that, as in the case of analytic geometry, if the Greeks had possessed enough symbolic material, they might have progressed further with the calculus. Also that despite the lack of a rigorous theory of limits, that which most agree to call "calculus" appeared nevertheless with Leibniz and Newton, but essentially as an operational apparatus. The theory of limits, and calculus as we know it, had still to evolve.

It is interesting to compare Coolidge's comments [5, p. 38]. He speaks of the Greek geometers' "procedure in attacking problems in computation which led naturally to the integral calculus. If we seek the root of the difference between their method of attack and ours, it lies partly in the greatly increased flexibility of our notation, partly in their lack of a clear concept of a limit." That is, a suitable notation was lacking—a suitable *tool*—and a necessary *concept*, that of limit, had yet to evolve. Since the concept of limit was not developed until after the work of Newton and Leibniz, this seems to place the emphasis again upon the Greek lack of an algebraic tool, as the root of the difference between the Greek calculus and that of Leibniz and Newton.

3d. *The curve concept.* Finally, I would like to view the evolution of a narrower, more specialized concept, that had its origin in antiquity, and which not only continued its growth in modern times but contributed in a noteworthy manner to a large portion of modern mathematics. It is not, I am sure, generally realized how great an influence this concept has had on the development of concepts which today contain hardly a trace of the notion. I refer to the concept of *curve.*

The accompanying chart, which may be referred to during the discussion, is supposed to set forth the principal lines of influence, and it is to these that I shall confine myself. It would be impossible to sketch here the history of the concept in full detail. It has been a common mathematical notion, as a rule purely intuitive, since the time of the Greeks, at least. And it is one of those mathematical notions that have been adopted by the layman. Usually it has been just a conventional categorical term, and seems to have continued so throughout the time when the study of curves was resumed analytically during the Renaissance. The first successful attempts at formulating precise definitions of the concept were not made by geometers, but by analysts! So far as I have been able to find, Georg Cantor was the first to be credited with a definition, but apparently what he really did [8, p. 194] was to determine those set-theoretic properties that constitute the *continuous,* as opposed to the discrete, aspect of a curve—or as a topologist would express it, he defined the topological notion of *continuum.* But a square together with its interior is a continuum, and we would hardly call this a curve! However, if the continuum lies in the plane, all we need to do is require that it contain no such square and we get what came to be called a "cantorian line." And although satisfactory only as a definition of the concept of plane curves, the cantorian line coincides exactly with what most topologists feel is the natural definition.

The next most notable attempt at definition was also not made by a geometer, but by C. Jordan in 1887. In the first (1882) volume of his *Cours d'analyse* [9], he speaks (as on p. 238, loc. cit.) of a plane curve as given by any two functions $x=f(t)$, $y=g(t)$, where the range of t is the real number interval $[0,1]$. Although no mention of continuity is made, if he was familiar with Cantor's proof [8, p. 122 ff.] published in 1878, 4 years earlier, of the (1-1)-correspondence be-



Evolution and influence of
CONCEPT OF CURVE
(simplified and incomplete)
(All dates only approximate)
(arrows indicate influence of any kind)

Greek Mathematics
Early Analysis
Early Algebra
Analytic Geometry (Fermat - Descartes, 1637)
17th - 19th Century Analysis
Topology of polyhedrals
Set Theory (Cantor, 1879)
Continua (Cantor, 1883)
Cantorian lines (Painlevé, 1900) (Zoretti, 1905) (Janiszewski, 1911) etc.
Continuous Curve Concept (1887)
Jordan Curve Theorem
"Space-filling" Curves (Peano, etc., 1890-1900)
Abstract Spaces (Frechet, 1908)
Dimension types (Frechet, 1909)
Geometric Characterization of Plane Continuous Curves (Schoenflies, 1908)
Closed Curve (Schoenflies)
Poincaré Dimension (1912)
Invariance (Brouwer, 1912)
Linked varieties (Lebesgue, 1911)
Invariance of Dimension (Cantor, Baire, Brouwer (1911), Lebesgue)
Topological characterization of continuous curves in general spaces (Mazurkiewicz-Hahn, 1914)
Brouwer Dimension (1913)
Jordan-Brouwer Theorem (1912)
Set-theoretic analysis of continuous curves (1914-1930)
Alexander Duality Theorem (1922)
Modern Algebra
Curve and dimension theory (Menger-Urysohn, 1921-1930)
Homology theory of general spaces (Vietoris, 1926; Cech, 1932, etc.)
Modern Topology

tween the points of such an interval and the points of the square plus its interior, he must have realized that without it, this definition could not be satisfactory. However, in an appendix ("Note") to the 3rd volume of this work, published 5 years later, in repeating this definition, he added [9, vol. 3, p. 587], "If the functions are continuous, the curve will be called *continuous*." (See also [9a, p. 92]; this reference to the first volume of the second edition, 1893, is the one customarily cited as the "origin" of the concept "continuous curve.") Cantor had noted that the (1-1)-correspondence just referred to was impossible in case the functions were continuous, and Jordan's definition of *continuous* curve lay between the two extremes—it does not require (1-1)-ness, although the functions are of course single-valued, and it does require continuity.

It is not difficult to surmise where Jordan got this notion; it was common practice to define curves parametrically in the books on calculus. For instance, the parametric representation of the cycloid is given in Cauchy's famous calculus [10, p. 52], published in 1826. Evidently what Jordan did was to formulate a general concept which was already current. I don't know who first attached Jordan's name to the notion, as a matter of fact. The Polish topologists were already doing so during the 'teens of the present century, and most writers of the twenties assume he was the first to formulate it explicitly. However, I have found no evidence that either Jordan's contemporaries or his immediate successors associated his name with the concept, which seems to be further evidence that the notion was generally current among analysts.

But to get on with the story: Geometers, possibly becoming suddenly conscious of their inadequacy, or perhaps jealous of their prerogatives, pounced upon the notion. I am confident that Jordan had no idea of usurping their natural rights; that all he had in mind was to make precise certain analytic notions that were already current, for his own special purposes. For instance, he showed that for a curve to be rectifiable, it is necessary and sufficient that the functions defining it be of bounded variation [9, vol. 3, p. 594 ff.]. And for the proof of his famous theorem, the Jordan Curve Theorem [9, vol. 3, p. 587 ff.; 9a, p. 92], he stipulated that the continuous curve was to have no double points except at the end points of the interval of definition of the variable t. For these purposes, his definition was a good one, and still is.

As is well known, it was only three years after Jordan's formulation until the Italian logician and geometer Peano published, in 1890, his famous space-filling curve. Hilbert and E. H. Moore followed suit soon after (for discussion and references, see [13, p. 943 ff.]). None of them mentions Jordan's name, incidentally,—perhaps further evidence that the concept was probably not considered as his personal property! Their examples show, of course, that Jordan's definition was not a satisfactory formulation of the precise notion of curve as it was intuitively conceived.

However, fortunately for the development of modern mathematics, the fact that the geometer found Jordan's analytic definition wanting did not mean that it was doomed to oblivion so far as geometry was concerned; even though it came to be recognized that in addition to including configurations we don't want to call "curves," it excluded some we do want to call curves (such as the well known continuum formed from sin $1/x$ and its limit points on the y-axis). That it included among its special cases the general euclidean space element—for of course the cube and its interior in three and higher dimensions are continuous curves in Jordan's sense—was of intense geometric interest. Thus it developed that interest in the continuous curve was focused on the nature of its geometric counterpart; if this counterpart was not what we call "curve" in geometry, just what configurations do come within its scope?

The first systematic investigation of this question was made by Schoenflies, who gave a complete answer for the planar continuous curves (for discussion and reference, see [13, p. 964 ff.]). His work, incidentally, shows a remarkable perception of the nature of the problem as well as ingenuity in selecting the right methods for solving it. Realizing that it could not be solved by any of the methods common to the older forms of geometry, he called upon the new set-theoretic ideas introduced by Cantor and the (apparently unrelated) topology of polyhedrals of Riemann and Poincaré.

Among the first to be influenced by Schoenflies' work was Brouwer, probably best known for his Intuitionism, but among topologists and analysts famous for the remarkable work which he did on the theory of manifolds, especially as regards mapping and fixed point theorems, during the second decade of the present century. Brouwer was impressed not only by Schoenflies' positive results, but by certain fundamental errors which, due to faulty intuition, subtly crept into Schoenflies' work. For in his study of the Jordan Curve Theorem, Schoenflies conceived the idea of separating, from its tie-up to the topological images of the circle, the property of a curve being "common boundary of two domains in the plane" and called any configuration having this property a "closed curve." That a closed curve so defined could not be the common boundary of *more than* two domains seemed perfectly natural to assume. Only it happens not to be true, as Brouwer was able to prove (for discussions and references, see [**13**, p. 924 ff.]). Also, Schoenflies was unable to give any proof of the invariance of the closed curve under topological transformations. Brouwer also supplied this [11], using a method of proof which contained the central idea of the extension, to general metric spaces, of the homology theory of polyhedrals which had been initiated by Riemann, Poincaré, and others, and extended in this country by Veblen and his colleagues.

But I must not get too far ahead. The threads of influence leading from Jordan's definition begin now to entangle with one another and with threads from other sources. About the time when Brouwer was writing his dissertation on the law of the excluded middle, with perhaps little thought of topology, Fréchet was working on his dissertation [12] on "abstract spaces." This tendency to abstract from the euclidean situation, which of course goes back, in its roots, to Fréchet's predecessors but was given clear-cut realization by him, was to influence the treatment of the continuous curve problem in the following manner: It had proved impossible, with the tools available at the time, to extend Schoenflies' methods to 3-space or higher dimensions. Here the point of view is relative to the imbedding space, the Euclidean space of 3 or higher dimensions, rather than intrinsic. The introduction of the abstract space idea led to the problem of *intrinsic characterization*, by geometric methods, of the continuous curve. In order to eliminate the separate functions defining the coordinates in Jordan's definition, one recognized that an equivalent idea is to consider the curve as defined by a single function, $f(t)$, single-valued and continuous over the interval [0, 1], but the values of the function to be in any space allowing of the definition of limit. From this point of view the problem became manageable and received a solution, independently, at the hands of Hahn and Mazurkiewicz. Using the topological notion of local connectedness, which had actually been introduced (presumably unknown to them) three years earlier by Pia Nalli in another connection (for reference, see [**13**, p. 922, footnote 224]), they characterized continuous curves as compact metric continua that are locally connected (for discussion and references, see [**13**, p. 947 ff.])—a thoroughly intrinsic definition, as contrasted with the results of Schoenflies.

This was about the time of the start of World War I, and the simultaneity of these results did not become known until after the War. There followed, not only in Poland and Austria, but in this country where the students of E. H. Moore had become interested in the new ideas in the foundations of geometry, a fairly complete analysis, especially during the period up to 1930, of the concept of continuous curve from the set-theoretic point of view.

Now it is characteristic of mathematical concepts that they often tend to be swallowed, as for instance in the process of generalization, by more embracing concepts. As a rule, this happens subsequent to the formulation and investigation of the concept. The number concept is a simple example of this. But in the case of the curve concept, the assimilation was to be coincident with the formulation of the larger concept. And so, for a few minutes, I shall turn to the concept of ***dimension***. I cannot take time to review its ancient history—remarks analogous to those which I made concerning the concept of curve hold here, such as its use by the Greeks, its adoption into popular parlance, etc. As we approach modern times, attempts to formulate a dimension concept begin to crop out in the literature, often in the works of analysts, whose formulation was usually based on the number of parameters involved in the description. Cantor gave the body blow to this when he showed [8, p. 122 ff.] that a single real parameter suffices, if one neglects continuity. The resulting investigations—and these included the Jordan formulation of curve as well as the Peano results already mentioned—especially the problem of the invariance of dimension under bicontinuous (1-1)-transformations, are too detailed for me to go into here. They played their part, however, in the evolution of the dimension concept. On the left-hand side of the chart will be found some indications of the miscellaneous work done in this direction—note the names of Baire, Brouwer, and Lebesgue accompanied by the date 1911. In addition certain dimension definitions or attempts at definitions are also indicated, and there is a notation regarding "cantorian lines," which were studied by both Zoretti and Janiszewski among others. (For references, see sections 12, 17 of [13], especially pp. 907 and 951–952.)

Now I do not maintain that any particular mathematician read any of the work of these men, and was directly influenced thereby. I call attention to their work chiefly as a proof that the notions of curve (or "line" as it was frequently called) and dimension were evolving in the mathematical culture stream.

To the right of the center of the chart you will notice the notation "Poincaré dimension, 1912." It seemed appropriate to place this over to the other side of the chart for several reasons; the most important being that it leads quite naturally to the concept which was finally generally accepted as embodying the precise formulation of the intuitive notion of curve and dimension. Another reason for setting this apart from the material on the left side of the chart is that although Poincaré did the most important kind of work in laying the foundations of the topology of polyhedrals, as is well known he had little acquaintance with or use for the theory of sets. Consequently, when he tried to indicate just what the dimension of a space is, in such essays as that on *The notion of space*, for instance [2, p. 243 ff.] (or see [14, p. 3]), he thought globally instead of in the small. And so his definition is an "in the large" definition—which may suffice for a homogeneous space, but most topological spaces are not homogeneous. However, the terminology in which it is formulated, such as "continua" and "separating" or "cutting," were actually set-theoretic. As a result, the definition was not given in precise mathematical terms. Nevertheless, anyone who happened to notice it and who had the suitable tools in hand could make it precise and a better approximation to a satisfactory definition. And this is exactly what Brouwer did the very next year, in 1913. (See [14, p. 4].)

Brouwer's paper attracted little attention, and when, in the early twenties, a satisfactory theory of curve and dimension was announced by Menger and Urysohn (citations may be found in [15, p. 83 ff.]), evidently neither of these gentlemen knew of the work of Brouwer, nor of each other's work. It is now known, of course, that all three theories are equivalent for a wide class of spaces—the locally connected, separable metric spaces.

The arrows on the chart, which, as the key in the upper right-hand corner states, are supposed to represent influence of any kind, need perhaps some further explanation. That the curve and dimension theory of Menger and Urysohn was influenced by the various set-theoretic activities of their predecessors is, I think, too obvious to need further comment. As I have already pointed out, it is probable that lack of contact with these could have prevented Poincaré from setting up some kind of a precise definition of dimension. The arrows coming in from the sides, however, are not so obvious. Just how much, if any, of the works on dimension problems which preceded them were actually read by Menger or Urysohn, I don't know. In the introduction to his formal presentation of his work published in 1925, Urysohn refers to the work of Zoretti on cantorian lines, for instance, and not only adopts the same term but entitles his monographs *Sur les multiplicités Cantoriennes*; he also remarks that his dimension definition is quite in conformity with the desires that motivated Poincaré. But even if no such direct and obvious influences existed, there is still the indirect influence that may have been exercised through the medium of others. Menger frequently made gracious acknowledgment to his teacher, Hans Hahn, not, so far as I know, for specific suggestions regarding definitions but for general guidance and inspiration; Urysohn's association with Alexandroff is well known.

But my time is running out, and I cannot give further details. I have already mentioned how the Brouwer proof of invariance of the Schoenflies closed curve contained the germ of the application of the homology theory of polyhedrals to general spaces—a fact which led Vietoris, who acknowledges his debt to Brouwer, to extend this theory to general metric spaces in order to establish an intrinsic form of the analogous higher dimensional invariance. (For citations and sketch of the developments discussed here, see [16, p. 14].) Let me also call attention to the arrows on the extreme right of the chart leading from the Jordan Curve Theorem to work of Lebesgue and Brouwer in higher dimensions, and thence to the Alexander Duality Theorem (which was motivated by the desire to obtain a general higher-dimensional extension of the Jordan-Brouwer separation theorem), and hence into algebraic aspects of modern topology. Incidentally, I have said very little about the role played by algebra in these matters; but its influence is mainly quite modern, and my intent has been not to sketch a history of curve theory, but to indicate as well as I can, in the time at my disposal, the manner in which the curve concept has influenced various other concepts right down to the inception of modern topology. Of course other concepts played a part too, but as I said before the threads of influence become quite tangled, and it is better to follow one at a time.

Before leaving the chart, let me point out, to prevent misunderstanding, the "simplified and incomplete" notation in the upper right-hand corner. Many things have been omitted, not because they are unknown (although there are many such), but to avoid complicating the diagram. There are, for example, geometric contacts that are omitted. If only in the way of providing certain perspective, geometric concepts exercised an influence on set theory, for example, and on curve and dimension theory.

4. **Some inferences from case histories.** Now what can be inferred from such observations as these? We can hardly *prove* anything, in the sense that we prove a mathematical theorem, any more than we can prove that certain methods of teaching mathematics are the best we might use. In such matters we are dealing with a kind of "metamathematics," although not in the technical sense in which the term is being used, to be sure. We can, however, make certain inferences of value.

4a. *Factors influential in concept formation.* For the sake of completeness we first note the existence of evolution of mathematical concepts from the general cultural environment—the non-mathematical environment, that is. This was not only operative in the origin of the primitive concepts concerning number and geometry, but has continued throughout mathematical history. It is going on today, as anyone who has worked on war or government projects can bear witness. And although it deserves closer study, I shall not give it any more attention here, but shall pass on to factors operative within mathematics. So from now on, in my conclusions, I shall treat mathematics as an organism which, although influenced by an outside environment, is going to be of interest only from the standpoint of its internal structure.

I have just used the term "organism." This was to emphasize the *cooperative* character of mathematical creation, as contrasted with the individual aspect. As Veblen observed, mathematics is "terribly individual." But we don't work in a vacuum. Neither, it is true, do we work in teams like the experimental scientists do, although we do collaborate in small groups of two or three—rarely in greater number. I have in mind, however, also those types of cooperation which are not so obvious—especially the type of cooperation I imply when I make the statement that a Newton can carry on only from the level which the mathematics of his time has reached. This is not peculiar to mathematics, of course—Beethoven was as much indebted to the musical developments of his predecessors as to the more obvious inventions of the musical scale in which he wrote and the instruments for which he composed. But I think that we mathematicians have not been properly conscious of the fact. When, for example, a genius of the stature of Leibniz expends so much energy in collecting evidence of the priority of his ideas over those of Newton (see [**17**], for instance), is it not clear that he is not properly aware of the debt owed to those on whose shoulders he stood? Or, when a concept has germinated and gradually emerged over a period of anywhere from 50 to 2000 years, and then is suddenly brought into full light through the inventive powers of two, or possibly three or four mathematicians —and when, moreover, historical research shows that others were nearly ready to bring out the idea—does it make sense to call the event a *coincidence*?

This is not to deprive brilliant intellects that make final syntheses of concepts of any credit. Without them, concepts would never get born. But it is equally true that without the cooperation which they have received from innumerable other workers, many of whom may be gone and forgotten, the material from which they made their syntheses, and the tools they used, would not have been available. And by properly giving credit where credit is due, perhaps more modesty in the effort to establish priority will make everyone concerned a little happier.

Of course many mathematicians have sensed the organic character of mathematics. One of the most recent examples I have noticed of this was in an article by H. Weyl entitled *A half-century of mathematics* [18]. When half-way through this article he pauses and exclaims, "The constructions of the mathematical mind are at the same time free and necessary. The individual mathematician feels free to define his notions and to set up his axioms as he pleases. But the question is, will he get his fellow-mathematicians interested in the constructs of his imagination? We can not help feeling that certain mathematical structures which have evolved through the combined efforts of the mathematical community bear the stamp of a necessity not affected by the accidents of their historical birth. Everybody who looks at the spectacle of modern algebra will be struck by this complementarity of freedom and necessity."

The individual mathematician cannot do otherwise than preserve his contact with the mathematical culture stream; he is not only limited by the state of its development and the tools which it has brought forth, but he must accommodate his desires to those concepts that have reached a state where they are ready for synthesis.

Analytic geometry could not be synthesized in the manner which Fermat and Descartes are usually credited with doing, until both the necessary geometry and algebra were at hand; nor until their predecessors had carried the synthesis nearly to fruition. The same remark holds for calculus. If we interpret the achievement of Newton and Leibniz as the creation of a symbolic method which synthesized those ideas of the calculus which were already in existence, then their dependence on the cooperation of others is clear. And it seems doubtful that Menger and Urysohn would have, independently, arrived at the precise notions of dimension and curve which they formulated, without the preliminary development of not only the topological tools necessary for their formulation, but of such concepts as that of *localization* which were not, apparently, possessed by Poincaré. And that a concept of dimension was trying to break through to the mathematical consciousness is evidenced by the related problems that had been treated during the preceding decades, as well as by its ultimate formulation by two independent workers (as in the case of analytic geometry and calculus).

As I said before, let me emphasize that I do not make these observations with any intent of taking away credit where credit is due. Furthermore, we must beware of looking backward from the heights which we have now scaled and concluding that the tasks of our predecessors were easy. Not long ago I overheard a group of young topologists discussing the state of topology about 1920, particularly the fertile field for research which was opening up at that time in the investigation of continuous curves. They seemed quite in agreement that they had been born 30 years too late—that it is a much harder task to find a subject for a dissertation now than it was then. As I was one of those who had been born 30 years sooner, I felt compelled to point out what the situation really was like at that time. I could tell them, for instance, of one young mathematician, who had just received the Ph.D. degree, and who decided to forsake topology since it was obviously all worked out! None of us really knew of the possibilities at that time. And much energy had to be expended on the invention of concepts which now seem almost trivial, and of new methods of attack.

The greatest factor in the evolution of concepts today is probably what I would call *conceptual contacts*; on the individual level, it is what we might call a "meeting of minds"; on the group level, it is the diffusion of concepts. Examples of the former, the meeting of individual minds, are to be found in the many cases of mathematicians, who, due to the political conditions prevailing in Europe the past 40 years, moved to new mathematical centers and established contacts with men whose ideas and methods often found fruitful syntheses with their own. Similar contacts are, of course, being made today through the medium of grants to foreign mathematicians enabling them to visit mathematical centers in this country—as well as in the reverse process wherein members of our group visit abroad. Diffusion of concepts on the group level is of that type which was exemplified in the past by the fusion of algebra and geometry to form analytic geometry; or by the diffusion into geometry of set-theoretic methods in order to provide satisfactory formulations of the curve and dimension concepts; or in the recent past by such fusions as those of algebra and topology to form algebraic topology or topological algebra, depending on which aspect of the fusion you place emphasis. Concepts such as these are not the result of the chance meeting of two mathematicians at a mathematical gathering or because of an invited lectureship, as was the case in some instances which I can recall; but resulted from the gradual building up, by many independent workers, of a host of component concepts. The manner in which such syntheses as these are brought about needs, I imagine, no further amplification. For the individual mathematical discipline they form the analogue, within the field of mathematics, of the interplay between mathematics and its environment in the general culture. They point up, of course, the desirability and the necessity, if we would encourage healthy mathematical growth, of inter-university cooperation on a more enduring basis than is provided by the meetings of societies; even though the latter do foster individual contacts and undoubtedly play an important part in developments on the higher group level.

Not far removed from the types of concept formation already mentioned is that which consists in the observation of similar patterns in several different branches of mathematics, or, at a lower level, in several special cases. The classic example is probably the concept of abstract group. The Jordan definition of continuous curve illustrates the formulation of a concept from observation of its special cases commonly used in analysis. And again this is a kind of concept-formation which, within mathematics, is the analogue of the type of concept-formation which has for its basis the observation of mathematical patterns in the external environment, particularly in physics.

Also of a related character is the introduction of new tools from other branches of mathematics. The concept of the general continuous curve space, formulated independently by Hahn and Mazurkiewicz, making use of the tools provided by abstract space concepts, is one example. Another example is the introduction of the axiomatic method into algebra with results which have been remarked upon by a number of writers, particularly Weyl [18]. A current example is the use of metamathematical tools to establish theorems about the theorems in a given branch or branches of mathematics.

This reminds one of certain methods, which, because of their wide applicability, have earned the right to special notice as concept-builders. The *axiomatic method* is one of the most notable of these. I cannot resist recalling here what Poincaré, in his article entitled *The future of mathematics*, conceived as the chief use to which the method would be put [2, p. 382]. He devoted less than half a page to it; remember that when he wrote, the method was still quite new in its modern sense. After commenting that Hilbert had obtained "the most brilliant results" with it, he observed that the problem of providing axiomatic foundations for various parts of mathematics would be very "restricted," and "there would be nothing more to do when the inventory should be ended, which could not take long. But when," he continued, "we shall have enumerated all, there will be many ways of classifying all; a good librarian always finds something to do, and each new classification will be instructive for the philosopher." It would seem that Poincaré believed that the method was of little importance for mathematical creation. I wonder what his opinion of it would be today, if he could see what a tool for research it has become?

Related to the axiomatic method, in that it provides a special field for its use, is the process of altering of concepts already formulated. This is the kind of thing we do when we replace an axiom, such as the parallel axiom of euclidean geometry for instance, by a contradictory—in the case mentioned replacing the parallel axiom by one of the types of non-parallel axioms. Here one has to be especially careful, of course, to preserve contact with established theories—or as Weyl put it, mindful of whether one will "get his fellow-mathematicians interested in the constructs of his imagination."

That this process operates on the group level as well as on the individual level is shown by the case already cited, that of non-euclidean geometry. When this began to break through to the individual mathematical consciousness, it had already smoldered for years in the fires of Saccheri's and others' determination to prove its logical impossibility. When it did find expression, it found it in several places. As Coolidge remarks [5, p. 73], "The outstanding effect of a comparison of the work of Lobachevski and Bolyai is surprise at their likeness. Both start with the parallel angle, both note the Euclidean nature of the geometry on the limiting surface, both develop the formula for corresponding lengths on parallel limiting surfaces, both note the relation to spherical trigonometry and the independence of this from the parallel axiom." And then when Gauss revealed that he had obtained almost identical results earlier, is it to be wondered that Bolyai suspected Lobachevski of plagiarism and exclaimed, "It can hardly be possible that two, even perhaps three, persons, knowing nothing of one another, have achieved almost complete the same results at about the same time, even if by different paths." It is a pity that he could not know that not only was it possible, but quite probable, since the evolution can be clearly traced in the mathematical and philosophical thought of the preceding centuries.

Then there is the well known and much abused generalization! That this operates on the individual level needs not to be elaborated on, since we all use it in our individual research. However, it does operate on the group level. For instance, that kind of generalization which results from the splitting of concepts may take place on the group level; I suppose the splitting of the concept of the real number system into its structural or topological aspects, and its operational or algebraic aspects, furnishes an example of this. Splitting of concepts, that is, may be due to the gradual evolution of the mathematical organism rather than to the conscious act of an individual mathematician—even though the final syntheses may be the acts of individuals.

4b. *Life span of a concept.* Having discussed some of the modes by which concepts originate, let us turn for a moment to the life of the concept, following its exact formulation. The examples of the precise formulation of the topological character of continuous curve by Hahn and Mazurkiewicz, and that of the general curve and dimension theory by Menger and Urysohn, are instructive here. It seems that the formulation of a new concept may result in a distinct branch, or sub-branch of mathematics as the case may be—a kind of "specialty," we might call it. This may become so popular that one is not in the "swing" in his particular field unless he joins in the investigation of the new concept. Mathematics is as subject to fashions as any other aspect of man's behaviour! Sooner or later, however, the so-called specialty is developed to such an extent that either it loses its popularity or achieves immortality through contact with other concepts. Thus we find in the cases mentioned that as soon as the related widespread investigations had resulted in solutions of the main problems, the fundamental parts of the theories were either absorbed in new concepts or made part of the general framework, both as to tools and materials, of general topology. The theory of continuous curves contributed not only such notions as that of local connectedness and its properties to the study of other theories, such as that of Lie groups, but was absorbed in a new concept, that of lc^* spaces (see [16, VI], for instance), whose definition became possible because of the extension of homology tools to general spaces in the decade succeeding the investigations on continuous curves. Like comments can be made concerning dimension theory, whose fundamental notions are now part of the general equipment of a topologist; and which was absorbed in a more general theory of dimensions utilizing homological notions [19; 14, VIII], and is constantly used in applications of topology. I presume that a more elementary example may be furnished by elementary geometry as the Greeks knew it. I imagine that it had already reached its zenith and was on the wane when Euclid wrote his encyclopaedic work—the historians can correct me if I am wrong about this. Here, if ever, was a case of a mathematical theory being embalmed. When mathematical archeologists, so to speak, resurrected it, it seems to have stimulated only futile attempts to prove the parallel postulate or to trisect the angle and square the circle. Only new perspective that comes with conceptual contacts and new tools could infuse life into it. And, of course, this came in the realization of the validity of the non-euclidean geometries, and the introduction of methods of algebra and analysis which resulted in the modern form of analytic geometry, and a surge upward in analysis following the rise of the calculus.

To summarize, then, a concept, having only a limited range of possibilities, seems to have a certain life span like that of a star, going through a period of tremendous activity and then waning. Without the infusion of, or fusion with, new conceptual material, it is probably destined to die. But there is insufficient evidence to judge of this, since fortunately the new material is usually ready. New tools may be introduced from other parts of mathematics or even from the same branch, or even suggested by cultural elements outside of mathematics; or the concept may join with other concepts to form new ones more fruitful than the old; I have mentioned examples of all these possibilities. I cannot, as a matter of fact, cite offhand any case of a mathematical concept that has died in the sense that it no longer has any use whatsoever. The accumulated body of knowledge to which it has led is sooner or later seized by workers in other branches of mathematics, or even outside of mathematics, and put to work—often in a way of which its devotees never dreamed.

An interesting fact, which is probably a result of this, is that whereas there may seem to emerge from all this the spawning of a host of narrow specialties, a process of unification is continually going on. It is as though mathematics strikes out in all visible directions, but conscious that it will never do to become too diverse, pauses periodically to consolidate and unify the gains made before making new advances. (For further discussion and case histories see [20] and citations therein.)

5. **Import to the individual mathematician.** Since I have spent so much time in this talk on the group level, I should like to make a comment or two about the implications to the individual. Some of these I mentioned in my introduction and will not repeat here. I should like to make one amplification of my previous remarks concerning the benefits that may accrue to the individual research worker, especially those who are starting their careers. I have been continually impressed, since beginning this study, with the manner in which it brings out not only the advisability and necessity of keeping in touch with the mathematical culture stream through the media of journals and personal contacts—something we already habitually do—but with the suggestive ideas that may be found by exploring the works of the past. I am not advocating here a mathe-

matical "great books' program," but just a little browsing now and then in the original attempts at formulations of commonly used concepts, as well as of concepts not yet satisfactorily defined. They often contain suggestions which were impossible of fulfillment at the time when written, because the tools for exploiting them were not available. It is not difficult to imagine that the dimension theory of Poincaré, which was published in a philosophical journal, might, if mathematics had developed in different fashion, not yet be susceptible to precise formulation; and would, in that case, be lying ready for the moment to arrive when the suitable tools become available.

For a similar reason, it seems that every new generation of mathematicians should give attention to the famous unsolved problems—for one never knows when the tools adequate for a solution may have evolved.

6. **Future developments.** It is time I brought this talk to a close. In doing so, I hope you will be so kind as to let me indulge in a little speculation. Up to this point I believe I have stuck to facts as well as one can in a domain where, as I said before, proofs are not possible in the mathematical sense. What I want to say now I shall freely label "speculation," and you can take it for what it seems worth to you.

Whenever I reflect on the changes that have occurred in mathematics during the past 40 years, I invariably recall that statement of Spengler whose *Decline of the West*, published in 1918, caused such a commotion in the intellectual world, namely: That in the concept of an abstract group mathematics achieved its "last and conclusive creation." And his prediction that "the time of the *great* mathematicians is past. Our tasks today are those of preserving, rounding off, refining, selection—in place of big dynamic creation, the same clever detail-work which characterized the Alexandrian mathematic of late Hellenism." Could any prophecy have been demonstrated fallacious more quickly and conclusively than this one has been? Also I recall a statement in Struik's little history that "toward the end of the 18th century some of the leading mathematicians expressed the feeling that the field of mathematics was somehow exhausted. The laborious efforts of Euler, Lagrange, D'Alembert, and others had already led to the most important theorems; the great standard texts had placed them, or would soon place them, in their proper setting; the few mathematicians of the next generation would only find minor problems to solve." And this was before 1800!

If I were given to the vice of prophecy, I would not hestitate to risk my reputation as a prophet with the prediction that mathematics, in this year of our Lord 1952, is only reaching out toward maturity, and that given encouragement it will, during the next 50 years, yield new concepts and methods which will revolutionize the subject. I envy those young men who are only on the threshold of their mathematical careers, for they will be possessed of powers that will put their elders to shame. I have already noticed, and no doubt some of you have, evidence of greater powers among recent recruits to the ranks of mathematical research. The future will no doubt see not only an increase in this, but a corresponding improvement will also be noticed in the ranks of undergraduates. I am one of those who believe that genius is not a rare occurrence—genius seems rare chiefly because of lack of opportunity. And in mathematics opportunity is measured to considerable extent by the quantity and magnitude of its concepts and the power of its methods. There is every indication that these are now approaching an all time high. To take a leaf from my own experiences: For some 30 years I have witnessed the growth of topology, sometimes halting, sometimes feverish, from a mere infant to a size where we don't even attempt to give it explicit definition any more. Yet today, I am amazed at the power and ingenuity of the new methods and concepts that are being introduced. And topology is not unique in this.

But it is not only for this reason that I feel as I do about the future of mathematics. I am also a firm believer in the evolutionary character of mathematical development. Concepts are not stable; they continually grow even though their outcroppings are discrete events. This we have noted in the evolution of the concepts of number and curve. And when a concept has evolved so far as to achieve a precise formulation in the mathematical framework then available, its further growth or evolution seems virtually determined by the existence of those logical laws and methods which every mathematician observes and uses. It is much as in the case of a particular axiomatic system, which, once set up, will lead almost inevitably to a certain set of theorems as its logical consequences. Or, to draw once more upon our case history of the concepts of curve and dimension, it is quite clear that the concepts of curve and dimension were already evolving in the mathematical culture stream before being precisely formulated. It is a nice experiment, by the way, to take a group of graduate students who have been trained in basic *modern* methods, especially in topology, but who have little or no acquaintance with dimension theory, and to give them Poincaré's notions about dimension. If they are any good, they will unfailingly grind out the concept of dimension which corresponds to these notions, just as Poincaré might have if he had had the necessary tools. This would not, of course, be the form which Menger and Urysohn gave. Similarly we can find, without reading too much into the prior facts, the evolving forms of concepts such as the calculus and analytic geometry. Their eventual emergence seems to be virtually a certainty. The ways of thought in which we are all trained, and which we all share more or less, seem as much of a guarantee of certain eventual conceptions being synthesized, from given basic material, as do the logical laws for the theorems of an axiomatic system.

As in any evolutionary process, however, the environment cannot be ignored. Just as the individual mathematician does not work in a vacuum, and is influenced by the work of his predecessors and coworkers, so does mathematics itself not evolve in a vacuum. If world crises occur, as we have ample reason to fear they may, they may negate entirely what I said above about the future of mathematical creation. That interplay between evolving concepts and the minds of those individual mathematicians who achieve their syntheses can be interrupted by forces from without. The process of mathematical creation can cease either from stagnation, as from lack of diffusion and the resultant mixing of concepts, or from stifling incidental to major disruptions of society. The former we have no longer any cause to fear; with modern means of communication, and other aids to cultural contacts, our field is more alive than ever before. With good fortune, we may escape the latter, and then I think the prophecy that I said I would make, if I were a prophet, will come true!

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UNIVERSITY OF MICHIGAN

THE JUNE MEETING IN MISSOULA

The four hundred ninety-third meeting of the American Mathematical Society was held at Montana State University, Missoula, Montana, on June 20, 1953. There were approximately seventy registrations, including the following forty-one members of the Society:

T. M. Apostol, B. H. Arnold, M. G. Arsove, Wilfred E. Barnes, R. A. Beaumont, L. G. Butler, Harold Chatland, P. A. Clement, C. M. Cramlet, R. Y. Dean, D. B. Dekker, F. E. Ehlers, Paul Erdös, R. M. Gordon, S. G. Hacker, M. E. Haller, C. A. Hayes, R. D. James, T. R. Jenkins, H. E. Kinerk, J. M. Kingston, M. S. Knebelman, R. B. Leipnik, A. T. Lonseth, A. S. Merrill, W. E. Milne, Leo Moser, W. M. Myers, Jr., Ivan Niven, A. R. Noble, I. L. Olson, T. G. Ostrom, W. T. Putney, R. A. Rosenbaum, R. G. Selfridge, A. J. Smith, W. M. Stone, J. R. Vatnsdal, R. M. Winger, R. J. Wisner, F. H. Young.

By invitation of the Committee to Select Hour Speakers for Far Western Sectional Meetings, Professor A. T. Lonseth of Oregon State College addressed the Society on *Approximate solutions of Fredholm-type integral equations*. The speaker was introduced by Professor R. M. Winger. The session for contributed papers began at ten o'clock Saturday morning. The presiding officer was Professor M. S. Knebelman. On June 19 there was the Pacific Northwest Sectional Meeting of the Mathematical Association of America. The hour speaker was Professor Leo Moser who addressed the Association on *The distribution of quadratic residues*. There was a joint dinner for members of the Society and the Association and their guests at the Hotel Florence, followed by a social hour at the Faculty Club.

Abstracts of the papers presented follow. Those whose abstract numbers are followed by "*t*" were presented by title. Paper number 530 was presented by Dr. J. E. Maxfield, number 534 by Professor Erdös, number 536 by Professor Hayes, and number 539 by Professor Leipnik. Professor Ohtsuka was introduced by Professor A. J. Lohwater.

ALGEBRA AND THEORY OF NUMBERS

526. T. M. Apostol: *Some series involving the Riemann zeta function.*

V. Ramaswami [J. London Math. Soc. vol. 9 (1934) pp. 165–169] has obtained three formulas which can be employed to obtain the analytic continuation of Riemann's zeta function $\zeta(s)$ over the whole s-plane. If $P_n(s)$ denotes the polynomial defined by $n!P_n(s)=s(s+1)(s+2)\cdots(s+n-1)$, then one of these formulas can be expressed as follows: $\zeta(s)(1-2^{-s}-3^{-s}-6^{-s})=1+2\sum_{n=1}^{\infty}P_{2n}(s)\zeta(2n+s)6^{-2n-s}$. In this paper simple properties of the Hurwitz zeta function are used to obtain the general identity: $\zeta(s)\sum_{d|k}\mu(d)d^{-s}=\phi_{-s}(k/2;\ k)+2\sum_{n=0}^{\infty}P_{2n}(s)\zeta(2n+s)k^{-2n-s}\phi_{2n}(k/2;\ k)$, where $k\neq 1$ is an integer, $\mu(n)$ is the Möbius function, and $\phi_\alpha(x;\ k)$ is the sum of the αth powers of those integers not exceeding x which are relatively prime to k. When $k\leq 6$, $k\neq 5$, the function $\phi_\alpha(k/2;\ k)$ becomes identically 1 and Ramaswami's formulas occur as the special cases $k=2, 3, 6$. The general formula can also be used to extend some of the other results in Ramaswami's paper. (Received May 7, 1953.)

527t. A. L. Foster: *Generalized "Boolean" theory of universal algebras; subdirect sums and normal representation theorem.*

In furtherance of earlier work [*On n-ality theories in rings*, etc., Amer. J. Math. (1950); *p-rings and their Boolean-vector representation*, Acta Math. (1950); etc.], dealing with extensions of the Boolean realm, the author here deals with a broader extension, to a comprehensive variety of universal algebras—of *class f*. The present paper defines and establishes the existence of a *normal expansion* for the functions of such universal algebras $\mathfrak{A}$, and only for such, as are *normal* subdirect sums of an algebra A of class f ($=$kernel). For appropriate kernels this incorporates the known normal expansions of Boolean-rings, p- and p^k-rings, and Post algebras. Various consequences of the existence of the normal expansion are studied, with emphasis on functionally complete kernels. A universal algebra A is of *class f* if for some $e_0, e_1\in A$ functions x' of the algebra exist such that $'$ is a permutation with $e_0'=e_1$, $e_1'=e_0$, and x is binary with e_0, e_1 as (both sided) null and identity. Rings, complemented lattices, Post algebras, etc., are of class f. A subdirect sum $\mathfrak{A}$ of A is *normal* if $\mathfrak{A}$ contains all $(\alpha, \alpha, \alpha, \cdots)$ $(\alpha\in A)$, and if with $\bar{a}\in\mathfrak{A}$ and for each $\mu\in A$, all projections $P_\mu\bar{a}$ are $\in\mathfrak{A}$. Here with $\bar{a}=(\alpha_1, \alpha_2, \cdots)$, $P_\mu\bar{a}$ replaces α_i by e_1 if $\alpha_i=\mu$ and by e_0 if $\alpha_i\neq\mu$. (Received April 27, 1953.)

528t. A. L. Foster: *Identities and subdirect sums of functionally complete universal algebras.* Preliminary report.

Among others the following results are established. *Theorem* 1. *If A and $\mathfrak{A}$ are universal algebras of the same species in which* (1°) *A is finite and functionally complete,* (2°) *every identity of A is an identity of $\mathfrak{A}$,* (3°) *neither A nor $\mathfrak{A}$ is the one-element algebra, then $\mathfrak{A}$ is a normal subdirect sum of A.* This result incorporates the familiar subdirect structure of such algebras as Boolean-rings, p- and p^k-rings, Post algebras, etc. A set I of identities is equationally saturated if I^* has only the trivial one-element algebra as a model; here $I^*=I$ augmented by any equation logically independent of I. *Theorem* 2. *If A is as in Theorem* 1, *the class I of all identities of A is equationally saturated. Moreover I has a finite basis.* A more general result, similar to Theorem 1, is established for the case where A is merely weakly functionally complete, i.e., where constants are permitted in both A-functions and equations. The methods employed borrow heavily from those used in the previous abstract. (Received April 27, 1953.)

529t. Mariano García: *On numbers with integral harmonic mean.*

The following results are obtained: (1) If n is an odd integer having an integral harmonic mean for its divisors, the prime factor decomposition of n cannot contain a prime of the form $(4k-1)$ raised to an odd power. (2) No odd number having a prime factor decomposition of the form $p_1^{2^{n_1}-1}\cdot p_2^{2^{n_2}-1}\cdots p_r^{2^{n_r}-1}$ can have an integral harmonic mean for its divisors. (3) Except for perfect numbers, no integer with prime factor decomposition $p^\alpha q$ can have an integral harmonic mean for its divisors.

These results are used to extend the table given by Ore [Amer. Math. Monthly vol. 55 (1948) pp. 615–619] of the positive integers less than 10,000 that have an integral harmonic mean for their divisors. The integers between 1 and 10,000,000 which have an integral harmonic mean are tabulated and these turn out to be all even. (Received May 7, 1953.)

530. J. E. Maxfield and Margaret W. Maxfield: *Sums of powers of numbers having a given period modulo m.*

In this paper the following theorem is proved: The sum of the positive integers less than p^s having period $\sigma = p^r\sigma'$ (mod p^s), where $0 \leqq r < s$, $p \nmid \sigma'$, $s \geqq 1$, and p is odd, is congruent to $\phi(p^r)\mu(\sigma')$(mod p^s), where $\phi(m)$ is the Euler ϕ-function and $\mu(\sigma')$ is the Moebius function. An analogous theorem is proved for the prime 2 and a solution of the problem modulo any composite is indicated. (Received May 11, 1953.)

531. Ivan Niven: *On the irrationality of trigonometric and hyperbolic functions for rational arguments.*

The method given by the writer to prove the irrationality of π [Bull. Amer. Math. Soc. vol. 53 (1947) p. 509] was extended by others to related problems of irrationality. For example, J. F. Koksma [Nieuw Archief voor Wiskunde (2) vol. 23 (1949) p. 39] treated e^r for nonzero rational r, and R. Rado [J. London Math. Soc. vol. 23 (1948) pp. 267–271] developed a generalization to solutions of certain systems of differential equations, with e^r and tan r as special cases. We here extend the method to cos r and cosh r, from which irrationality of all the trigonometric and hyperbolic functions for nonzero rational arguments follows readily. (Received April 22, 1953.)

532*t*. Alex Rosenberg: *Finite-dimensional and nil subalgebras of certain primitive algebras.*

Let M, N be a pair of dual vector spaces over a division ring D with center Z, and let M and N both be of countable dimensions. Let $A = L(M, N)$ be the ring of all linear transformations on M with adjoints on N and let $S = F(M, N)$ be its socle. The following theorems are proved: Let B be a simple subalgebra of A containing the unit of A and finite-dimensional over Z, then if B' is the commutator of B in A, $B'' = B$. If B and C are two isomorphic subalgebras satisfying the same hypotheses as B in the previous theorem, the isomorphism can be extended to an inner isomorphism of A. Let R be a maximal nil subring of S, then if R annihilates a vector in M (R^* annihilates a vector in N) there exists a pair of biorthogonal bases of M, N such that R can be written as the ring of matrices zero on and below (zero on and above) the diagonal. This generalizes a theorem of Shoda-Köthe-Levitzki. (Received May 7, 1953.)

533*t*. Alex Rosenberg and Daniel Zelinsky: *On Nakayama's extension of the $x^{n(x)}$ theorems.*

Nakayama showed (Canadian Journal of Mathematics vol. 5 (1953) pp. 242–244) that if A is a division ring with center Z such that (1): $\alpha_1 a^{n_1(a)} + \cdots + \alpha_r a^{n_r(a)}$ is in Z for fixed central elements $\alpha_1, \alpha_2, \cdots, \alpha_r$ and $n_1(a) < n_i(a)$, $i = 2, \cdots, r$, then $A = Z$. We show that a primitive ring satisfies (1) if and only if it is commutative or an algebraic algebra over a finite field. In the latter case the center is an absolutely algebraic field of characteristic $p \neq 0$, and (1) implies the condition (2): $a^{n(a)} - a^{m(a)} = 0$. If A is a semi-simple algebra satisfying (1) with bounded $n_i(a)$ it is a subdirect sum of a commutative algebra and an algebra R. Here R is a subdirect sum of finite simple rings and satisfies (2) with fixed $n(a)$, $m(a)$. If the $n_i(a)$ in (1) are no longer assumed bounded the algebra R need no longer satisfy (2). (Received May 11, 1953.)

ANALYSIS

534. Frederick Bagemihl, Paul Erdös, and Wladimer Seidel: *Some boundary properties of analytic functions defined by certain infinite products in the unit circle.*

It is shown by direct and completely elementary methods that, if the sequence of positive integers $n_j \to \infty$ rapidly enough, $f(z) = \prod_{j=1}^{\infty} \{1 - z^{n_j}(1 - n_j^{-1})^{-n_j}\}$ is regular in $|z| < 1$ and possesses numerous boundary properties, including some established for other functions by Fatou (Bull. Soc. Math. France vol. 48 (1920) pp. 208–314), Lusin and Priwaloff (Ann. École Norm. (3) vol. 42 (1925) pp. 143–191), and others; e.g.: there exists a circle C_n in $|z| < 1$ about each zero of $f(z)$ such that the C_n are mutually exterior to one another and $|f(z)| \to \infty$ as $|z| \to 1$ outside the C_n; $|f(z)| \to \infty$ uniformly on a sequence of circles with the origin as center; $|f(z)| \to \infty$ along almost all radii of $|z| < 1$, and along spirals in $|z| < 1$ which approach $|z| = 1$ asymptotically; $|f(z)| + |f'(z)| \to \infty$ as $|z| \to 1$ in $|z| < 1$; $|f(z)|$ grows arbitrarily slowly as $|z| \to 1$. With suitable modifications of the above product, further examples are obtained concerning the radial limits of functions regular in $|z| < 1$; e.g.: a $g(z)$ for which $|g(z)|$ has no radial limit but $\to \infty$ uniformly on a sequence of circles with the origin as center; an $h(z)$ with $|h(z)| \to 0$ along almost all radii. (Received May 4, 1953.)

535*t*. A. P. Calderón and Antoni Zygmund: *On singular integrals.*

Let $P = (x_1, x_2, \cdots, x_n)$, $K(P)$ a homogeneous function of degree $-n$, $K_\lambda(P) = K(P)$ for $|P| = (x_1^2 + x_2^2 + \cdots + x_n^2)^{1/2} > \lambda$ and $K_\lambda(P) = 0$ otherwise, and $f(P) \in L^p$. The authors showed in a previous paper that if $K(P)$ satisfies certain continuity conditions, then the integral $\int_{E_n} K_\lambda(P - Q) f(Q) dQ$ converges in the mean of order p and pointwise almost everywhere as $\lambda \to 0$. The continuity conditions on $K(P)$ are removed and these results are shown to hold if: (i) $K(P)$ is odd, that is, $K(P) = -K(-P)$, and the restriction of $|K(P)|$ to the spherical surface $|P| = 1$ is integrable; (ii) $K(P)$ is even, that is, $K(P) = K(-P)$, and the restriction of $|K(P)| \log^+ |K(P)|$ to $|P| = 1$ is integrable. These results are the best possible of their type and are obtained by a method entirely different from the previous one. (Received April 29, 1953.)

536. C. A. Hayes and C. Y. Pauc: *The problem of the differentiation of set functions.* Preliminary report.

This paper complements the papers *Vitalische Systeme in booleschen σ-Verbanden* by O. Haupt and C. Y. Pauc, Sitzungsberichte der Mathematisch-Naturwissenschaftlichen Klasse der Bayerischen Akademie der Wissenschaften zu Munchen (1950) pp. 187–207, *Differentiation with respect to φ-pseudo-strong blankets* by C. Hayes, Proc. Amer. Math. Soc. vol. 3 (1952) pp. 283–296, and *Differentiation of some classes of set functions* by C. Hayes, Proc. Cambridge Philos. Soc. vol. 48 (1952) pp. 374–382. The writers treat several types of differentiation bases (of which *blankets* are a special case) endowed with various properties which are sufficient to permit the differentiation of suitable corresponding classes of set functions. The present paper deals with the converse problem; it is shown for each of the different types of bases that the possession of differentiation power implies a corresponding covering property. The relationship between the present paper and some earlier efforts of R. de Possel, H. Busemann, W. Feller, and B. Younovitch are also explored. (Received April 6, 1953.)

537*t*. V. E. Hoggatt, Jr.: *The inverse Weierstrass p-function: numerical solution and related properties*. Preliminary report.

The inversion integral for the Weierstrass elliptic p-function is numerically evaluated by expanding the integrand in a power series and applying the contour integral term by term. The only assumption is a knowledge of the periods ω and ω' of $p(z)$. If $g_3 \neq 0$, the transformation $g_2 = 12\lambda^4$; $g_3 = 4(2+k)\lambda^6$; $t = \lambda^2 x$ reduces the expression under the radical in the integrand from $(4t^3 - g_2 t - g_3)$ to $[4\lambda^6\{x^3 - 3x - (2+k)\}]$ which is expanded in a Maclaurin's expansion in k. For $p(z) = 0$ the series converges absolutely and uniformly for $|k| < 2$. The resulting contour integrals, which are the coefficients of powers of k, are reducible to a finite number of elementary functions. The series is valid for finding $z = p^{-1}(\alpha)$ for all complex α except in three regions around the singularities of the coefficient integrals and applies to the entire rectangular case and a large part of the skew case. (Received May 6, 1953.)

538*t*. R. E. Lane: *Note concerning the Stieltjes integral.*

A modification (based on the trapezoidal rule) of the definition of the Riemann-Stieltjes integral is used to define a more general integral which has most of the desirable properties of the Stieltjes integral, including integration by parts. Under this definition the integral $\int_a^b u(x)dv(x)$ exists if, for example, u is *any* function which has only discontinuities of the first kind in $[a, b]$ and v is *any* function which is of bounded variation in $[a, b]$. The definition can be extended to functions of more than one variable. (Received April 8, 1953.)

539. R. B. Leipnik and A. P. Morse: *Heaviside calculus of difference operators.*

The Heaviside calculus of power fields of operators developed by the authors is applied to the simple special case of difference operators. Strengthened forms of classical results of difference calculus are obtained with ease. (Received June 20, 1953.)

540*t*. Makoto Ohtsuka: *Capacity of product sets.*

Let Ω be a metric space and $U_\alpha^\mu(P)$, $\alpha \geqq 0$, the potential $\int_\Omega \phi(r_{PQ})\, d\mu\ (Q)$, where $\phi(r_{PQ}) = r_{PQ}^{\alpha}$ if $\alpha > 0$ and $\phi(r_{PQ}) = -\log r_{PQ}$ if $\alpha = 0$, and where μ is a non-negative distribution of unit mass on a bounded closed set F in Ω. Let $V = \inf_\mu(\sup_{P \in \Omega} U_\alpha^\mu(P))$, and define $C^{(\alpha)}(F) = \phi^{-1}(V)$. For a set X in Ω, define $C_i^{(\alpha)}(X) = \sup\ C^{(\alpha)}(F)$ over all closed bounded sets $F \subset X$. A bounded set X is called an α-polar set if there exists μ such that $U_\alpha^\mu(P) = \infty$ for every $P \in X$. Let $X \times Y$ be a product set in the metric space $\Omega_1 \times \Omega_2$, where $X \subset \Omega_1$, $Y \subset \Omega_2$. Then a lower bound for $C_i^{(\alpha+\beta)}(X \times Y)$ is given in terms of $C_i^{(\alpha)}(X)$ and $C_i^{(\beta)}(Y)$. If Ω_1 is Euclidean n-space, then the lower bound for $C_i^{(n+\beta)}(X \times Y)$ can be expressed in terms of $C_i^{(\beta)}(Y)$ and $m_n(X)$, the n-dimensional inner Lebesgue measure of X. It is shown that $C_i^{(n+\beta)}(X \times Y) > 0$ if $m_n(X) > 0$ and $C_i^{(\beta)}(Y) > 0$, and that $X \times Y$ is an $(n+\beta)$-polar set if X is bounded and if Y is a β-polar set. Applications are made to the regularity criteria of boundary points for the Dirichlet problem. (Received May 6, 1953.)

541. R. G. Selfridge: *Generalized Walsh transforms.*

Complex-valued Walsh functions have been introduced by Chrestenson (see Bull. Amer. Math. Soc. Abstract 59-4-503) and shown to form an orthonormal, complete system over $[0, 1]$. For the complex-valued Walsh function one can define a Walsh function with arbitrary subscript by $\psi_y(x) = \psi_{[y]}(x)\psi_{[x]}(y)$. The resulting class includes the ordinary Walsh functions (see Fine, Trans. Amer. Math. Soc. vol. 69 (1950) pp. 66–77). With this Walsh function one can develop a transform theory for functions in $L_p(0, \infty)$. For functions in L_2 a Plancherel theorem is developed. For functions in L_p, $1 < p < 2$, a transform exists that maps L_p into $L_{p/(p-1)}$. If $f(x)$ is in L_1 and $F(y) = \int_0^\infty f(x)\psi_y(x)dx$, then many of the theorems of ordinary Fourier transforms carry over. In particular one has $f(x) = \lim_{R\to\infty}\int_0^R F(y)\overline{\psi_x(y)}dy$ if $f(y)$ is of bounded variation in a neighborhood of x and x is either a point of continuity or a point of right-hand continuity and has a finite expansion in powers of α, where $\psi_x(x)$ has been developed in terms of α. Further one has that $f(x) = \lim_{R\to\infty}\int_0^R(1-[y]/R)F(y)\overline{\psi_x(y)}dy$ for almost every x at which $f(y)$ is locally essentially bounded. (Received April 20, 1953.)

542t. R. N. Tompson: *Areas of k-dimensional non-parametric surfaces in k+1 space.*

Suppose k is a positive integer greater than one, X is a k cell in Euclidean k space E_k, g is a continuous real-valued function on X, and the function f is the k-dimensional nonparametric surface in E_{k+1} defined by $f(x) = (x_1, \cdots, x_k, g(x))$ for $x \in X$. It is shown that the k-dimensional Lebesgue area of f = the k-dimensional integralgeometric stable area of $f = F^k_{k+1}$ (range f), where F^k_{k+1} is the k-dimensional integralgeometric (Favard) measure over E_{k+1} (see H. Federer, *Measure and area*, Bull. Amer. Math. Soc. vol. 58 (1952) pp. 306–378). It is also proved that F^k_{k+1} (range f) is finite if and only if g is of bounded variation in the sense of Tonelli. If g is of bounded variation in the sense of Tonelli, then $\int_X Jf(x)dL_k x \leq F^k_{k+1}$ (range f) where Jf is the Jacobian associated with f by means of its approximate differential and L_k is the k-dimensional Lebesgue measure. Equality holds if and only if g is absolutely continuous in the sense of Tonelli. (Received March 17, 1953.)

543t. Alexander Weinstein: *On the Cauchy problem for the Euler-Poisson-Darboux equation.*

Let k be a real parameter, $k \neq -1, -3, \cdots$, and let x denote a system of m variables $x_1, \cdots, x_m$. Let $u^k(x, t)$ satisfy the equation (1) $\Delta u^k = u^k_{tt} + kt^{-1}u^k_t$. A solution of the Cauchy problem with the initial data $u^k(x, 0) = f(x)$, $u^k_t(x, 0) = 0$, given in a previous paper (C. R. Acad. Sci. Paris vol. 234 (1952) pp. 2584–2585) is replaced by a new and simpler formula which is valid for $k < m-1$. Let the integer n satisfy the inequality $k+2n \geq m-1$. By the results obtained previously, u^{k+2n} can be determined by the initial conditions $(k+1)\cdots(k+2n-1)u^{k+2n}(x, 0) = f(x)$, $u^{k+2n}_t(x, 0) = 0$. A repeated application of the formulas $u^k = t^{1-k}u^{2-k}$, $u^k_t = tu^{k+2}$ yields the new formula $u = t^{1-k}(\partial/t\partial t)^n(t^{k+2n-1}u^{k+2n})$ which solves Cauchy's problem for $k < m-1$. Another solution of the same problem which includes the exceptional values $k = -1, -3, \cdots$ will be published by Diaz and Weinberger in the Proceedings of the American Mathematical Society. This work has been sponsored by the Office of Naval Research. (Received May 6, 1953.)

APPLIED MATHEMATICS

544t. Abraham Charnes: *Constrained games and linear programming.*

Constrained games are viewed as "semi-normalization" of unconstrained games. Equivalence of optimal play for player 1 vs. player 2 with (resp.) dual linear programming problems is established in a fixed-point-free proof of the saddle-point property of a pair of dual optimal solutions. Fixed-point-free proof of the fundamental theorem for convex games is a corollary. (Received April 30, 1953.)

545. F. E. Ehlers: *The induced velocity field of an axially symmetric supersonic jet at an angle of attack in a supersonic stream.* Preliminary report.

The linearized equations for the perturbation velocity potentials inside an axially symmetric jet and outside in the supersonic stream are solved by the method of the Laplace transform. The angle of incidence and the pressure difference are assumed small. As a consequence the conditions of continuous pressure and velocity inclination which must hold on the free boundary of the jet are satisfied instead on a circular cylinder which nearly represents the jet. The results are obtained in the form of infinite integrals which can be evaluated numerically. (Received May 4, 1953.)

546t. P. G. Hodge, Jr.: *The effect of strain-hardening on an annular slab.*

A procedure is outlined for obtaining the stresses and strains in a circular slab with a cutout, subject to uniform biaxial tension. An arbitrary stress-strain curve in tension is approximated by any number of straight-line segments. For biaxial states of stress the material is assumed to satisfy a flow law based on the maximum shear stress, and to be incompressible throughout. The general equations are given and then simplified by assuming that boundary motions can be neglected if the strains are small, and that elastic strain components may be neglected if the strains are large. For the case of linear strain-hardening a complete solution is given in closed form. If the rate of strain-hardening is small, these results may be further simplified. (The results presented in this paper were obtained in the course of research conducted as a Consultant under Contract N7onr-35810 between the Office of Naval Research and Brown University.) (Received May 6, 1953.)

547t. Eugene Levin: *Symmetric reinforcements of circular cutouts.*

A slab with a circular cutout and a reinforcement of arbitrary symmetric cross section is subjected to stresses in the plane of the slab. The slab and the reinforcement are assumed to be of a perfectly plastic material which satisfies Tresca's yield criterion. Under the assumption of generalized plane stress and using a limit design theorem of Drucker, Greenberg, and Prager (Quarterly of Applied Mathematics vol. 9 (1952) pp. 381–389) a method is described for the determination of a lower bound on the collapse load. The results agree with those previously obtained by Weiss, Prager, and Hodge (Journal of Applied Mechanics vol. 19 (1952) pp. 397–402) for the special case of a cylindrical reinforcement. (The results presented in this paper were obtained in the course of research conducted under Contract N7onr-35810 between the Office of Naval Research and Brown University.) (Received May 6, 1953.)

548. W. M. Stone: *On the effect of a randomly modulated carrier in pulsed radar theory.*

J. I. Marcum (Report 314, Rand Corporation) has exhaustively discussed the probability of detection for a pulsed radar, assuming a constant amplitude of signal return from target. The theory has been extended to include a randomly modulated signal by means of an averaging process. A considerable number of curves are constructed by using either an Edgeworth asymptotic expansion or by the numerical evaluation of a certain definite integral. (Received May 4, 1953.)

GEOMETRY

549*t*. H. W. E. Schwerdtfeger: *On Jacobsthal's circle matrices.*

Let K be a matrix with complex elements a and b in the first row and c and d in the second row. It is said to be a circle matrix (E. Jacobsthal, *Zur Theorie der linearen Abbildungen*, Berliner Math. Ges. Sitzungsber. vol. 33 (1934) pp. 15–34) if it satisfies the condition $K\overline{K}=\mu E$ ($E=$ a matrix with 1 and 0 in the first row and 0 and 1 in the second row). This is necessary and sufficient for the transformation $Z=(az+b)/(cz+d)=K(z)$ to be an inversion, with respect to a real circle (denoted by $\mathfrak{K}$) if the multiplier $\mu>0$, with respect to an imaginary circle if $\mu<0$. If $c=0$ the circle $\mathfrak{K}$ becomes a straight line; if $\mu=0$, a point circle. The equation of the circle $\mathfrak{K}$ is given by $z=K(z)$. Jacobsthal suggests (loc. cit. p. 28) the study of the circles $\mathfrak{K}^m$ for all integral m which corresponds to the circle matrices K^m. Instead we consider the continuous iteration $\mathfrak{K}^s$ for all real s, corresponding to the iteration K^s of the Moebius transformation K, i.e. $Z=K(z)$. It is found that for $\mu>0$ the transformation K can only be either hyperbolic (then the circle $\mathfrak{K}$ has no point in common with the real axis) or parabolic (then the circle $\mathfrak{K}$ touches the real axis at the fixed point of the Moebius transformation K) or elliptic (when the circle $\mathfrak{K}$ cuts the real axis at the fixed points of the transformation K); further that the circle $\mathfrak{K}$ is an element of the pencil P of all circles which are interchanged by the transformation K, thus orthogonal to the invariant circles. Then it is shown that the continuous iteration K^s $(-\infty<s<+\infty)$ corresponds to the continuous development of the pencil P out of the circle $\mathfrak{K}$. (Received May 21, 1953.)

STATISTICS AND PROBABILITY

550*t*. Ernest Ikenberry: *The probability of convergence of Gram-Charlier expansions of distribution functions.*

This paper gives the probabilities of convergence of Gram-Charlier expansions exp. $\{-k^2(x-X)^2\}\cdot\sum_{r=0}^{\infty}C_rH_r\{k(x-X)\}$ of the Gaussian distribution function $f(x)=((1/(2\pi)^{1/2})\sigma)\cdot\text{exp.}\ \{-(x-\mu)^2/2\sigma^2\}$, where X (the mean) and $2/k^2=s^2$ (the variance) are evaluated for a random sample $\{X_i\}$ of n from a normal parent population $N(\mu,\sigma^2)$ with mean μ and variance σ^2. It has recently been shown that the series expansion exp. $(-k^2w^2)\cdot\sum_{r=0}^{\infty}a_rH_r(kw)$ of exp. $(-\alpha w^2)\cdot P(w)$, where the $H_r(kw)$ are Hermite polynomials of degree r and $P(w)$ is any entire function of order less than 2, or of minimal type if of order 2, converges provided that k^2 lies inside of the circle $|k^2-\alpha|=|\alpha|$ in the k^2-plane [E. Ikenberry and W. A. Rutledge, Journal of Mathematics and Physics vol. 31 (1952) pp. 180–183]. Since the computed k^2 is always real and positive, the probability of the convergence of the expansion of $f(x)$ is given by $P=P(k^2<1/\sigma^2)=P(\nu s^2/\sigma^2<\nu/2)$, where $\nu=n-1$ is the number of degrees of freedom. It is known that $(\nu s^2/\sigma^2)$ is distributed as χ^2 with $\nu=n-1$ degrees of freedom. Hence P, for a given ν, may be obtained by locating in a table of percentage points of the chi-square distribution the tabular value of χ^2 which is equal to $\nu/2$, interpolating where necessary. Values of P range from .342 for $\nu=1$ to .988 for $\nu=29$. (Received March 20, 1953.)

551t. Jacob Wolfowitz: *Estimation of a linear relationship when both variables are subject to error.*

The estimators obtained by the author (Skandinavisk Aktuarietidskrift, 1952, pp. 132–151) converge with probability one no matter what the covariance matrix of u and v is. Under weak regularity conditions the results are valid when the incidental parameters are constants (and not chance variables). When the incidental parameters are independent, identically distributed chance variables, these regularity conditions are fulfilled with probability one. In the notation of the paper cited let $D_n(x)$ be a normal distribution function with mean zero such that $\delta(B_n(x),\ D_n(x))<1/n+\delta(B_n(x),\ N^*)$. The variance of $D_n(x)$ converges to the variance of $(v-\beta u)$ with probability one. (Received April 30, 1953.)

552t. Jacob Wolfowitz: *The estimation of structural parameters.*

Let $\{X_{ij}\}$, $i=1, \cdots, n$; $j=1, \cdots, m_i$, be independently distributed chance variables. Let $F_i(x|\theta, \alpha_i)$ be the distribution function of $X_{i1}, \cdots, X_{im_i}$, and $B^n(x)$ be the empiric distribution function of $\{X_{ij}\}$, for all i and j. Let $C^n(x|\theta', \alpha')$ be $(\sum_{i=1}^{n} m_i)^{-1} \sum_{i=1}^{n} m_i F_i(x|\theta', \alpha_i')$. Let $\theta_n^*, \alpha_{1n}^*, \cdots, \alpha_{nn}^*$ be Borel measurable functions of $\{X_{ij}\}$ such that, writing $\alpha_n^*=(\alpha_{1n}^*, \cdots, \alpha_{nn}^*)$, $\sup_x |C^n(x|\theta_n^*, \alpha_n^*)-B^n(x)|<1/n+\inf_{\theta',\alpha'} \sup_x |C^n(x|\theta', \alpha')-B^n(x)|$. Under mild regularity conditions θ_n^* converges to θ with probability one. (Received April 30, 1953.)

TOPOLOGY

553t. B. J. Ball: *A note on paracompactness.*

It is shown that every linearly ordered space is countably paracompact and that a necessary and sufficient condition that a locally compact linear space be paracompact is that it be covered by a collection of mutually exclusive, closed, open sets each of which is the sum of a countable number of intervals. A sufficient condition that a normal Hausdorff space H be paracompact is that it have the Lindelöf property and in case every uncountable subset of H has a limit point, this condition is also necessary. (Received May 7, 1953.)

554t. Armand Borel: *The ends of homogeneous spaces of Lie groups.*

G denotes a connected Lie group, H a closed subgroup of G. It is first proved that if H is connected, the homology groups of the coset space G/H are isomorphic to those of K/L, where $K\supset L$ are maximal compact subgroups of G and H respectively; from that and from a theorem of E. Specker one deduces that if H is connected, G/H has at most two ends (in the sense of Freudenthal). It is moreover established that if G/H has two ends, it is homeomorphic to the topological product of K/L by a line. Simple 3-dimensional examples (manifolds of unit tangent vectors to certain Riemann surfaces) show the existence of homogeneous spaces G/H (H non connected) having an infinite number or any finite number of ends. The second part of the paper deals with the determination of spaces which are called n-homogeneous, i.e. which admit a Lie group of homeomorphisms transitive on m-tuples of points. The results are as follows: There is no simply connected space which is either non compact and m-homogeneous for $m\geqq 3$, or compact and m-homogeneous for $m\geqq 4$. The simply connected 2-homogeneous spaces are the euclidean spaces, the spheres, the complex and quaternionian projective spaces, the projective plane over the Cayley numbers, and, among them, only the spheres are 3-homogeneous. (Received May 4, 1953.)

555*t*. S. D. Liao: *On topology of cyclic products of spheres.*

The p-fold cyclic product Σ_{np} of an n-sphere S_n, $n\geqq 2$, has vanishing integral cohomology groups $H^s(\Sigma_{np}, Z)$ for $1\leqq s<n$, $s=n+1$, and has $H^n(\Sigma_{np}, Z)\approx Z$. We calculate for a prime p the iterated cyclic reduced powers (mod p) of the generator of $H^n(\Sigma_{np}, Z)$. We study also some homotopy properties of Σ_{np} for $p=2$, $p=3$. We have that the 2-primary subgroup of $\pi_s(\Sigma_{n2})$ vanishes for $5\leqq n<s\leqq n+3$. Let S_n be oriented, and let it be imbedded in Σ_{np} for an arbitrary p such that S_n represents the generator of $H_n(\Sigma_{np}, Z)$. Let E_{n+4} be an oriented $(n+4)$-cell. Then, a representative map $f:\partial E_{n+4}\to S_n$ of an element α of $\pi_{n+3}(S_n)$ extends over E_{n+4} in Σ_{n2} for $n\geqq 5$ if α is a 2-primary element and in Σ_{n3} for $n\geqq 3$ if α is a 3-primary element. In the former case, $\pi_{n+3}(S_n)/2\pi_{n+3}(S_n)$ is characterized by the square operation Sq^4. In the latter case the 3-primary subgroup of $\pi_{n+3}(S_n)$ is characterized by the reduced power P^1(mod 3). These results are useful for the study of fourth obstructions of sphere bundles. (Received May 18, 1953.)

556*t*. R. F. Williams: *Local contractions of compact metric sets which are not local isometries.*

The purpose of this paper is to answer a question raised by A. Edrei in a paper entitled *On mappings which do not increase small distances*, which appeared in the September 1952 issue of the Proceedings of the London Mathematical Society. The author demonstrates the existence of a compact subset of the plane M and a non-local-isometric mapping f of M onto M such that each point of M is a point of contraction under f (thus f might be called a local contraction). It is further pointed out that minor alterations of M and f give such examples where f is a homeomorphism and (1) M is in the plane, (2) M is a connected subset of Euclidean three-space, (3) M is totally disconnected. (Received May 4, 1953.)

J. W. GREEN,
Associate Secretary

HAROLD CHATLAND

A COMMUNICATION FROM THE TREASURER

To the Board of Trustees of the
American Mathematical Society:

Gentlemen:

I have the honor to submit herewith a special report of the Treasurer for the six-month period December 1, 1952–May 31, 1953. This is necessitated by the decision of the Trustees to change the fiscal year so as to begin on June 1 and end on May 31 annually instead of running from December 1 to November 30.

Caution should be used in drawing conclusions from the accompanying schedules, as they are not directly comparable with earlier annual reports. It cannot be assumed that figures should normally be approximately half of the analogous figures cited from the annual report dated November 30, 1952, as income is not received nor are expenses incurred uniformly throughout the year. Furthermore, the balances on hand include grants for special purposes, largely unexpended, which were not available to the Society in the previous fiscal period.

During the six-month period one $1000 Texas and Pacific Railway Company bond was called, $16,600 Series G Savings Bonds matured or about to mature were exchanged for the new United States Government 3¼% bonds, and $5,000 Province of Ontario 3¼% bonds were purchased. Also, $500 United States Treasury 2½'s were sold and $400 additional United States 3¼'s were acquired to round out our holding.

Our return on our invested funds during the six-month period has been at the rate of 4.8% per annum.

Amortization of the cost of moving to Providence continues as originally scheduled.

Respectfully submitted,
Albert E. Meder, Jr.
Treasurer

EXHIBIT A

BALANCE SHEET

Assets

	May 31, 1953	November 30, 1952 (adjusted)
CURRENT FUNDS:		
Cash	$ 68,660.23	$ 19,430.02
Accounts Receivable (United States Government)	5,665.10	4,667.27
Investments	67,208.26	67,329.96
	$141,533.59	$ 91,427.25
INVESTED FUNDS:		
Cash	$ 81.53	$ 1,109.03
Investments	291,892.57	288,090.07
	$291,974.10	$289,199.10
TOTAL ASSETS	$433,507.69	$380,626.35

Liabilities

	May 31, 1953	November 30, 1952 (adjusted)
CURRENT FUNDS:		
Publications	$ 22,257.16	$ (5,042.85)
Other Current Funds	26,610.80	23,918.66
Miscellaneous (Credits)	2,496.03	1,809.67
Surplus	90,169.60	70,741.77
	$141,533.59	$ 91,427.25
INVESTED FUNDS:		
Endowment Fund Principal	$ 94,000.00	$ 94,000.00
Library Proceeds Fund	52,800.00	50,000.00
Other Expendable Funds	28,551.47	28,551.47
Prize Funds	6,575.00	6,575.00
Life Membership and Subscription Reserve	3,822.48	3,822.48
Mathematical Reviews	80,000.00	80,000.00
Reserve for Investment Losses	5,550.00	5,550.00
Profit/Loss on Sale of Securities	20,675.15	20,700.15
	$291,974.10	$289,199.10
TOTAL LIABILITIES	$433,507.69	$380,626.35

EXHIBIT B

GENERAL RECEIPTS AND GENERAL DISBURSEMENTS

GENERAL RECEIPTS:			
Dues from Ordinary Memberships		$44,875.28	
Dues from Contributing Memberships		629.00	
Dues from Institutional Memberships		5,250.00	
Publication Charges from Non-Member Institutions		440.00	
Initiation Fees		45.00	
Income From			
Henderson Estate	$ 3,525.00		
Current Funds Investments	765.74		
Invested Funds	4,819.61	9,110.35	
Miscellaneous			
Donations	$ 35.00		
International Congress (Net)	834.10	869.10	
Overhead on Contracts		1,460.71	
TOTAL GENERAL RECEIPTS			$62,679.44
GENERAL DISBURSEMENTS:			
General Activities			
Accounting Department	$ 4,913.02		
Executive Director	10,393.73		
New Haven Office	2,440.20		
Officers' Expenses	766.94		
Meetings	4,135.70		
Policy Committee	134.89	$22,784.48	
Publications			
Bulletin	$ 5,505.96		
Proceedings	8,521.21		
Transactions	2,798.71		
Directory	533.88		
American Journal	7.37		
Canadian Journal	1,000.00		
Annals	2,100.00	20,467.13	
TOTAL GENERAL DISBURSEMENTS			43,251.61
EXCESS OF GENERAL RECEIPTS OVER GENERAL DISBURSEMENTS			$19,427.83

EXHIBIT C

ANALYSIS OF SURPLUS

SURPLUS, DECEMBER 1, 1952	$70,741.77
EXCESS OF GENERAL RECEIPTS OVER GENERAL DISBURSEMENTS	19,427.83
SURPLUS, MAY 31, 1953	$90,169.60

SUMMARY OF CURRENT FUNDS

	Balance 12/1/52	Additions	Deductions	Balance 5/31/53
CURRENT FUNDS				
Special Publication Funds				
Mathematical Reviews	$ 352.43	$ 39,957.75	$37,351.93	$ 2,958.25
Colloquium	17,349.89	4,209.24	1,085.59	20,473.54
Mathematical Surveys	(4,155.16)	798.07	233.39	(3,590.48)
Birkhoff's Collected Papers	(577.59)	139.83	10.26	(448.02)
Symposia on Applied Mathematics	(520.71)	153.42	40.10	(407.39)
Memoirs	651.51	707.63	374.96	984.18
Russian Vocabulary	(82.40)	8.25	1.47	(75.62)
Translations	(135.57)	265.02	266.75	(137.30)
Transactions Reprinting	2,500.00			2,500.00
	$15,382.40	$ 46,239.21	$39,364.45	$22,257.16
				[To Exhibit A]
Other Funds				
Office Transfer Fund	$ 2,865.83	$ 1,750.00	$ 2,800.00*	$ 1,815.83
Policy Committee	858.01	316.99	486.22	688.78
Combined List of Members	(4,685.55)	69.00	3,245.41	(7,861.96)
Prize Funds Accumulated Income	2,405.27	155.78		2,561.05
Profit/Loss on Sale of Securities	2,049.85		51.40	1,998.45
Summer Institute		20,000.00	38.01	19,961.99
Publication Project		6,300.00	7.24	6,292.76
Navy Overhead Accrual		3,385.28	2,231.38	1,153.90
International Congress 1950		952.34	952.34	
	$ 3,493.41	$ 32,929.39	$ 9,812.00	$ 26,610.80
				[To Exhibit A]
General Publications				
Directory			$ 533.88	$ (533.88)
Bulletin		$ 2,796.54	8,302.50	(5,505.96)
Proceedings	($13,568.85)	16,522.19	11,474.55	(8,521.21)
Transactions	(6,856.40)	14,951.27	10,893.58	(2,798.71)
	($20,425.25)	$ 34,270.00	$31,204.51	($17,359.76)
				[To Exhibit B]
TOTAL	($ 1,549.44)	$113,438.60	$80,380.96	$31,508.20

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* Transferred to Invested Funds (Library Proceeds)

SUMMARY OF INVESTED FUNDS

	PRINCIPAL				INCOME			
	Balance 12/1/52	Additions	Deductions	Balance 5/31/53	Balance 12/1/52	Additions	Deductions	Balance 5/31/53
INVESTED FUNDS								
Permanent Funds								
Endowment	$ 94,000.00			$ 94,000.00		$2,217.31	$2,217.31*	
Expendable Funds								
Boyle	$ 250.00			$ 250.00		$ 6.17	$ 6.17	
Brown	1,000.00			1,000.00		24.02	24.02	
Henderson	1,000.00			1,000.00		24.02	24.02	
Hutchinson	1,000.00			1,000.00		24.02	24.02	
Merrill	650.00			650.00		15.78	15.78	
Reilly	23,651.47			23,651.47		557.93	557.93	
Whittemore	1,000.00			1,000.00		24.02	24.02	
Library Proceeds	50,000.00	$2,800.00		52,800.00		1,217.43	1,217.43	
	$ 78,551.47	$2,800.00		$ 81,351.47		$1,893.39	$1,893.39*	
Prize Funds								
Bocher	$ 1,450.00			$ 1,450.00	$ 868.70	$ 34.31		$ 903.01
Cole	2,550.00			2,550.00	1,186.17	60.39		1,246.56
Moore	2,575.00			2,575.00	350.40	61.08		411.48
	$ 6,575.00			$ 6,575.00	$2,405.27	$ 155.78		$2,561.05
Mathematical Reviews	$ 80,000.00			$ 80,000.00		$1,887.22	$1,887.22†	
Reserves								
Life Membership	$ 3,726.96			$ 3,726.96		$ 87.84	$ 87.84	
Life Subscriptions	95.52			95.52		2.06	2.06	
Against Investment Losses	5,550.00			5,550.00		131.08	131.08	
Profit/Loss on Sale of Securities	20,700.15		$25.00	20,675.15		487.93	487.93	
	$ 30,072.63		$25.00	$ 30,047.63		$ 708.91	$ 708.91*	
TOTAL	$289,199.10	$2,800.00	$25.00	$291,974.10	$2,405.27	$6,862.61	$6,706.83	$2,561.05

* Transferred to General Receipts
† Transferred to Mathematical Reviews Current Fund

BOOK REVIEWS

Sur quelques propriétés des valeurs caractéristiques des matrices carrées. By M. Parodi. (Mémorial des Sciences Mathématiques, no. 118.) Paris, Gauthier-Villars, 1952. 64 pp. 800 fr.

This book deals in the first place with the applications of the following determinant theorem: Let $A=(a_{ik})$ be a square matrix of order n with real or complex elements and let $|a_{ii}| > \sum_{k\neq i}|a_{ik}|$ $(i=1, \cdots, n)$. It follows that the determinant $D=|a_{ik}|\neq 0$. This theorem has varied applications and is being rediscovered continually. Parodi mentions some of its discoverers and some of its generalizations without making any attempt at completeness. In particular, Price's recent improvement of Ostrowski's lower bound for $|D|$ is not mentioned.

A most important application of this determinant theorem lies in its usefulness for the bounding of characteristic roots of matrices. For, it implies that the characteristic roots of A lie inside or on the boundaries of the n circles $|a_{ii}-z| = \sum_{k\neq i}|a_{ik}|$. A number of applications and generalizations are treated, in particular that due to A. Brauer in which the n circles are replaced by $n(n-1)/2$ Cassini ovals. The references are again incomplete; in particular, the pioneering work of Gershgorin is not mentioned.

Parodi exploits a theorem of Ostrowski which deals with the variations of the elements which preserve the nonsingularity of a matrix.

The book concludes with the study of the determinantal equation of the form

$$\left| a_{ik}^{(0)} z^{n} + a_{ik}^{(1)} z^{n-1} + \cdots + a_{ik}^{(n)} \right| = 0$$

where $(a_{ik}^{(j)})$ are square matrices; the case $n=2$ which turns up in electrical networks is treated earlier and upper bounds for the real parts of the roots z are determined.

Apart from various applications to circuit theory, the book deals with general stability problems, with the reducibility of polynomials, and with methods of finding bounds for the modules of the zeros of polynomials. It contains a number of worked numerical examples of matrices of small orders.

O. TAUSSKY

Complex analysis. By L. V. Ahlfors. New York, McGraw-Hill, 1953. 12+247 pp. $5.00.

In American universities the course in complex analysis is often used to review advanced calculus with modern standards of rigor. This procedure, which clearly has certain disadvantages, is to some extent forced on our graduate schools, and it is for such a course that this text is written. In complex analysis the temptation to base proofs on intuitive arguments is especially strong. It is a challenge to make the proofs simple and intuitively clear, and at the same time such that the reader can easily fill in the steps of a formal argument. The author has met this challenge in a masterful way.

Roughly the first third of the book is concerned with notions preliminary to complex integration. In it complex numbers are defined by means of an extension field, after which they are plotted in the plane, which is in turn projected onto a sphere. The elementary functions log x, arc sin x, etc., are defined for real values of the variable by means of appropriate integrals, from which the properties of these functions and their inverses are derived. These functions can then be defined for complex values of the variable in terms of their real and imaginary parts. The complex derivative is defined and the notion of conformality introduced. By this time the reader has studied several mappings in the large, including the set of bilinear functions, and has some notion of the local properties of analytic functions. This part of the book ends with an intuitive discussion of Riemann surfaces which, as the author states, is only for purposes of illustration.

The treatment of complex integration, which has been influenced by the work of Artin, is novel in several respects. If the continuous function $z(t)$ has a piecewise continuous derivative in the interval $a \leq t \leq b$, then $z(t)$ describes an arc γ when t describes $[a, b]$. If $f(z)$ is continuous on γ, then the integral on $f(z)$ over γ is defined in terms of a real integral by

$$\int_\gamma f(z)dz = \int_a^b f(z(t))z'(t)dt.$$

In this definition it is not assumed that γ is a Jordan arc nor that $z'(t)$ is different from zero. It is then shown that the line integral $\int_\gamma(pdx+qdy)$ defined in a region (open connected set) depends only on the end points of γ if and only if there is a function $U(x, y)$ such that $p = U_x$, $q = U_y$. In particular the integral of the functions 1, z around a rectangle is zero. This result is then used to prove Cauchy's theorem for a rectangle by the method of dissection. Cauchy's theorem for any closed curve γ (with the same smoothness conditions as above) in an open disk is a consequence of these results. For it can now be shown that if $f(z)$ is regular in an open disk, then the function

$$F(z) = \int_\sigma f(z)dz$$

is well defined in the disk. Here σ is a path from the center of the disk to a point z, consisting of one vertical and one horizontal line segment. There are two such paths corresponding to a single point z, and their difference is a rectangle.

In the case of Cauchy's integral formula some notion of the topology of the path of integration must be present, implicitly or explicitly. In order to separate the topology from the analysis so far as possible the author makes use of the winding number. If γ is a closed curve and a is a point not on γ, then the winding number $n(\gamma, a)$ is what the name implies, the number of times that γ winds around the point a. If γ is a closed curve lying in a disk in which $f(z)$ is analytic, then Cauchy's integral formula is

$$n(\gamma, a)f(a) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z - a}\, dz.$$

Criteria can be given that $n(\gamma, a)$ is equal to 1 (or zero) which are easier to prove, although less general, than the Jordan curve theorem.

This information is then used to investigate a variety of topics, which include Taylor's series, the maximum principle, and the local mapping of analytic functions. Cauchy's theorem is then proved under more general conditions, and the chapter on complex integration ends with a discussion of the calculus of residues.

The topics which are now discussed include infinite products and entire functions through Hadamard's factorization theorem, Harnack's principle, the gamma function, Riemann's mapping theorem, and subharmonic functions. The discussion of subharmonic functions is carried far enough to solve the Dirichlet problem for regions of finite connectivity. Harmonic measure is discussed briefly and regions of finite connectivity are mapped onto canonical regions.

The last chapter is concerned with multiple-valued functions. It includes a precise discussion of Riemann surfaces, branch points, homotopic curves, and algebraic functions. The book concludes with a lucid discussion of the hypergeometric function.

Among the topics not included are elliptic functions and, surprisingly in view of the author's own work, the length-area principle and the Phragmén-Lindelöf principle.

The reviewer noted only two places where the proof involves more than the text implies, namely p. 170, line 8 and p. 188, line 18. Each gap can be filled by well known methods but the first is the more serious because the simplest proof (that occurred to the reviewer) depends on ideas developed between pages 170, 188. The book is an important contribution to mathematical literature. At every turn one sees the care and ingenuity which the author has used to make his proofs rigorous and readable. The book is intended for the conscientious student, and it will repay him well for the hours that he may spend with it.

A. C. SCHAEFFER

Foundations of the nonlinear theory of elasticity. By V. V. Novozhilov. Trans. from the first (1948) Russian ed. by F. Bagemihl, H. Komm, and W. Seidel. Rochester, Graylock, 1953. 6+233 pp. $4.00.

Students of mechanics will be grateful to the translators and the publishers for making available the second of the three[1] existing monographs on the general theory of elasticity—the more so, since the Russian original is in this country at least a very rare book.

The translation is unusually good English (except for "compatability") and the translators have taken unusual care that the exposition of this elaborate subject shall make sense, although they are not always familiar with the terms used in mechanics (e.g. on p. 58 they use "components of a vortex" for "components of the curl"). Despite its being planographed, and thus repulsive to the eye, the text is readable.

The author's approach is straightforward, honest, and vigorous. There is little or no nationalism, rhetoric, or pedagogery. The author gives every evidence of his earnest competence and his respect for a difficult and important group of problems. The book is not scholarly, however; most of the some ninety items in the bibliography are not cited in the text, part of which presents material first published in important papers not listed in the bibliography. It is quite possible that many of the results in this book are rediscoveries by the author himself.

This is a serious work, deserving detailed notice. The author's preface is dated 1947, and the book is on the whole a careful, accurate, and reliable exposition of some of the mechanical aspects of the classical nonlinear theory of elasticity as it stood at that date. It was in 1948 that the numerous publications of Rivlin, which have enlivened the subject and changed the whole view of it, began to appear.[2] Thus the book, through no fault of its author, cannot be called a definitive exposition of general elasticity, nor even an adequate introduction to it. The term "foundations" in the title is justified by the usually careful treatment of principles: the author's objective is to set up the governing equations and various approximations to them, without any attempt at solutions in special cases.

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[1] The other two are *Théorie des corps déformables* by E. and F. Cosserat, Paris, 1909, and *Finite deformation of an elastic solid* by F. D. Murnaghan, New York, 1952. The latter was reviewed in Bull. Amer. Math. Soc. vol. 58 (1952) pp. 577–579.

[2] These are briefly summarized in Chap. IV of my paper, *The mechanical foundations of elasticity and fluid mechanics*, Journal of Rational Mechanics and Analysis vol. 1 (1952) pp. 125–300; corrections and additions, vol. 2 (1953) pp. 593–616.

Nearly two thirds of the text, comprising the first four chapters, is devoted to the basic equations of three-dimensional finite elasticity. In Chapter I the concept of strain is developed with especial care, and various levels of approximation are carefully distinguished (esp. §§13–15). In speaking of "small elongations and shears," however, the author fails to remark that shear is not an invariant concept; what he intends is "small principal extensions."

The treatment of stress in Chapter II is not only rather slipshod but also depressingly elaborate, while to follow the fifteen page derivation of stress-strain relations in Chapter III, even though it begs the main question at issue, would require an iron resistance to boredom not easily bred in temperate climes. It is in Chapter III, §29, that we find the author's most serious oversight. While he reduces the stress-strain relations for isotropic bodies to a material ("Lagrangian") form which is rather simple in appearance, he does not mention the possibility of using spatial ("Eulerian") strain measures [Finger, Sitzber. Akad. Wiss. Wien (IIa) vol. 103 (1894) pp. 1073–1100]. It is this possibility which renders problems of large strain manageable. It seems unfortunate that Joseph Finger, the first to notice this simple but centrally important fact and to obtain the stress-strain relations whose rediscovery has made possible the striking progress in general elasticity since 1947, is unknown in the history of mechanics.

On p. 113 the author states without proof the following invariant form for the "generalized" stress matrix S:

$$S = \Psi_2\mathrm{II}_0 + \Psi_1\mathrm{II}_1 + \Psi_0\mathrm{II}_2,$$

"where Ψ_2, Ψ_1, Ψ_0 are functions of the strain invariants, II_0 [is] the unit tensor, II_1 [is] a tensor whose components are linear combinations of the [material] strain components, and II_2 [is] a tensor whose components are quadratic combinations of the strain components." Here the author has found a portion of the basic invariance theorem established simultaneously by Reiner [Amer. J. Math. vol. 70 (1948) pp. 433–446]: S may be taken as the true stress matrix, II_1 as any matrix whose proper values are analytic functions of the principal extensions and whose principal axes are the principal spatial strain axes, while II_2 may be taken as $(\mathrm{II}_1)^2$.

In §31 the author writes "It follows from the above that for every material a range of small deformations can be established for which Hooke's law is approximately valid." If one wonders how a mathematical theory could ever establish such a result, one must turn back to §30, where one finds that the strain energy has been assumed analytic because "no negative powers can appear in the series." There is no statement that an assumption has been made, not even a discussion of why fractional powers, for example, might not be appropriate, although a considerable engineering literature devoted to this possibility exists [cf. e.g. R. Mehmke, Z. Math. Phys. vol. 42 (1897) pp. 327–338]. In fact, the only experimental justification of the assumption of analyticity is the *experimental* validity of Hooke's law for many [by no means all!] materials under sufficiently small loads—but this is the direct opposite of the author's reasoning.

The author's analysis (§32) of Hencky's theory of plasticity (in this country usually considered with respect to strain-hardening, and often called "the theory of Roš, Eichinger, and Schmidt") was obtained also by C. Weber [Zeitschrift für Angewandte Mathematik und Mechanik vol. 28 (1948) pp. 189–190; vol. 29 (1949) p. 256].

In this book all results are written out at length in rectilinear coordinates. Some sets of formulae cover most of a page. In the preface we find an explanation: "To make the book as accessible to as wide a circle of readers as possible, the author has attempted to carry out all deductions in the simplest and most intuitive manner, avoiding, in particular, tensor calculus . . . ." In fact, at the top of p. 67 the word "tensor" or the idea behind it is avoided with comical precaution: apparently the reader is assumed not to have studied the classical treatises of Voigt and Love. I believe that an intelligent student completely untutored in geometry on reading this page would set himself the problem of formulating and exploring the geometric concept which the two obviously connected results so forcibly separated by the author most plainly suggest. But on p. 111 the word "tensor" suddenly appears without explanation and is used several times later. It is somewhat similar with Green's theorem, which is carefully avoided in the creaking development of the properties of the stress tensor but appears later on p. 106 (where, however, it is regarded as so extraordinary as to need two of the five references given in the first 200 pages, the others being to works on orthogonal curvilinear coordinates).

It is not in disrespect to Euler and Cauchy that I say their methods in continuum mechanics are now unnecessarily elaborate; in fact, it is in their papers on continuous media that some of the earliest discoveries in the theory of differential invariants occur, and tensor analysis is in part an elaboration of their work. But I think the student who follows in this book the endless pages of dreary resolutions and projections in the Euler-Cauchy style could better spend his time learning tensor analysis, which would enable him to reproduce four fifths of the author's work in twenty pages, while freeing his attention for the important questions and ideas which are scattered through the remaining one fifth.

Although the author founds all his analysis in the fully general theory, his main interest is in the case next in order of generality past the fully linear one, when the extensions are small, but the displacements and rotations may be large. The cause of this restriction, on which he lays considerable emphasis, is his desire to furnish structural engineers with the basic theories needed for rational solution of their nonlinear elastic problems. Since typical structural materials, such as steel, fail to retain their elastic reversibility when subjected to extensions as great as 1%, there are essentially only two such nonlinear problems: (1) elastic stability, which the author interprets as determining the smallest load at which Kirchhoff's uniqueness theorem breaks down, and (2) bending of "flexible" bodies, such as thin rods, plates, and shells. The last two chapters, the most important in the book though occupying only about eighty pages, are devoted to these two problems.

While the author's points are well taken, it is instructive to consider a simple analogy. Suppose we are to clarify the problems besetting horological engineers when their pendulums swing in a range beyond that in which the approximation $\sin\theta \approx \theta$ is sufficiently accurate. Doubtless then $\sin\theta \approx \theta - \theta^3/6$ is quite sufficient for all practical problems of this type. Accordingly, if we were to follow the practice which was nearly universal in nonlinear elasticity up to 1948 and is recommended by the author, we should devote ourselves to the differential equation

$$\ddot{\theta} + k^2(\theta - \theta^3/6) = 0.$$

In so doing, we should lose all the simplicity of the linear theory, having to face at once all the complications of nonlinear mechanics—but even if completely successful, all we should have, at great cost, would be a somewhat better approximation. It is common knowledge that it is no harder to settle the whole matter rigorously by studying the exact solutions of the exact equation

$$\ddot{\theta} + k^2 \sin\theta = 0.$$

Now a very similar thing has happened in elasticity theory. The work of Rivlin has shown us that it is quite a feasible and practical thing to work directly with the *exact* equations for arbitrarily large strain of a material characterized by an arbitrary strain energy function. There is not only the scientific satisfaction of solving a really general problem for its own sake (cf. the last paragraph of the "Historical Introduction" to Love's *Treatise on the mathematical theory of elasticity*, 4th ed., Cambridge, 1927), but also the precision of a general analysis leads to simplicity and certainty in the end. While only relatively simple problems can be solved explicitly within the fully general theory, these particular cases are very important, and it was the light they cast upon the nonlinear theory which pointed the way to an approximate procedure valid for *all* problems of prescribed loading.[3] An example of the defectiveness of the approach usual in nonlinear elasticity is furnished by the author's formulation of the problem of elastic instability within an approximate nonlinear theory of elasticity. In the absence of a mathematical approximation theorem, we cannot assert with confidence that critical loads obtained from the author's equations approximate the critical loads which would be obtained from the general theory. But these remarks must not be taken as criticism of the author's work, which presents in a few pages a relatively simple and cogent development of the problem of elastic instability in the usually received sense.

There is some question also about the author's distinction between "geometrical" and "physical" nonlinearity (§34, and again on p. 197). For example, whether or not the rotations are large cannot be determined by "geometric considerations" *a priori*; the rotations result from loading, and (unless one is using an inverse method) one cannot know in advance whether for given loading of a material defined by a given strain energy function the nonlinear terms in the strain components will need to be retained or not. True, *after* the problem is solved the question becomes purely geometric, but if we have the exact solution then it is no longer very important whether we can neglect certain terms or not. The question of whether certain approximations are valid *in advance* is avoided by the author; its treatment would require a new type of approximation theorem for partial differential equations.

The excellent last chapter is summarized in the author's conclusion (§54):

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[3] R. S. Rivlin, Journal of Rational Mechanics and Analysis, vol. 2 (1953) pp. 53–81.

"Ordinarily, the theory of deformation of flexible bodies (plates, shells, rods) is developed by making certain assumptions which immediately reduce the problem to a two-dimensional one (in the case of plates and shells) or a one-dimensional problem (in the case of rods). However, with such assumptions one necessarily loses sight of the connection between the theory of plates, shells, and rods and the general theory of elasticity. In view of this, many people consider the theory of flexible bodies as a kind of hypothetical superstructure over the general theory of elasticity, as a foreign element in it.

"Only in this manner can one probably explain why most contemporary books on the theory of elasticity omit all mention of the problem of deformation of flexible bodies, which is of such practical importance. An attempt was made in Love's book to relate the 'hypotheses' of the theory of flexible bodies to the general theory of deformation.

"But special work in this direction was carried out by B. G. Galerkin, in whose papers the classical theory of shells and plates truly became a branch of the general theory of elasticity.

"The basic idea championed by B. G. Galerkin was that the problems of the bending of plates and shells must always be examined in the context of the general theory. This simple but profound idea was responsible to a large extent for the successful development of the theory of plates and shells in the Soviet Union and turned out to be fruitful not only in the case of thick plates and shells, but also in the case of thin plates and shells.

"It is natural to extend this idea to the nonlinear theory of elasticity, since one can expect that many results of this theory may be systematized by starting out from the general equations. The present chapter was an attempt to give a uniform method for investigating the deformation of flexible bodies on the basis of the general nonlinear theory of deformations. It was our aim to clarify, with the aid of the general equations, those 'hypotheses' on which the theory of plates, shells, and rods is ordinarily based, and to examine, from a uniform point of view, all these problems, which are ordinarily treated separately in spite of their common features."

The "basic idea championed by B. G. Galerkin" goes back to Cauchy and Poisson for the theory of plates, while for slight bending of shells it is the author himself [C. R. (Doklady) Acad. Sci. URSS. vol. 38 (1943) pp. 160–164] who has given us the first adequate treatment based on the three-dimensional theory.[4] In the present work he carefully derives from nonlinear three-dimensional elasticity several of the nonlinear theories of rods, plates, shells, taking pains to show that the special hypotheses used are consistent to the degree of approximation considered. The reader not already familiar with this subject, where in the past outright inconsistent assumptions have often been made, may not realize that the author's treatment deserves the description "simple but profound."

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[4] Similar treatments were constructed independently by R. Byrne (1941) [Sem. Repts. Math. Univ. Calif. (n.s.) vol. 2 (1944) pp. 103–152] and in my Princeton

C. TRUESDELL

The higher arithmetic. By H. Davenport. London, Hutchinson's University Library, 1952. Text ed. $1.80, Trade ed. $2.25.

This book is an introduction to the theory of numbers which is suitable for a very wide class of readers. On the one hand, no extensive mathematical knowledge is required of the reader; in fact, a good high-school training in mathematics would be sufficient. On the other hand, the author discusses subjects of real mathematical interest and treats them in a very readable way, so that a person of considerable mathematical maturity would find much enjoyable and profitable reading in this work.

The titles of the seven chapters are as follows: Factorization and the primes, Congruences, Quadratic residues, Continued fractions, Sums of squares, Quadratic forms, Some Diophantine equations. As can be seen from the list, a fairly wide range of material is covered. No attempt is made to treat each topic exhaustively, but the author goes far enough to enable the reader to get some appreciation of the main ideas and problems in each area. A few of the more noteworthy things to be found in the book are as follows: (1) a good presentation of the method of mathematical induction and a proof of the unique factorization theorem by this method, (2) a proof of Chevalley's theorem that an algebraic congruence in several unknowns to a prime modulus always has a nontrivial solution if the constant term is zero and the degree is less than the number of unknowns, (3) a proof of the theorem on the number of positive integers n between 1 and $p-2$ (inclusive) for which n and $n+1$ have prescribed quadratic character modulo the odd prime p, (4) a rather thorough treatment of the continued fractions of quadratic irrationals, (5) a presentation of various constructions for the two squares into which a prime of the form $4k+1$ can be decomposed, (6) a discussion (without proof) of Dirich-

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dissertation (1943) [Trans. Amer. Math. Soc. vol. 58 (1945) pp. 96–166], and W. Z. Chien has asserted in a letter that the similar material in his paper [Sci. Rep. Tsing Hua Univ. vol. A 5 (1948) pp. 240–251] derives from his Toronto Thesis (1942). The idea does not appear to have taken hold in this country.

let's class-number formula, and (7) an exposition (without proof) of the Thue-Siegel theorem.

Although this book is not written as a textbook but rather as a work for the general reader, it could certainly be used as a textbook for an undergraduate course in number theory and, in the reviewer's opinion, is far superior for this purpose to any other book in English. Admittedly there are no formal lists of problems, but there are plenty of problems implicit in the text in the form of easy proofs and calculations left to the reader; also there are many hints for further discussion or further reading. Students will certainly like the author's facility in presenting new concepts and proofs clearly without introducing elaborate notations.

Finally the reviewer believes that this book should be in every college library worthy of the name, regardless of whether or not there is a course in number theory in the curriculum. It is hard to think of a better book to suggest to an interested undergraduate for independent reading.

P. T. BATEMAN

Éléments de mathématique. By N. Bourbaki. Book II, *Algebra.* Chaps. I–VII. (Actualités Scientifiques et Industrielles, nos. 934, 1032, 1044, 1102, 1179.) Paris, Hermann, 1942, 1947, 1948, 1950, 1952.

Our time is witnessing the creation of a monumental work: an exposition of the whole of present day mathematics. Moreover this exposition is done in such a way that the common bond between the various branches of mathematics becomes clearly visible, that the framework which supports the whole structure is not apt to become obsolete in a very short time, and that it can easily absorb new ideas. Bourbaki achieves this aim by trying to present each concept in the greatest possible generality and abstraction. The terminology and notations are carefully planned and are being accepted by an increasing number of mathematicians. Upon completion of the work a standard encyclopedia will be at our disposal. The volume on *Topologie générale* which is complete is already being used enthusiastically, especially by the younger generation. A comparison with the "Encyclopädie der mathematischen Wissenschaften" should not be made. The aim was different; proofs were omitted and each article was written by a different author.

I hope that this work will continue in the same spirit and with the same vigor. I would suggest an English translation.

The volumes on algebra that have appeared show the same general features as the rest of Bourbaki. Numerous exercises, many of them of highest interest, are found at the end of each paragraph. From time to time excellent historical notes explain the development of the ideas. It is inevitable that much of the material is of standard nature. In the following more detailed discussion I intend to underline mainly the novel ideas that appear in the work.

A few general remarks must precede this discussion. We all believe that mathematics is an art. The author of a book, the lecturer in a classroom tries to convey the structural beauty of mathematics to his readers, to his listeners. In this attempt he must always fail. Mathematics is logical to be sure; each conclusion is drawn from previously derived statements. Yet the whole of it, the real piece of art, is not linear; worse than that its perception should be instantaneous. We all have experienced on some rare occasions the feeling of elation in realizing that we have enabled our listeners to see at a moment's glance the whole architecture and all its ramifications. How can this be achieved? Clinging stubbornly to the logical sequence inhibits the visualization of the whole, and yet this logical structure must predominate or chaos would result. Bourbaki is quite aware of this dilemma. The fact that his work is subdivided into books, the fact that exercises are given which utilize more advanced parts of the theory show this awareness. However I feel that in some instances the subdivision into books is not enough. This inadequacy is strongly felt in the course of Chapter V as we shall see later.

Chapter I acquaints us with the fundamental concepts of abstract algebra: groups, rings, fields, vector spaces. To avoid repetitions the notions of an internal and of an external composition in a set are introduced. The internal composition is patterned after the addition or multiplication in a ring, the external one after the multiplication of a vector by a scalar or the product of an operator of a group with an element of the group. In this very general setup Bourbaki discusses topics like the associative and commutative laws, the question of the existence of a neutral element (0 element in case of addition) and that of a symmetric element (the negative in case of addition). Symmetrization is the abstract counterpart to the introduction of the negative integers.

Bourbaki calls a set together with several internal or external compositions an algebraic structure. The notion of a quotient structure generalizes that of factor group or residue class ring. Finally laws like the distributive (in the case of several compositions) are discussed.

These general concepts are now applied to groups, rings and fields. The discussion of groups, which includes the usual elementary notions and theorems, culminates in the Jordan-Hölder theorem and is followed by a detailed study of transformation (permutation) groups. It seems to be Bourbaki's intention not to discuss some finer points of the theory of finite groups. I would deplore this since some of the topics like p-groups and Sylow groups might easily find their place in some later volume and are of great importance for algebraic number theory.

The discussion of rings, ideals and fields leads to the usual elementary theorems. As a deeper property of fields (which solves a classical problem of projective geometry) a future edition might include the following beautiful theorem of Hua.

Let σ be an additive map of one field (not necessarily commutative) into another which satisfies $\sigma(a^{-1}) = (\sigma(a))^{-1}$ for all $a \neq 0$ and $\sigma(1) = 1$. Then σ is either an isomorphism or an anti-isomorphism into this field. By the way, the connection with Hua's work follows from the following amusing noncommutative identity:

$$a - (a^{-1} + (b^{-1} - a)^{-1})^{-1} = aba$$

which shows $\sigma(aba) = \sigma(a)\sigma(b)\sigma(a)$.

Chapter II deals with linear algebra. As could be expected, Bourbaki puts the geometric concepts in the foreground. He begins with the definition of a module over a ring (not necessarily commutative), that of submodules, factor modules, free modules, bases, linear maps and their properties. The usual notion of dimension (supplemented by the notion of codimension) for vector spaces over a field is introduced. The main properties are derived from the exchange theorem. Then follows a paragraph discussing the dual of a module and the adjoint (transposée) of a map.

A most delightful part is §5 and I wish to call it to the attention of the algebraists. By means of the novel notion of a primordial element of a subspace with respect to a given basis of the whole space, the theory of linear equations with coefficients in a subfield is quickly developed and yields powerful results.

The computational aspect is not neglected; §6 gives a complete discussion of matrices. The chapter ends with a preliminary study of algebras. The algebra arising from a semigroup (monoïde) prepares the way for the definitions of group rings, polynomials, and power series.

One of the basic ideas of Chapter III is expressed by the following theorem: Given a finite number of modules E_i over a commutative ring A. An A-module M (unique up to isomorphisms) can be constructed which has two properties: (1) There exists a canonical map ϕ of the cartesian product of the E_i into M, and the image generates M.

(2) Every multilinear map of the E_i into any A-module N is of the form $g\phi$, where g maps M linearly into N. The module M is called the tensor product of the E_i. This idea is used again later to define the exterior pth power of a module E by means of multilinear alternating maps of E. The existence is proved by taking a suitable factor module of the pth tensor power of E.

The abstract idea underlying these constructions is taken up in an appendix on "universal maps."

After a deviation to tensor products of algebras a tensor of a module E (which is partly contravariant and partly covariant) is defined to be any element of a tensor product whose factors are either E or its dual. The foundation of tensor algebra, the exterior powers which we mentioned before, and the Grassmann algebra follow.

The definition of a determinant has been deferred up to this moment. The reason is clear: Let E be an A-module known to have n basis elements and ϕ a linear map of E into E. One wishes to define the determinant of the map ϕ without explicit reference to the basis. Let F be the nth exterior power of E. Then F is one-dimensional and ϕ induces on F a map whose "stretching factor" is the desired determinant. The usual rules for determinants can be quickly derived. The last paragraph is devoted to duality in the Grassmann algebra.

One may ask whether Chapter II or III should not be enlarged so as to contain the algebraic parts of homology and cohomology theory. It is becoming increasingly clear that this theory represents a very basic universal mechanisn of mathematics with applications in many fields. The use of diagrams of mappings would also enhance clarity.

Chapter IV centers around polynomials, power series, and general derivations. The question of unique factorization is deferred to later chapters.

Chapter V entitled *Commutative fields* includes Galois theory. The form of the existence proof of an algebraic closure has obviously arisen from the desire to avoid any finite existence theorems. It seems to me too complicated. With a simple argument on polynomial identities (without previous existence theorems) one can prove that an extension exists in which every polynomial of the ground field has at least one root. Repeating this construction denumerably many times one obtains an algebraic closure. Degree of transcendency is based on the exchange theorem.

The best part of the chapter is the thorough discussion of separability in general extension fields, and its relation to the important notion of linear disjointness and to the derivations. But I was greatly amused to see that one has to quote 3 propositions, 2 corollaries, and one theorem in order to prove that an extension is separable if and only if every finitely generated subextension can be separably generated. I think that new editions should improve on this situation.

The proofs of Galois theory proper are quite efficient. Yet a heavy price has to be paid for the fact that one is not permitted to use number theory or the notion of the dual of a finite abelian group (which can be obtained without any arithmetic or roots of unity). One of the results is the amusing footnote on page 168. To be more serious, almost no example over the rational field can be given, since it is next to impossible to show the irreducibility of a polynomial without some arithmetic. Because of this the irreducibility of the cyclotomic equation has to be deferred.

Another beautiful application of Galois theory has to be omitted for a similar reason. Let k be a field which contains the nth roots of unity and E an abelian extension field of exponent n. There exists a canonical description of the dual of the Galois group in terms of the ground field. Bourbaki has to restrict himself to the cyclic case, and obscure the main fact of the natural duality. One can scarcely believe that this compromise came from the pen of Bourbaki.

But these are questions of taste and the reader receives compensations by other extremely interesting examples as for instance those on pages 176 and 177.

The chapter ends with appendices on symmetric functions and on infinite Galois theory.

Chapter VI is entitled *Ordered groups and ordered fields*. But one has to understand that Bourbaki's ordered groups are usually called partially ordered groups in contrast to his totally ordered groups. After some preliminary investigations the first paragraph is dedicated, to the study of lattice ordered groups and their arithmetic. It culminates in Theorem 2 describing the necessary and sufficient condition for uniqueness of factorization into "primes." This theorem is to be used later on groups of ideals and will yield the usual uniqueness statements.

§2 is devoted to (totally) ordered fields and the main theorems are derived by means of the very elegant method of Serre.

The essential content of Chapter VII (in classical terminology) is the basis theorem of abelian groups. A few pages suffice to derive the elementary arithmetic in principal ideal rings, especially in ordinary integers and polynomial rings of one variable over a field (making use of the results of the previous chapter). The remaining part of the chapter is clear. Bourbaki investigates modules over a principal ideal ring, proves the "basis theorem" for finitely generated modules and introduces the elementary divisors. The theory is finally applied to the classical elementary divisor theory and yields the classification of the endomorphisms of a vector space.

In concluding I wish again to emphasize the complete success of the work. The presentation is abstract, mercilessly abstract. But the reader who can overcome the initial difficulties will be richly rewarded for his efforts by deeper insights and fuller understanding.

E. ARTIN

Inhalt und Mass. By K. Mayrhofer. Vienna, Springer, 1952. 8+269 pp. $9.30.

This is a very careful and detailed presentation of the Carathéodory measure theory, with special emphasis on Lebesgue measure in R^n. The first chapter, after a short paragraph on rings and fields of sets, defines and studies abstractly the notions of content and measure, the former being simply additive and defined on a field of sets, the latter countably additive and defined on a σ-field of sets (only positive set functions are considered). The usual notions related to content and measure (measurability, exterior and interior content or measure, measurable hull and measurable kernel of a set) are investigated with perhaps greater detail than in any other treatise on the subject; so are the relationship between content and measure, and the well known process of "completion" by which a completely additive content can be extended to a measure (on a larger field of sets). Also treated in this chapter are the products of two contents or measures, although one misses the corresponding facts for infinite products (this is probably the only important part of abstract measure theory which is not covered by the book).

The second chapter is devoted to Jordan content and quarrable sets ("quarrable" is to content as "measurable" is to measure), the third to Borel and Lebesgue measures in R^n; among the special features that should be mentioned are examples of nonquarrable Jordan curves, the Vitali covering theorem and the density theorem, and a study of nonmeasurable sets (for Lebesgue measure). Chapter IV deals with transformation of content and measure by a linear mapping in R^n; as an application the measures of various "elementary volumes" are computed. The general notion of measurable mapping from R^n into R^m is also considered, but, surprisingly enough, the author's definition is not the usual one: he defines a measurable mapping as one which sends measurable sets into measurable sets, with the consequence that a continuous function is not always measurable! Chapter V takes up Carathéodory's theory of abstract "exterior measures," i.e., set functions containing as particular cases the exterior measures deduced from a measure; these are axiomatically characterized among all "general exterior measures," and a similar characterization is given for interior measures. The chapter also includes the Carathéodory criterion for measurability of closed sets in a metric space; finally the connection is made between this theory and that developed in chapter I. The last chapter treats measure theory in boolean algebras and σ-complete boolean algebras. An appendix gives the "transfinite" generation of Borel sets and their main properties in finite-dimensional spaces.

The author's claim that he has "reached or even gone beyond" the limit of what is known today on the subject can hardly be accepted without restriction; for instance, no mention is made, in the last chapter, of the Stone and Loomis representation theorems, although they make the developments of that chapter practically pointless! No mention is ever made of characteristic functions of sets, which would at times make proofs much easier (for instance, the well known derivation of the "boolean ring" structure of a boolean algebra). Finally, the reviewer wants to take exception to the author's statement that measure theory (as understood in this book) is the foundation of the theory of integration. This *was* undoubtedly true some years ago, but is fortunately no longer so, as more and more mathematicians are shifting to the "functional approach" to integration. It is always rash to make predictions, but the reviewer cannot help thinking that, despite its intrinsic merits, this book, as well as its brethren of the same tendency, will in a few years have joined many an other obsolete theory on the shelves of the Old Curiosity Shop of mathematics.

J. DIEUDONNÉ

Lectures in abstract algebra. Vol. II. *Linear algebra.* By N. Jacobson. New York, Van Nostrand, 1953. 12+280 pp. $5.85.

Linear algebra is now universally recognized as perhaps the most important tool of the modern mathematician; its concepts and methods, moreover, when properly reduced to their essential features, are among the simplest and most straightforward imaginable. Nevertheless, it is still not uncommon to find graduate students who are totally unfamiliar with some of the fundamental notions of linear algebra, such as, for instance, the theory of duality. This may perhaps be attributed to the scarcity of good textbooks on the subject; if so, the present volume will undoubtedly do much to remedy this situation. Although this is the second part of a work which will ultimately be a treatise covering the whole of what may be called "elementary modern algebra," the author has taken great pains to make the book as self-contained and as easy to read as possible. Indeed, pedagogical intentions are apparent throughout: for instance, definitions which are particular cases of others already given in the first volume are often stated again, sometimes twice or more, and with increasing generality. In addition, the text is accompanied by many well chosen examples and exercises, which provide ample opportunities for the student to test his understanding of the theory.

The concept of finite-dimensional vector space over a division ring is introduced in the first chapter, first for the example of *n*-tuples, and then more abstractly. The fundamental theorems on linear dependence, bases and dimension are proved; matrices (defined in vol. I) are connected with sets of vectors considered in relation with a given basis, and this connection is immediately used to link properties of matrices with linear dependence. The chapter ends with a study of the lattice of subspaces of a finite-dimensional space, and in particular of the notion of (finite) direct sum.

Chapter II is devoted to the general theory of linear transformations between finite-dimensional vector spaces. Matrices are now connected with linear transformations, as well as with systems of linear equations. The second part of the chapter takes up the theory of duality, following rather closely the corresponding treatment in the books of Halmos and Bourbaki on linear algebra.

Chapter III is almost as long as the first two chapters together, and develops the theory of elementary divisors of a linear transformation in a vector space over a field. From the pedagogical motives mentioned above, the author does not begin by giving the theory of finitely generated modules over a principal ideal ring; this only comes in the middle of the chapter, after the link between the two theories has been made perfectly natural. The theory of invariant factors is then used to complete the reduction of a square matrix over a field; the usual results are proved, in particular the Hamilton-Cayley theorem; at the end of the chapter the centralizer of a matrix is entirely determined, and related to the notion of a ring of endomorphisms of a module.

A shorter chapter follows, on sets of linear transformations, where the notions of invariant subspace, decomposability, reducibility and complete reducibility are discussed, and illustrated in particular for a set consisting of a single transformation, thus linking this chapter with the preceding one; the common reduction of a commutative set of linear transformations is given at the end of the chapter.

The next three chapters are concerned with bilinear forms. They are defined in the general case of two vector spaces, one left and one right, over the same division ring, and their relation to linear transformations and duality theory is duly emphasized. The general theory is then specialized to a bilinear form over the product of a left vector space by itself, when the division ring possesses an anti-automorphism; this leads to the theory of hermitian and alternate bilinear forms, whose reduction is treated in detail. The final part of the chapter contains a proof of Witt's theorem (the first to appear in a textbook) in its full generality (including the case of characteristic 2 when every element of the division ring is a "trace" $\xi+\bar{\xi}$); some fragmentary results are also given on symmetric non-alternate forms over a field of characteristic 2.

The classical theory of euclidean and unitary spaces (of finite dimension) over the real field is developed in chapter VI, including the usual reduction and factorization theorems for normal, symmetric, hermitian, orthogonal, and unitary matrices. A short paragraph on analytic functions of matrices yields the exponential relation between hermitian and unitary transformations.

The next chapter is much more abstract, and gives the basic facts on tensor products of vector spaces and of linear transformations. The notion of tensor product is introduced in connection with the theory of duality, the tensor product $E'\otimes F$ of the dual E' of a left vector space E and of a left vector space F being in a natural correspondence with the group $\mathcal{L}(E, F)$ of all linear mappings from E to F. It is shown how for two-sided vector spaces the tensor product can again be made into a two-sided vector space; the rest of the chapter is concerned with the most important case, in which the vector spaces are over a field. There is a short description of the principal algebraic features of tensors and some of their symmetry classes; also included in this chapter are the extension of the field of scalars of a vector space and the definition of the tensor product of two algebras.

The last two chapters develop the beginning of the theory of rings of linear transformations. The finite-dimensional case is treated in chapter VIII, which in a very short space gives a neat description of the fundamental relationship between subspaces and ideals, as well as the characterization of automorphisms of the ring of linear transformations, all this being done without writing down a single matrix! Finally, chapter IX extends the previous notions and results to infinite-dimensional vector spaces; it is definitely on a higher level than the rest of the book, and is probably its most attractive feature, because of the variety of new topics which are here again presented in book form for the first time, and the elegance of their presentation; as an introduction to the modern theory of rings (which will be treated much more exhaustively in a forthcoming book of the author), it should make fascinating reading for any advanced student. Among the topics treated are dimensionality in infinite-dimensional vector spaces, including a determination of the dimension of the dual space; the "finite" topology on spaces of linear transformations, with the notion of total subspaces and of dense rings; the determination of two-sided ideals and of isomorphisms of certain rings of linear transformations; a discussion of dense rings of linear transformations possessing involutorial anti-automorphisms; and finally the author's well known density theorem, with classical applications to irreducible algebras of linear transformations.

The choice of notations and terminology might at times, in the reviewer's opinion, be definitely better. For instance, the author sticks to his habit of writing linear transformations on the right of the variable; there is no objection against this as long as one remains in pure algebra; but linear algebra is now the daily bread of mathematicians from every part of the horizon, and its results should be immediately available to them, without having to use mirrors to restore the operator to what is considered by the majority of analysts as its proper place! It is amusing, by the way, to see how the author himself violates his own convention when it comes to writing linear forms, for instance! The term of "direct product" and the notations $x \times y$ and $E \times F$ for the tensor product are quite unfortunate, since they already have several other meanings; why not adopt the von Neumann symbol $\otimes$, now almost universally used? Writing $\langle x, f\rangle$ or $\langle f, x\rangle$ for $f(x)$ in duality theory would clarify many a formula and emphasize the symmetry of the results. The fundamental notion of characteristic vector of a matrix is relegated to an exercise; so is the relation $\dim (E+F)+\dim (E\cap F)=\dim E+\dim F$, although a whole page is devoted to the "modular" law in the lattice of subspaces, which is never used any more (and is there apparently for the sake of those who still believe lattice theory is an important part of mathematics!).

But these are very minor criticisms of a remarkable book, which has every chance of becoming a classic in its field for many years to come.

J. DIEUDONNÉ

NEW JOURNAL

The Michigan Mathematical Journal. Vol. 1. University of Michigan Press, Ann Arbor, 1952. 197 pp. $2.00 to individuals ordering directly. $4.00 to others.

This volume consists of two numbers, containing a total of 18 papers on very diverse topics, reproduced by photo-offset from typescript. The function of the journal appears to be to provide rapid publication of papers by residents of Ann Arbor and vicinity—so rapid, indeed, that the second number, carrying the date July 1952, and actually issued, as far as I can ascertain, in May 1953, contains papers dated as late as January 1953. This suggests experimental confirmation of the hypothesis (which I accept with reluctance) that the desire for speedy publication does not completely overcome the distaste for seeing one's work appear in a less than maximally attractive format.

R. P. BOAS, JR.

BRIEF MENTION

The collected works of Bernhard Riemann. Ed. by H. Weber, with the assistance of R. Dedekind. With a supplement ed. by M. Noether and W. Wirtinger. A new introduction by Hans Lewy. New York, Dover, 1953. 10+10+558+8+116 pp. $2.55 paperbound, $4.95 clothbound.

This is a photographic reprint of the second ed. (1892) of the Gesammelte Mathematische Werke and of the Nachträge (1902); full bibliographical information is included. There is an 8 page introduction by H. Lewy.

Topologie, vol. I. By C. Kuratowski. 3d ed. (Monografie Matematyczne, vol. 20.) Warsaw, Polish Mathematical Society, 1952. 12+450 pp. $7.50.

This appears to be a photographic reprint of the 2d edition (1948; reviewed in this Bulletin, vol. 58, p. 265) except for the correction of some errors.

Tables of Chebyshev polynomials $S_n(x)$ and $C_n(x)$. (National Bureau of Standards Applied Mathematics Series, no. 9.) Washington, U. S. Government Printing Office, 1952. 30+161 pp. $1.75.

The particular Chebyshev polynomials tabulated here are $C_n(x) = 2 \cos (n \cos^{-1} x/2)$ and $S_{n-1}(x) = 2(4-x^2)^{-1/2} \sin (n \cos^{-1} x/2)$.

Construction and applications of conformal maps. Ed. by E. F. Beckenbach. (National Bureau of Standards Applied Mathematics Series, no. 18.) Washington, U. S. Government Printing Office, 1952. 6+280 pp. $2.25.

This volume contains 32 papers presented at a symposium in 1949.

Mémoires sur la théorie des systèmes des équations différentielles linéaires. By J. A. Lappo-Danilevsky. Three volumes in one. New York, Chelsea, 1953. 14+254+4+208+4+204 pp. $10.00.

This is a photographic reprint of the author's collected papers as published in Trav. Inst. Math. Stekloff, vols. 6–8 (1934–36).

Theory and application of the gamma function, with a table of the gamma function for complex arguments significant to the sixth decimal place. By W. Sibagaki. Tokyo, Iwanami Syoten, 1952. 6+202 pp.

The text is in Japanese. The tables occupy pp. 135–187 and give the real and imaginary parts of log $\Gamma(z)$ for x from -10 to more than $+10$ and y from 0 to $+10$ at varying intervals.

The meaning of relativity. By Albert Einstein. 4th ed. Princeton University Press, 1953. 4+168 pp. $3.50.

The third edition was reviewed in this Bulletin, vol. 57, p. 207. The fourth edition differs by a new version of Appendix II, now entitled *Generalization of gravitation theory.*

Les transformations birationnelles du plan. By L. Godeaux. (Mémorial des Sciences Mathématiques, no. 122.) Paris, Gauthier-Villars, 1953. 71 pp. 900 fr.

"Seconde édition entièrement refondue." The first edition was no. 22 of the same series, published in 1927.

Fluid dynamics. (Proceedings of Symposia in Applied Mathematics, vol. 4.) New York, McGraw-Hill, 1953. 5+186 pp. $7.00.

This volume, published for the Society by McGraw-Hill, contains 13 papers and one abstract. The symposium was held in June, 1951.

NOTES

The American Mathematical Society has received a grant from the National Science Foundation for a survey to determine operating procedures to be followed in compilation and maintenance of a register of mathematicians.

New research contracts have been negotiated with the following institutions by the Air Research and Development Command, to support research by the professor indicated: University of Chicago, Marcel Riesz; University of Cincinnati, Konrad Knopp, G. A. Moore; Columbia University, C. C. Chevalley, Samuel Eilenberg, B. O. Koopman, E. R. Lorch; Cornell University, R. Bechhofer; Emmanuel Missionary College, E. J. Specht; Emory University, H. E. Campbell; Johns Hopkins University, W. L. Chow, D. C. Lewis; Iowa State College of Agriculture and Mechanic Arts, D. L. Holl; University of Maryland, Alexander Weinstein; Massachusetts Institute of Technology, Warren Ambrose; University of New Hampshire, H. G. Rice; New York University, Morris Kline; Ohio State University, H. B. Mann; Stanford University, Max Shiffman; University of Tennessee, F. A. Ficken; University of Wisconsin, W. F. Eberlein; Yale University, Einar Hille.

Assistant Professor F. H. Brownell of the University of Washington has been awarded a Ford Foundation Fellowship and will study at the Institute for Advanced Study.

Assistant Professor David Gale of Brown University has received a Fulbright award for research in mathematical economics to be carried on at the University of Copenhagen and the Applied Mathematics Institute, Technical University of Denmark.

Dean Gillie A. Larew of Randolph-Macon Woman's College was honored with a Doctor of Humane Letters degree on June 8. This was the first honorary degree to be awarded by Randolph-Macon Woman's College.

Professor M. A. A. Al-Bassam of Tillotson College has accepted a position with the ministry of Education, Iraq.

Mr. Robert Croisot of the University of Poitiers has been appointed a lecturer at the University of Besançon.

Assistant Professor A. G. Anderson of Oberlin College has been appointed to an assistant professorship at Duquesne University.

Professor E. T. Bell of the California Institute of Technology has retired with the title professor emeritus.

Associate Professor J. H. Bell of Michigan State College has accepted a position as Mathematician with the Bendix Aviation Corporation, Detroit, Michigan.

Mr. Jonas Beraru of the Ford Instrument Company has accepted a position as mathematician with the Reeves Instrument Company, New York, New York.

Mr. A. S. Cahn of the National Bureau of Standards has accepted a position with the Analyzer Corporation, Los Angeles, California.

Mr. E. D. Carey has accepted a position as Senior engineer with the Federal Telecommunications Laboratories, Nutley, New Jersey.

Associate Professor Paul Civin of the University of Oregon is on leave of absence at the Institute for Advanced Study.

Associate Professor L. W. Cohen of Queens College is on leave of absence and has been appointed program director for mathematical sciences for the National Science Foundation.

Mr. A. C. Downing of the University of Michigan has accepted a position as mathematician with the Oak Ridge National Laboratory.

Dr. H. P. Edmundson of the University of California, Los Angeles, has accepted a position as mathematician with the Defense Department, Washington, D. C.

Professor Howard Eves of Champlain College, State University of New York, has been appointed to a professorship at Harpur College, State University of New York.

Mr. A. G. Eyles of Arthur G. McKee and Company has accepted a position as design engineer with the Foster-Wheeler Corporation, Boston, Massachusetts.

Dr. C. D. Firestone has accepted a position as mathematician with the Applied Physics Laboratory, Johns Hopkins University.

Dr. J. E. Flanagan of the University of Illinois has been appointed to an assistant professorship at the Carnegie Institute of Technology.

Dr. M. M. Flood has been appointed to a professorship of industrial engineering at Columbia University.

Assistant Professor M. K. Fort, Jr. of the University of Illinois has been appointed to an associate professorship at the University of Georgia.

Assistant Professor W. C. G. Fraser of Dartmouth College has accepted a position as defence research service officer with the Computation Center, McLennan Laboratories, University of Toronto.

Dr. H. D. Friedman of Pennsylvania State College has accepted a position with the General Electric Company, Syracuse, New York.

Dr. Leonard Geller of Union Carbide and Carbon Corporation has accepted a position as associate mathematician with the Cornell Aeronautical Laboratory, Buffalo, New York.

Assistant Professor D. C. Gerneth of Mississippi Southern College has accepted a position as senior aerophysics engineer with the Consolidated Vultee Aircraft Corporation, Fort Worth, Texas.

Mr. Herbert Goertzel of the Oak Ridge National Laboratory has accepted a position as programmer-coder with the AEC Computing Facility, New York University.

Assistant Professor Samuel Goldberg of Lehigh University has been appointed to an assistant professorship at Oberlin College.

Mrs. Irene M. Goldsmith of United Aircraft Corporation has accepted a position as aerodynamics engineer with the Northrop Aircraft Company.

Dean P. H. Graham of New York University has retired.

Professor Emeritus F. L. Griffin of Reed College has been appointed to a visiting professorship at Newcomb College, Tulane University of Louisiana.

Dr. J. F. Hannan of Catholic University of America has been appointed to an assistant professorship at Michigan State College.

Dr. C. M. Hebbert of the Bell Telephone Laboratories has been appointed to a professorship at the Polytechnic Institute of Brooklyn.

Dr. R. T. Herbst has accepted a position as mathematician with the Applied Physics Laboratory, Johns Hopkins University.

Assistant Professor I. R. Hershner of the University of North Carolina has been appointed to a professorship at the University of Vermont.

Mr. S. B. Hobbs of the University of New Hampshire has accepted a position as assistant project engineer with the Sperry Gyroscope Company.

Mr. E. A. Hoy of the Naval Reserve Training Publications Center has accepted a position as chief of the Interindustry and Industrial Mobilization Section, Planning Research Branch, Office of the Comptroller of the Army.

Assistant Professor S. L. Isaacson of Iowa State College of Agriculture and Mechanic Arts is on leave of absence and has been appointed to a visiting assistant professorship at Stanford University.

Professor Solomon Lefschetz of Princeton University has retired with the title Henry Burchard Fine professor emeritus.

Mr. H. T. McAdams of the Aluminum Research Laboratories has accepted a position as research physicist with the Cornell Aeronautical Laboratory, Buffalo, New York.

Mr. C. W. McArthur of the University of Maryland has been appointed to an assistant professorship at the Alabama Polytechnic Institute.

Assistant Professor K. M. McMillin of the United States Air Force Institute of Technology has been appointed to an assistant professorship at the College of St. Thomas.

Dr. E. A. Michael of the University of Chicago has been appointed to an assistant professorship at the University of Washington.

Assistant Professor D. S. Miller of the University of Rochester has accepted a position with the Eastman Kodak Company, Rochester, New York.

Dr. J. T. Moore has been appointed to an assistant professorship at the Georgia Institute of Technology.

Mr. S. I. Neuwirth of the Schering Corporation has accepted a position as biometrician with the Committee on Research, American Medical Association, Chicago, Illinois.

Dr. J. A. Nohel of the Massachusetts Institute of Technology has been appointed to an assistant professorship at the Georgia Institute of Technology.

Dr. J. V. Norris of the Ballistic Research Laboratories has been appointed to an associate professorship at the University of New Mexico.

Assistant Professor M. J. Norris of the College of St. Thomas has accepted a position as staff member with the Sandia Corporation.

Mr. L. R. Norwood of the Signal Corps, Ft. Monmouth, New Jersey, has accepted a position as senior engineer with the Electronics Defense Laboratory, Sylvania Electronics Products, Incorporated, Mountain View, California.

Dr. W. R. Orton of Oberlin College has been appointed to an assistant professorship at the University of Arkansas.

Assistant Professor F. D. Parker of St. Lawrence University has been appointed to an associate professorship at the Clarkson College of Technology.

Dr. Pasquale Porcelli of the University of Texas has been appointed to an assistant professorship at DePaul University.

Professor L. J. Reed of Johns Hopkins University, who had retired with the title professor emeritus, has been appointed president of the university.

Mr. J. D. Rice of Rice Institute has been appointed to an assistant professorship at Lamar State College of Technology.

Assistant Professor R. A. Roberts of West Virginia University has been appointed to an assistant professorship at the University of Miami.

Associate Professor G. de B. Robinson of the University of Toronto is on leave of absence and has been appointed to a visiting professorship at Michigan State College.

Dr. Raymond Sedney of the Douglas Aircraft Company has accepted a position with the Exterior Ballistics Laboratory, Aberdeen Proving Ground, Maryland.

Dr. H. D. Sprinkle of the University of Florida has been appointed to an assistant professorship at the Alabama Polytechnic Institute.

Dr. R. A. Struble of the Douglas Aircraft Company has been appointed to an assistant professorship at the Illinois Institute of Technology.

Dr. G. L. Thompson of Princeton University has been appointed to an assistant professorship at Dartmouth College.

Associate Professor R. J. de Vogelaere of Laval University has been appointed to a visiting professorship at the University of Notre Dame.

Mr. D. W. Wall of the University of Michigan has accepted a position as analyst with the Defense Department.

Professor Emeritus L. T. Wilson of the United States Naval Academy has been appointed to a professorship at Jacksonville State Teachers College, Jacksonville, Alabama.

Dr. Oswald Wyler of Northwestern University has been appointed to an assistant professorship at the University of New Mexico.

Miss Irma M. Wyman of the Willow Run Research Center, University of Michigan has accepted a position as mathematician with the Raytheon Manufacturing Company, Waltham, Massachusetts.

The following promotions are announced:

J. G. Adshead, Dalhousie University, to a professorship.

M. G. Arsove, University of Washington, to an assistant professorship.

Samuel Borofsky, Brooklyn College, to a professorship.

C. B. Boyer, Brooklyn College, to a professorship.

R. S. Burrington, Bureau of Ordnance, Navy Department, to special assistant to the Director of Research and Development.

P. L. Butzer, McGill University, to an assistant professorship.

D. G. Chapman, University of Washington, to an associate professorship.

K. L. Cooke, State College of Washington, to an assistant professorship.

M. L. Curtis, Northwestern University, to an assistant professorship.

N. J. Fine, University of Pennsylvania, to an associate professorship.

A. B. Finkelstein, Long Island University, to an assistant professorship.

H. J. Hamilton, Pomona College, to a professorship.

T. C. Holyoke, Northwestern University, to an assistant professorship.

H. T. Karnes, Louisiana State University and Agricultural and Mechanical College, to a professorship.

L. M. Kelly, Michigan State College, to an associate professorship.

M. S. Klamkin, Polytechnic Institute of Brooklyn, to an assistant professorship.

Lee Lorch, Fisk University, to a professorship.

E. R. Mullins, Jr., Swarthmore College, to an assistant professorship.

Andrewa R. Noble, Mills College, to an assistant professorship.

W. V. Parker, Alabama Polytechnic Institute, to Dean of the Graduate School.

W. D. Peeples, Jr., Howard College, to an associate professorship.

Walter Prenowitz, Brooklyn College, to a professorship.

David Rosen, Swarthmore College, to an assistant professorship.

L. A. Schmidt, Armour Research Foundation, Illinois Institute of Technology, to assistant supervisor in charge of the flow and aerodynamics section.

H. M. Schwartz, University of Arkansas, to a professorship.

D. C. Spencer, Princeton University, to a professorship.

B. M. Stewart, Michigan State College, to a professorship.

Ruth W. Stokes, Syracuse University, to an associate professorship.

J. S. Stubbe, Clark University, to an associate professorship.

J. T. Tate, Princeton University, to an assistant professorship.

Margaret Y. Woodbridge, Brooklyn College, to an assistant professorship.

The following appointments to instructorships are announced: Cornell University: Dr. J. K. Goldhaber, Dr. C. S. Herz; Illinois Institute of Technology: Mr. R. H. Oehmke; University of Maryland: Dr. J. W. Brace; Michigan State College: Dr. J. W. Gaddum; University of Nevada: Dr. L. E. Ward, Jr.; Oberlin College: Dr. J. D. Baum; Ohio State University: Dr. W. E. Deskins; Ohio Wesleyan University: Mr. P. C. Stanger; University of Oregon: Mr. E. C. Smith; Polytechnic Institute of Brooklyn: Dr. Seymour Schuster; University of Washington: Dr. R. F. Tate.

Dr. B. L. Falconer of Marlin, Texas died on April 26, 1953 at the age of eighty-two years. He had been a member of the Society for thirty-three years.

Professor A. D. Michal of the California Institute of Technology died on June 14, 1953 at the age of fifty-four years. He had been a member of the Society for thirty years.

Professor R. v. Mises of Harvard University died on July 14, 1953 at the age of seventy years. He had been a member of the Society for thirteen years.

The note concerning Murray Rosenblatt in the May issue of the Bulletin is in error. It should read "Assistant Professor" instead of "Associate Professor."

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RECENT DEVELOPMENTS IN THE THEORY OF LOCALLY CONVEX VECTOR SPACES

J. A. DIEUDONNÉ

1. Introduction. Although the theory of Banach spaces has been very popular among American mathematicians during the last twenty years, comparatively little attention seems to have been given, in this country, to its generalizations, except in the very last few years. With the exception of the outstanding work of G. W. Mackey [47; 48], most contributions to the general theory of locally convex spaces have been made by European mathematicians. There may be some interest, therefore, in a survey in broad outline of the most recent advances in that field, some of which have not yet appeared in print.

The principal motivation behind the general theory is the same as that of Banach himself: namely, a search for general tools which might be applied successfully to functional analysis. Two different sectors contributed the main influences: the first originated in the work of G. Köthe, O. Toeplitz, and their students on sequence spaces [32–46; 26; 50; 11], which began around 1934 and was partly related to the theory of functions of a complex variable [62]; many of the ideas which were to become fundamental in the later development of the general theory appeared there for the first time, and also a great wealth of illuminating examples and counter-examples. For unknown reasons, this remarkable pioneering work has to this day remained practically ignored in this country, in spite of its intrinsic importance and usefulness.

The other influence was exerted by the developments of the theory of integration, and chiefly through the efforts to free that theory from the shackles of the Carathéodory measure theory and turn it into a mere chapter of the general theory of topological vector spaces [6]. These efforts culminated in L. Schwartz's theory of distributions (1945), which could be expressed only in the language of locally convex vector spaces [56]; it turned out that for that theory, Banach spaces were an utterly inadequate tool, and the realization of that fact led to very active research on more general spaces, to which most of the results obtained in the last few years owe their origin.

This recent work has led in particular to a new classification and

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reëvaluation of the whole subject. Part of the interest which attached itself to the study of Banach spaces was certainly due to their particularly simple definition by means of a norm, and it is quite natural that this simple case should have been attacked first. But analysts are more interested in the properties of a space than in the way it is defined; hence the idea, first clearly formulated by L. Schwartz, of classifying topological vector spaces according to their behavior with regard to the validity of the main theorems of functional analysis. For instance, two of the most important of these, the Banach-Steinhaus theorem and the closed graph theorem, are valid in Banach spaces; it is quite natural to ask in what other spaces they are still true. This program has been very actively pushed forward within the last few years; its main outcome is that, surprisingly enough, practically no important property is really special to Banach spaces, although the latter still constitute a very useful technical tool in the study of the general theory.

2. The fundamental notions: I. Locally convex spaces (see [4]). We shall be exclusively concerned with vector spaces over the real field: the passage to complex spaces offers no difficulty. We shall assume that the definition and properties of *convex sets* are known. A convex set A in a vector space E is *symmetric* if $-A=A$; then $0\in A$ if A is not empty. A convex set A is *absorbing* if for every $x\neq 0$ in E, there exists a number $\alpha\neq 0$ such that $\lambda x\in A$ for $|\lambda|\leqq\alpha$; this implies that A generates E.

A *locally convex space* is a topological vector space in which there is a fundamental system of neighborhoods of 0 which are convex [54; 63]; these neighborhoods can always be supposed to be symmetric and absorbing. Conversely, if any filter base is given on a vector space E, and consists of convex, symmetric, and absorbing sets, then it defines one and only one topology on E for which $x+y$ and λx are continuous functions of both their arguments.

A *semi-norm* on a vector space E is a function $p(x)$ defined on E, such that $0\leqq p(x)<+\infty$ for all $x\in E$, $p(\lambda x)=|\lambda|p(x)$, and $p(x+y)\leqq p(x)+p(y)$; the sets defined by any of the relations $p(x)<\alpha$, $p(x)\leqq\alpha$ $(\alpha>0)$ are convex, symmetric and absorbing; conversely, for every such set A, there exists one and only one semi-norm p such that A contains the set $p(x)<1$ and is contained in $p(x)\leqq 1$. From these remarks it follows that the topology of a locally convex space can also be defined by a family (p_α) of semi-norms, to which correspond the neighborhoods of 0 defined by $p_\alpha(x)<\lambda$ $(\lambda>0)$; and conversely, such a family always defines a locally convex topology. This topology is Hausdorff if and only if, for every $x \neq 0$, there is an α such that $p_\alpha(x) \neq 0$; it is metrizable if the family (p_α) is denumerable.

3. The fundamental notions: II. Bounded sets (see [5]). The concept of bounded set is easily defined in a normed space: it is a set contained in *some* ball $\|x\| \leq R$. To extend this notion when no metric is at hand, we may reformulate it as follows: B is bounded if, given *any* ball $\|x\| \leq r$, there exists a $\lambda > 0$ such that λB is contained in that ball. If we say that a set A *absorbs* a set B if there exists $\lambda > 0$ such that $\lambda B < A$, we can therefore say that a bounded set is one which is absorbed by every ball. Hence the general definition of a bounded set in a locally convex space E: it is a set B which is *absorbed by every neighborhood of* 0 *in* E [31; 54]. An equivalent definition is that *every* semi-norm which defines the topology of E is *bounded* on B. If B is bounded, so is μB for any μ; the convex hull of B is bounded, as well as its closure. The union of a finite number of bounded sets is bounded; so is $A + B$, if both A and B are bounded. Precompact sets (in particularly Cauchy sequences) are bounded.

The notion of bounded set is not very important in normed spaces, because it is then equivalent to the notion of (arbitrary) subset of a ball; in other words, there is a fundamental system of *bounded neighborhoods* of 0. This turns out to be exceptional among locally convex spaces: indeed, a Hausdorff locally convex space possesses bounded neighborhoods of 0 *if and only if* its topology can be defined by means of a norm [31].

On a locally convex space (as on any abelian topological group) there is a uniform structure determined by its topology, and such a space E is said to be *complete* if every Cauchy *filter* (for that uniform structure) converges in E; for any Hausdorff locally convex space E, there is a well determined locally convex space $\hat{E}$ which is complete and in which E is dense (the *completion* of E). There are important locally convex vector spaces (for instance, all infinite-dimensional vector spaces with "weak" topologies; see §6 below) which fail to be complete; but most spaces which occur in functional analysis have at least the weaker property that *bounded closed sets are complete*; they are called *quasi-complete* spaces. A still weaker property, which suffices for many applications, is that every Cauchy *sequence* is convergent, in which case the space is said to be *semi-complete*. The three notions coincide of course for metrizable spaces.

4. The $\mathfrak{S}$-topologies on the spaces $\mathcal{L}(E, F)$ [5; 7]. The most important applications of locally convex spaces to functional analysis deal with *linear operators*, that is, linear mappings from a functional space E into a functional space F, subject in general to conditions related to the topologies of E and F. One is thus led, in particular, to study the set $\mathcal{L}(E, F)$ of *all continuous linear mappings* of a locally convex space E into a locally convex space F. This is itself a vector space, and one of the main problems of the theory is to define and to study on $\mathcal{L}(E, F)$ topologies related in a natural way to those of E and F. The known methods of defining topologies on functional spaces by conditions of "uniform smallness" on certain subsets [3] lead to the following tentative definition: for every subset A of E and every neighborhood V of 0 in F, let $T(A, V)$ be the set of all $u \in \mathcal{L}(E, F)$ such that $u(A) \subset V$; one takes as a fundamental system of neighborhoods of 0 in $\mathcal{L}(E, F)$ the sets $T(A, V)$, where A runs through a family $\mathfrak{S}$ of subsets of E and V through a fundamental system of neighborhoods of 0 in F. It turns out that this in fact defines a locally convex topology (called *$\mathfrak{S}$-topology*) on $\mathcal{L}(E, F)$, provided the sets $A \in \mathfrak{S}$ are *bounded* in E. If E and F are Hausdorff, $\mathcal{L}(E, F)$ is Hausdorff if the union of the sets of $\mathfrak{S}$ is dense in E. The family $\mathfrak{S}$ can always be supposed to consist of closed, convex, and symmetric sets, and to be such that the closed convex hull of the union of any finite number of sets of $\mathfrak{S}$ belongs to $\mathfrak{S}$. Among all $\mathfrak{S}$-topologies for which the union of the sets of $\mathfrak{S}$ is E, the finest is the topology for which $\mathfrak{S}$ is the set of *all* bounded, convex, closed symmetric sets of E (topology of *bounded convergence* on $\mathcal{L}(E, F)$; when E and F are normed spaces, it is the usual norm or "uniform" topology on $\mathcal{L}(E, F)$); the coarsest is the topology for which $\mathfrak{S}$ is the set of all bounded convex closed finite-dimensional subsets of E (topology of *pointwise convergence* on $\mathcal{L}(E, F)$).

A subset H of $\mathcal{L}(E, F)$ is bounded for the $\mathfrak{S}$-topology, or *$\mathfrak{S}$-bounded*, if and only if for every set $A \in \mathfrak{S}$, the union of the sets $u(A)$, where $u \in H$, is bounded in F. This notion depends in general on the family $\mathfrak{S}$; however, if E is *semi-complete* (see §3) any set which is bounded for the topology of pointwise convergence is also bounded for every $\mathfrak{S}$-topology [48; 7].

In the next three sections, all spaces will be supposed to be *Hausdorff* locally convex spaces.

5. **t-spaces and the Banach-Steinhaus theorem** [5; 7]. Particularly important subsets of a space $\mathcal{L}(E, F)$ are the *equicontinuous* subsets: such a set H is characterized by the property that for every neighborhood V of 0 in F, there is a neighborhood U of 0 in E such that $u(U) \subset V$ for *all* $u \in H$. Equicontinuous subsets are $\mathfrak{S}$-bounded for every family $\mathfrak{S}$, but the converse need not hold. A locally convex

space E is called a *t-space* (French: "espace tonnelé") if for every locally convex space F, the subsets H of $\mathcal{L}(E, F)$ which are *bounded for the topology of pointwise convergence* (that is, such that for every $x \in E$, the set of $u(x)$, where $u \in H$, is bounded in F) are *equicontinuous* (hence bounded for every $\mathfrak{S}$-topology). This property implies the *Banach-Steinhaus theorem*: for every sequence (u_n) of continuous linear mappings of E into F, such that $u_n(x)$ tends to a limit $v(x)$ for every $x \in E$, v is a linear continuous mapping of E into F, and u_n converges uniformly to v on every precompact subset of E. Moreover, if the sequence (u_n) is such that the set of $u_n(x)$ $(n=1, 2, \cdots)$ is bounded in F for every $x \in E$, then the same conclusion follows if convergence of $(u_n(x))$ to a limit is assumed only in a dense subset of E, provided F is semicomplete. These properties, as is well known, are among the most often used in functional analysis.

The definition of t-spaces can be given an equivalent formulation in which only "internal" properties of E intervene. In a locally convex space E, a *barrel* is a *closed*, *convex*, *symmetric*, and *absorbing* set; closed convex symmetric neighborhoods of 0 are of course barrels. Now E is a t-space if and only if, conversely, *all barrels are neighborhoods of* 0 [7].

The most important of t-spaces are the *Baire spaces*, that is, spaces in which every denumerable intersection of open dense subsets is dense [7]; this is of course the case for Banach spaces and more generally (F)-spaces (locally convex, metrizable, complete spaces). But there are metrizable spaces which are t-spaces without being Baire spaces [14]; there are also nonmetrizable complete spaces which are t-spaces without being Baire spaces, and complete spaces which are not t-spaces.

The completion of a t-space is a t-space; products and quotients of t-spaces are t-spaces [7]. However, a closed subspace of a t-space is not necessarily a t-space [20; 15; 61].

6. **Duality in locally convex spaces** [5]. The general results on spaces $\mathcal{L}(E, F)$ of continuous linear operators apply in particular to the space $E' = \mathcal{L}(E, R)$ of continuous linear *forms* (also called linear *functionals*); this space is called the *dual* (or *conjugate*) of E. The $\mathfrak{S}$-topologies in E' can here be described in the following simple way: for any $x \in E$, $x' \in E'$, write $\langle x, x' \rangle = x'(x)$; for any subset A of E, the *polar* A^0 of A in E' is the set of all $x' \in E'$ such that $|\langle x, x' \rangle| \leq 1$ for *all* $x \in A$; it is a convex, symmetric, and absorbing set. The $\mathfrak{S}$-topology on E' has a fundamental system of neighborhoods which consists of finite intersections of the *polars of the sets* $A \in \mathfrak{S}$. The most im-

portant $\mathfrak{S}$-topologies on E' are (as in general on spaces $\mathcal{L}(E, F)$) the topology of pointwise convergence, also called *weak* topology*, and denoted by $\sigma(E', E)$, and the topology of bounded convergence, also called *strong topology*, and denoted by $\beta(E', E)$ (when E is a normed space, $\beta(E', E)$ is simply the usual norm topology on E').

Equicontinuous subsets of E' are those which are contained in *polars* V^0 *of neighborhoods of* 0 in E; these sets V^0 are *weakly* compact* by Tychonoff's theorem. Equicontinuous sets are strongly bounded, and strongly bounded sets are weakly* bounded; in general these three classes of sets are distinct. However:

(1) every weakly* bounded set in E' is equicontinuous if and only if E is a t-space;

(2) if E is semi-complete, every weakly* bounded set in E' is strongly bounded;

(3) in order that every strongly bounded set in E' be equicontinuous, it is necessary and sufficient that every barrel in E, which absorbs all bounded subsets of E, be a neighborhood of 0; E is then said to be a *quasi-t-space*; the completion of such a space is a t-space (see §5).

From the last results, it follows easily that E' is *weakly* quasi-complete* (see §3) if E is a t-space, and strongly *quasi-complete* if E is a quasi-t-space (for conditions insuring completeness of E' (with the strong topology) see §9).

Let E'' be the dual of the space E', when the latter is given the strong topology; E'' is called the *bidual* (or *second conjugate*) of E; not much is known in general about its properties (for instance, it is not known whether E'', with the strong topology $\beta(E'', E')$, is always complete, even if E is a t-space).

For every $x \in E$, consider the linear form $x' \rightarrow \langle x, x' \rangle$ on E'; this linear form $\tilde{x}$ is weakly* (hence also strongly) continuous, and therefore $x \rightarrow \tilde{x}$ is a linear mapping of E into E'', which is easily seen to be 1-1. E can thus be imbedded (algebraically) into E''; the topology $\sigma(E, E')$ induced on E by the weak* topology $\sigma(E'', E')$ is called the *weak topology* on E; it is coarser than the initial topology on that space, but *bounded sets* and *closed convex* sets are the same for both topologies [48; 9]. In general, the imbedding $x \rightarrow \tilde{x}$ is not onto; in order that $E'' = E$, it is necessary and sufficient that every bounded subset of E be relatively compact for the weak topology $\sigma(E, E')$; E is then said to be *semi-reflexive*. The topology induced on E by the strong topology $\beta(E'', E')$ is in general distinct (and finer) than the initial topology on E; it is identical to it if and only if E is a quasi-t-space (see above); the weak* closures (in E'') of the neighborhoods of 0 in E for the initial topology constitute in that case a fundamental system of neighborhoods of 0 for $\beta(E'', E')$. In order that $E''=E$ *and* that the strong topology $\beta(E'', E')$ coincides with the initial topology on E, it is necessary and sufficient that E be a *semi-reflexive t*-space; such spaces are called *reflexive*. The strong dual of a reflexive space is a reflexive space.

Let F be a closed subspace of a locally convex vector space E; the dual F' of F can be identified in a natural way with the quotient space E'/F^0, where F^0 is the orthogonal subspace to F in E', and the dual of the quotient space E/F can be naturally identified with the subspace F^0 of E'; moreover, the weak topologies $\sigma(F^0, E/F)$ and $\sigma(E'/F^0, F)$ are respectively the induced and the quotient topology of $\sigma(E', E)$ [9]. However, when it comes to strong topologies, the corresponding results (which are true for Banach spaces) do not hold any more, even for (F)-spaces or Montel spaces (see §7). In particular, a closed subspace of a semi-reflexive space is still semi-reflexive, but a closed subspace of a reflexive space need not be reflexive, and a quotient space of a reflexive space is not even necessarily semi-reflexive (see, however, §12).

7. **Duality in special spaces.** (F)-spaces (see §5) are next to Banach spaces in the applications to functional analysis, and their theory has been recently the subject of several papers [13; 15], the deepest results being due to A. Grothendieck [20]. The dual E' of an (F)-space is always complete for the strong topology (see §8) but is never metrizable unless E is a Banach space. Moreover, it is not always a *t*-space (for the strong topology) even if E is separable [20]; however, when E is reflexive, or when E' itself contains a denumerable dense subset for the strong topology (in which case E is necessarily separable itself), E' is a *t*-space [20]. In every case, the bidual E'' is an (F)-space for the strong topology [20; 15].

A locally convex space E is called a *Montel space* if it is a *t*-space (§5) and if every bounded closed subset of E is *compact*. A Banach space cannot be a Montel space unless it is finite-dimensional; but a great many functional spaces are Montel spaces, for instance the space of holomorphic functions in a domain, with the compact-open topology, or most spaces which occur in the theory of distributions (see §12). A Montel space is reflexive, and its strong dual is again a Montel space; but a closed subspace of a Montel space need not be a Montel space [20]; and an example has been given [20] of a Montel space E, which is also a separable (F)-space, such that there is a quotient space E/F of E which is isomorphic to the Banach space l^1 (hence fails to be reflexive).

8. Projective and inductive limits of spaces [4]. Let (F_α), $\alpha \in A$, be a family of locally convex spaces, and let E be a vector space, initially without any topology. Suppose first that for each α there is given a linear mapping f_α *of* E *into* F_α; then a locally convex topology can be defined on E by the condition that it be the *coarsest* of all locally convex topologies on E for which all the f_α are continuous; with that topology, E is called the *projective limit* of the F_α (relative to the maps f_α). If all F_α are Hausdorff spaces, E is Hausdorff if and only if for every $x \neq 0$ in E there is an α such that $f_\alpha(x) \neq 0$; the mapping $x \rightarrow (f_\alpha(x))$ is then a topological imbedding of E into the product space $\prod_{\alpha \in A} F_\alpha$. It is easily shown that every Hausdorff locally convex space can thus be imbedded into a product of *Banach spaces*. In order that a linear mapping g from a locally convex space G into a projective limit E be continuous, it is necessary and sufficient that $f_\alpha \circ g$ be continuous for each $\alpha \in A$.

The family (F_α) and E being as above, suppose now that for each α there is given a linear mapping g_α *of* F_α *into* E; then a locally convex topology can be defined on E by the condition that it be the *finest* of all locally convex topologies on E for which all the g_α are continuous; with that topology, E is called the *inductive limit* of the F_α (relative to the maps g_α). In order that a linear mapping f from E into a locally convex space G be continuous, it is necessary and sufficient that $f \circ g_\alpha$ be continuous for each $\alpha \in A$.

When all the F_α are Hausdorff, no simple general condition is known which will guarantee that the inductive limit E is Hausdorff; this property can, however, easily be checked for all the most important cases of inductive limits reviewed below. An inductive limit of t-spaces (respectively quasi-t-spaces) is a t-space (respectively quasi-t-space).

An obvious example of inductive limit is a quotient space F/M (with only one f_α, namely the natural mapping). Less obvious but quite important is the *direct sum* of an arbitrary family $(E_\alpha)_{\alpha \in A}$ of locally convex vector spaces [29; 27]: this is the subspace E of the (nontopologized) product $\prod_{\alpha \in A} E_\alpha$, consisting of elements having only a finite number of coordinates $x_\alpha \neq 0$; consider in E the *finite* products $F_J = \prod_{\alpha \in J} E_\alpha$, each with the product topology, and the injection g_J of F_J in E; the direct sum topology on E is the inductive limit of the topologies of the F_J (relative to the injections g_J); we shall write $E = \sum_{\alpha \in A} E_\alpha$. Every inductive limit of the E_α (for some family of maps) which is the union of the images of the E_α is isomorphic to a quotient space of $\sum_{\alpha \in A} E_\alpha$. If all E_α are Hausdorff spaces, so is their direct sum. Bounded sets in E are those which are contained in one of the products F_J. If all E_α are complete Hausdorff spaces, so is E [27]. An interesting particular case is that in which all the E_α are one-dimensional: this gives on the direct sum E the *finest* locally convex topology.

There is a natural duality between direct sum and product [29]. More precisely, the dual of a direct sum $E=\sum_{\alpha\in A}E_\alpha$ is the product $E'=\prod_{\alpha\in A}E'_\alpha$; the weak* topology $\sigma(E', E)$ is the product of the topologies $\sigma(E'_\alpha, E_\alpha)$, and the strong topology $\beta(E', E)$ is the product of the strong topologies $\beta(E'_\alpha, E_\alpha)$. Conversely, the dual of the product $F=\prod_{\alpha\in A}E_\alpha$ is the direct sum $F'=\sum_{\alpha\in A}E'_\alpha$, and the strong topology $\beta(F', F)$ is the direct sum of the strong topologies $\beta(E'_\alpha, E_\alpha)$ (this is not true for the weak* topologies). In order that E or F be semi-reflexive (respectively reflexive, or a Montel space) it is necessary and sufficient that each of the E_α be semi-reflexive (respectively reflexive, a Montel space).

9. **(LF)-spaces, bornological spaces and the closed graph theorem.** Another important instance of inductive limit is the case of a strictly increasing sequence (E_n) of subspaces of a vector space E, such that E is the union of the E_n; each E_n is given a locally convex topology $\mathcal{T}_n$, such that $\mathcal{T}_n$ induces $\mathcal{T}_{n-1}$ on E_{n-1}. The space E, with the inductive limit topology $\mathcal{T}$ of the $\mathcal{T}_n$ (for the inclusion maps) is called the *strict* inductive limit of the E_n; $\mathcal{T}$ induces $\mathcal{T}_n$ on each E_n, and if each E_n is closed in E_{n+1}, E_n is closed in E; moreover, in this last case, E is Hausdorff if the E_n are, and bounded sets in E are those which are contained in some E_n and bounded in E_n; E is never metrizable nor a Baire space [13]. Always in the case in which E_n is closed in E_{n+1}, E is complete (respectively semi-reflexive, reflexive, a Montel space) if and only if each E_n has the same property [41].

The most interesting case is that in which the E_n are (F)-spaces; E is then called an *(LF)-space* [13]; many functional spaces are of this type, in particular, the space $(\mathcal{D})$ of Schwartz. Unfortunately, these spaces fail in general to possess many of the nice properties of (F)-spaces; for instance, a closed subspace F of an (LF)-space E is not necessarily an (LF)-space, nor is the quotient E/F necessarily an (F) or (LF)-space [20].

The third important example of inductive limit consists of spaces E which are inductive limits of *normed* spaces E_α with respect to maps g_α such that E is the union of the $g_\alpha(E_\alpha)$; these spaces are called *bornological* [7; 15; 48; 61]. Such a space can also be characterized by any of the two following properties; (a) any convex, symmetric set in E which absorbs (see §3) all bounded subsets of E is a neighborhood of 0 in E; (b) any linear mapping u of E into a locally convex space F, which transforms bounded sets into bounded sets, is continuous [48]. Every metrizable locally convex space is bornological [48]; every bornological space is a quasi-t-space (see §7), but can fail to be a t-space (see §5); whether there are t-spaces which are not bornological is an unsolved problem, as is also the question whether the completion of a bornological space is bornological. The dual of a bornological space is strongly complete [7]. The strong dual of an (F)-space is bornological if and only if it is a t-space [20].

An inductive limit of bornological spaces is bornological. A product of bornological spaces is bornological if the set of factors has a cardinal at most equal to that of the continuum; whether the result extends to arbitrary products is an open question, equivalent to a problem of Ulam in measure theory [48; 49; 15]. Examples are known of bornological spaces having non-bornological closed subspaces [20; 15; 61].

The notion of inductive limit is linked with the most extensive generalization which has been given thus far of the *closed graph theorem* of Banach [13; 42; 24]: suppose E is a bornological semi-complete (see §4) space, and F is defined as the inductive limit of a sequence (F_n) of (F)-spaces, relative to a sequence (g_n) of linear mappings, such that F is the union of the $g_n(F_n)$. Then, if u is a linear mapping of E into F such that the graph of u is closed in $E\times F$, u is continuous. It follows from this that any linear continuous mapping of F *onto* E is an open homomorphism.

10. Spaces of bilinear functionals and tensor products. As already stressed above (see §4), the study of spaces of operators, and more generally of spaces $\mathcal{L}(E, F)$, is a central theme in modern functional analysis: well known examples are the theory of rings of operators in Hilbert space, and the more recent theory of kernels developed by L. Schwartz in connection with his theory of distributions [57]. For deeper results on spaces $\mathcal{L}(E, F)$, a knowledge of their duality theory is of great value; this is the main starting point of the very recent theory of *topological tensor products*, which we are now going to consider.

Let E and F be t-spaces, E' and F' the duals of E and F, and write E_s, E'_s for the spaces E, E' with the weak topologies $\sigma(E, E')$ and $\sigma(E', E)$ respectively, and similarly for F and F'. Every continuous linear mapping u from E to F is also a continuous linear mapping from E_s to F_s, and conversely, in other words, $\mathcal{L}(E, F)=\mathcal{L}(E_s, F_s)$. Now, for every $x\in E$ and every $y'\in F'$, consider the number $v(x, y')=\langle u(x), y'\rangle$; this is a bilinear functional on $E\times F'$, and it is continuous with respect to *each* variable, when considered as defined over the topological space $E_s \times F'_s$; conversely, any such *separately continuous* bilinear functional over $E_s \times F'_s$ has the form $(x, y') \to \langle u(x), y' \rangle$, where u is a well determined element of $\mathcal{L}(E, F)$; in other words, there is an algebraic *isomorphism* between the space $\mathcal{L}(E, F)$ and the space $\mathfrak{B}(E_s, F'_s)$ of separately continuous bilinear functionals over $E_s \times F'_s$.

Changing slightly our point of view, let us now consider generally the space $\mathfrak{B}(E, F)$ of bilinear separately continuous functionals over a product $E \times F$ of two locally convex spaces. We observe that this space contains, of course, the subspace $\mathcal{B}(E, F)$ of all *continuous* bilinear functionals (with respect to both variables), but is not generally identical with it (although it is well known that the two spaces are identical if E and F are (F)-spaces). We also observe that $\mathfrak{B}(E, F)$ and $\mathfrak{B}(E_s, F_s)$ are obviously identical, which provides the link with the preceding considerations.

Now, looking for a duality theory adapted to $\mathfrak{B}(E, F)$ (or $\mathcal{B}(E, F)$) means that we are trying to determine a space such that the bilinear functionals over $E \times F$ become *linear* functionals over that space. In pure algebra, this has been achieved long ago by the consideration of the *tensor product* (or *Kronecker product*) $E \otimes F$ of the two spaces [2]; this is defined as the set of all formal (finite) linear combinations $\sum_i x_i \otimes y_i$ $(x_i \in E, y_i \in F)$, where identifications have been performed, such that $(x+x') \otimes (y+y')$ is identified with $x \otimes y + x' \otimes y + x \otimes y' + x' \otimes y'$, and $(\alpha x) \otimes (\beta y)$ with $\alpha\beta(x \otimes y)$. Linear forms on that space are exactly in one-to-one correspondence with all bilinear forms on $E \times F$, the correspondence associating to the linear form u the bilinear form $(x, y) \to u(x \otimes y)$. When E and F are locally convex spaces, we can still consider the algebraic tensor product $E \otimes F$, and $\mathfrak{B}(E, F)$ (or $\mathcal{B}(E, F)$) can then be identified with a space of linear forms on $E \otimes F$; but we want to define $E \otimes F$ as a *topological* vector space in such a way that the elements of $\mathfrak{B}(E, F)$ (or $\mathcal{B}(E, F)$) will become *continuous* over $E \otimes F$. This problem was first successfully attacked by R. Schatten when E and F are Banach spaces [55]; it has now been extensively treated in the general case by A. Grothendieck [23; 24]. It turns out that, even in the case of Banach spaces, there are several "natural" solutions.

Let us suppose, for simplicity's sake, that E and F are complete t-spaces. The spaces $E \otimes F$ and $\mathcal{B}(E, F)$ are put in a natural duality by the "scalar product" $\langle x \otimes y, u \rangle = u(x, y)$ as seen above. If we recall that, in the dual of a locally convex space G, polars of the neighborhoods of 0 in G are equicontinuous sets (see §6), we have a natural way of defining neighborhoods of 0 in $E\otimes F$: for each *equicontinuous* set M of continuous bilinear functionals, consider the set M^0 in $E\otimes F$ consisting of all $z=\sum_i x_i\otimes y_i$ such that $|\langle z, u\rangle| = |\sum_i u(x_i, y_i)| \leqq 1$ for *all* $u\in M$. A fundamental system of neighborhoods of 0 in $E\otimes F$ will then be taken as consisting of all sets of the form M^0. There are two equivalent definitions: (a) take a convex neighborhood V of 0 in E, a convex neighborhood W of 0 in F, and consider the convex hull $\Gamma(V\otimes W)$ of all elements $x\otimes y$, where $x\in V$ and $y\in W$; these sets form a fundamental system of neighborhoods of 0; (b) equivalently, take a semi-norm α on E, a semi-norm β on F, and define, for $z\in E\otimes F$,

$$\gamma(z) = \text{g.l.b.}\left(\sum_i \alpha(x_i)\beta(y_i)\right)$$

the g.l.b. being taken over all possible representations of z in the form $\sum_i x_i\otimes y_i$; then the semi-norms γ define the topology of $E\otimes F$. For that topology, linear continuous functionals on $E\otimes F$ are *exactly* identified with the bilinear continuous functionals on $E\times F$; more generally, for any locally convex space H, the space $\mathcal{L}(E\otimes F, H)$ is identified with the space of *continuous bilinear mappings* of $E\times F$ into H.

The preceding operations, however, still give in general "too small" a vector space for the tensor product. This is best understood when one of the spaces E, F is a space of *numerical functions* over some set S; it is then easily verified that the space $E\otimes F$ (without topology) can be identified with the set of mappings $t\rightarrow\sum_i x_i(t)y_i$ (finite sum), where $x_i\in E$ and $y_i\in F$; these are mappings of *finite rank*, and it is clear that interesting results will be obtained only if we enlarge that functional space by adding to it the "limits" of its elements in some sense. This leads to the final step in the definition of the tensor product: we consider the space $E\otimes F$ with the topology defined above and take its *completion* $E\otimes_1 F$, which is then called the *projective tensor product* of E and F.

Similar definitions may be given when $\mathcal{B}(E, F)$ is replaced by $\mathfrak{B}(E, F)$; we take here as sets $M\subset\mathfrak{B}(E, F)$ those which are *separately equicontinuous* (that is, for every $x\in E$, the set of linear functionals $y\rightarrow u(x, y)$ is equicontinuous when u runs through M, and the same condition holds when E and F are exchanged). The polar sets M^0 then define a new topology on $E\otimes F$, and the *completion* of $E\otimes F$ for that topology is a second tensor product, written $E\otimes_0 F$ and called the *inductive tensor product*. It is identical to $E\otimes_1 F$ when both E and F are (F)-spaces. Continuous linear mappings of $E\otimes_0 F$ into a complete locally convex space H are here identified with the *separately continuous* bilinear mappings of $E\times F$ into H.

A third "natural" topology is finally defined in the following way. Let us consider the duals E' and F' of E and F, and their tensor product $E'\otimes F'$; this space (without topology) can be identified in a natural way with a subspace of $\mathcal{B}(E, F)$, by associating to the tensor product $x'\otimes y'$ the bilinear form $(x, y)\rightarrow\langle x, x'\rangle\langle y, y'\rangle$ (*product* in the classical sense of x' and y'). We can now take as sets M in the general mèthod outlined above the *products* $P\otimes Q$ (set of all $x'\otimes y'$, where $x'\in P$, $y'\in Q$), where P and Q are *equicontinuous* subsets of E' and F' respectively. The completion of $E\otimes F$ under this topology is called the *biprojective tensor product* of E and F, and written $E\otimes_2 F$.

Of these three topologies on $E\otimes F$, the biprojective is the coarsest, the inductive the finest, and the projective is between these two extremes. Hence the "largest" tensor product is the biprojective one, and the "smallest" the inductive one, the projective tensor product being "intermediate."

11. Properties of the tensor products [24]. We shall only be concerned with the projective and biprojective tensor products. A "concrete" interpretation can be given of these products in several important cases. For instance, if E and F are (F)-spaces, then $E\otimes_1 F$ and $E\otimes_2 F$ are (F)-spaces; the elements of $E\otimes_1 F$ can be described as sums of convergent series $\sum_k \lambda_k x_k\otimes y_k$, where $\sum_{k=0}^{\infty} |\lambda_k|$ is finite, and the sequences (x_k) and (y_k) tend to 0 in E and F respectively. The space $l^1\otimes_1 E$ can be identified with the space of *absolutely convergent series* in E (that is, series $\sum_{n=0}^{\infty} x_n$ such that for every seminorm p on E, the series $\sum_{n=0}^{\infty} p(x_n)$ is absolutely convergent), whereas the space $l^1\otimes_2 E$ can be identified with the space of all *unconditionally convergent series* in E. Similarly, if $C(K)$ is the space of real continuous functions on a compact space K, $C(K)\otimes_2 E$ can be identified with the space of continuous mappings from K to E, with the topology of uniform convergence.

Other important interpretations of tensor products stem from the initial remark of §10. Every element $\sum_i x_i'\otimes y_i$ of the (algebraic) tensor product $E'\otimes F$ can be identified with the linear mapping $x\rightarrow\sum_i\langle x, x_i'\rangle y_i$ of E into F; in other words, $E'\otimes F$ can be considered as the space of all linear continuous mappings of *finite rank* of E into F. The tensor products $E'\otimes_1 F$ and $E'\otimes_2 F$ (where E' is given its strong topology) will, in the most important cases, be identified with spaces of linear mappings of E into F which are "limits" in some sense of mappings of finite rank. For instance, if E is a reflexive Banach space with a basis, $E'\otimes_2 E$ will be identified with the space of *com-*

pletely continuous mappings of E into itself, whereas, in general, $E'\otimes_1 E$ will be a proper subspace of that space (for instance, when E is a Hilbert space, a completely continuous hermitian operator will belong to $E'\otimes_1 E$ only if the series of its eigenvalues is absolutely convergent [14; 55]). The dual of $E'\otimes_2 E$ is then identified with $E\otimes_1 E'$; hence, its bidual with $\mathcal{B}(E', E)$ [14].

This shows that in general the dual of $E\otimes_2 F$ will not be equal to $\mathcal{B}(E, F)$. It is much smaller, and can be determined directly from the duals E', F' in the following explicit way. The (algebraic) spaces $E\otimes F$ and $E'\otimes F'$ are in a natural duality, defined by $\langle x\otimes y, x'\otimes y'\rangle = \langle x, x'\rangle\langle y, y'\rangle$; the topology of $E\otimes_2 F$ is such that $x'\otimes y'$ can be extended by continuity to a continuous linear functional on $E\otimes_2 F$. Now, all continuous linear functionals on $E\otimes_2 F$ can be obtained from these "elementary" ones by a process of *integration*: more precisely, any such functional can be written

$$u \to \int \langle u, x' \otimes y'\rangle d\mu(x', y')$$

where μ is a positive Radon measure defined on a product $P\times Q$ of an equicontinuous (hence compact) subset P of E' and an equicontinuous subset Q of F'. This, applied to $E'\otimes_2 F$, leads to important classes of linear mappings of E into F, which have many useful properties.

12. **Nuclear spaces** [24]. Comparison of the two tensor products $E\otimes_1 F$ and $E\otimes_2 F$ has led A. Grothendieck to the investigation of a new class of locally convex spaces, namely those spaces E for which $E\otimes_1 F$ and $E\otimes_2 F$ are (topologically) *identical* for *every* locally convex space F. Such spaces are called *nuclear*; they possess many remarkable properties, which make them closer to finite-dimensional spaces than any other known category of infinite-dimensional spaces.

A space is nuclear if and only if its completion is nuclear; we shall therefore consider only complete nuclear spaces. In such a space, a bounded set is always relatively compact, hence the space is always semi-reflexive (§6), but it need not be a *t*-space; if it is a *t*-space, it is of course a Montel space; moreover, if it is an (F)-space, its (strong) dual is also a nuclear space. A closed subspace F of a nuclear space E is nuclear and so is the quotient space E/F. Any projective limit of nuclear spaces (§8) is nuclear; so is the inductive limit of a sequence of nuclear spaces.

If both E and F are (complete) nuclear spaces, so is their tensor product $E\otimes_1 F = E\otimes_2 F$: moreover, $E\otimes_1 F$ can be identified with the whole space $\mathfrak{B}(E', F')$ of separately continuous bilinear functionals on $E' \times F'$. As seen above, the dual of $E \otimes_2 F$ is then $\mathfrak{B}(E, F)$ (hence, *all* continuous bilinear functionals on $E \times F$ have the "integral" representation given at the end of §11); moreover, if E and F are (F)-spaces, $\mathfrak{B}(E, F)$ can be identified (topologically) with the tensor product $E' \otimes_1 F'$ of the (strong) duals. In the same case, the space $\mathcal{L}(E, F)$, with the topology of bounded convergence (§4) is nuclear, and its strong dual is also nuclear.

The (F)-spaces which are nuclear spaces can be characterized by the following property: they are the only (F)-spaces in which every *unconditionally convergent* series is also *absolutely convergent*: for Banach spaces, this gives of course, in particular, an entirely new proof of the Dvoretzky-Rogers theorem [16].

The importance of nuclear spaces lies chiefly in the fact that most spaces which occur in the theory of distributions, or in the theory of analytic functions, are nuclear spaces (for instance, the spaces $(\mathcal{D})$, $(\mathcal{D}')$, $(\mathcal{E})$, $(\mathcal{E}')$ of Schwartz [56; 57], as well as the space $\mathcal{H}(G)$ of holomorphic functions in a domain, with the topology of compact convergence (compact-open topology) [58; 8; 45; 46; 25]). The simple properties of nuclear spaces will undoubtedly prove very useful in later developments of these theories.

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UNIVERSITY OF MICHIGAN

THE SIXTH SYMPOSIUM IN APPLIED MATHEMATICS

The Sixth Symposium in Applied Mathematics of the American Mathematical Society was held at Santa Monica City College, Santa Monica, California, on August 26–28, 1953. The topic of the Symposium was *Numerical analysis*, and it was co-sponsored by the National Bureau of Standards. The following 185 persons, including 107 members of the Society, attended:

A. A. Albert, O. W. Albert, F. L. Alt, L. H. Amaya, F. R. Anderson, M. M. Andrew, L. A. Aroian, K. L. Austin, W. S. Bagby, T. Bauer, E. F. Beckenbach, M. M. Beenken, F. Bell, J. C. Bell, J. L. Bell, Stefan Bergman, E. J. Betinis, N. M. Blachman, J. Brandstatter, Paul Brock, L. F. Brown, R. H. Bruck, G. Bullock, K. A. Bush, A. S. Cahn, Jr., S. S. Cairns, J. W. Calkin, H. H. Campaigne, R. E. Carr, D. S. Carter, B. H. Colvin, G. B. Dantzig, C. E. Diesen, H. P. Doole, R. D. Dotts, Jim Douglas, Jr., R. C. Douthitt, A. Downing, Worthie Doyle, M. Dresler, C. E. Duncan, H. L. Durham, Jr., L. A. Elrod, M. Elveback, Arthur Erdélyi, Paul Erdös, G. W. Evans, II, J. H. Evans, A. B. Farnell, H. G. Ferris, G. E. Forsythe, F. G. Foster, S. P. Frankel, D. R. Fulkerson, R. S. Gardner, I. M. Garfunkel, Wallace Givens, A. M. Gleason, Max Goldstein, R. A. Good, J. W. Green, H. R. J. Grosch, W. D. Gutshall, D. W. Hall, Cecil Hastings, Jr., R. M. Hayes, M. R. Hestenes, A. J. Hoffman, D. S. Hoffman, J. Hogen, O. H. Hoke, A. S. Householder, M. Howard, L. Hulthen, J. A. Hummel, H. F. Hunter, Rufus Isaacs, Eugene Isaacson, J. S. Isenberg, R. E. Jackson, Fritz John, D. L. Johnson, S. M. Johnson, R. E. Kalaba, William Karush, E. Katz, M. Kennedy, R. E. Kidder, S. I. Klein, Erwin Kleinfeld, H. G. Kolsky, M. Krakowski, Harold W. Kuhn, S. H. Lachenbruch, R. Landshaff, L. R. Langden, J. E. LeBel, R. S. Lehman, D. H. Lehmer, Emma Lehmer, I. J. Lieberman, J. W. Lindsay, H. A. Linstone, B. J. Lockhart, A. T. Lonseth, J. F. Ludloff, J. L. McGregor, F. B. MacKenzie, Irwin Mann, J. Marschak, E. Maso, F. B. Meek, H. H. Meek, W. A. Mersman, H. I. Meyer, C. J. Mifsud, K. L. Miller, E. H. Mookini, A. Moskin, T. S. Motzkin, M. E. Muller, A. B. Neale, A. B. Novikoff, L. J. Paige, Gordon Pall, C. L. Perry, Jr., E. P. Perry, L. L. Philipson, M. Posin, A. G. Prisson, R. Radner, L. Regelson, Edgar Reich, R. R. Reynolds, D. E. Richmond, S. M. Rock, R. M. Roelun, J. H. Rogers, P. C. Rosenbloom, H. D. Ruderman, R. W. Rumsey, Arthur Sard, H. H. Schatz, R. H. Sehnert, J. L. Selfridge, Seymour Sherman, I. M. Singer, Judson C. Smith, S. S. Smith, T. H. Southard, O. S. Spears, K. Spierman, M. L. Stein, M. Steinberg, Robert Steinberg, A. C. Sugar, J. D. Swift, A. H. Taub, Olga Taussky, A. E. Taylor, D. Teichroew, C. J. Thorne, John Todd, C. B. Tompkins, C. H. Tross, J. R. Vine, S. Voris, H. M. Wagner, D. D. Wall, J. L. Walsh, M. Walter, Jack Warga, W. R. Wasow, E. B. Weinberger, R. L. Westhafer, H. W. Wielandt, J. E. Wilkins, Jr., N. Z. Wolfsohn, H. A. Wood, J. L. Wood, Arthur Wouk, J. W. Wrench, Jr., T. T. Wu, D. M. Young, Jr., J. W. T. Youngs.

There were five half-day sessions, at which five hour addresses and sixteen shorter papers were presented. Those presiding at the various sessions were A. M. Gleason, Seymour Sherman, G. B. Dantzig, Fritz John, M. R. Hestenes, and Arthur Erdélyi. One half day was reserved for visits to nearby computing facilities. Visits were arranged to the RAND Corporation, U.C.L.A. Engineering Department, Douglas Aircraft Company, Telecomputing Corporation, California Institute of Technology, U.S. Naval Air Missile Test Center, Lockheed Aircraft Corporation, Bendix Computer Division, and the Institute for Numerical Analysis of the National Bureau of Standards.

The following papers were presented, all of which are scheduled to appear in the Proceedings of the Symposium, edited by John Curtiss and published by the McGraw-Hill Book Company. In the absence of Messrs. Bellman, Warschawski, and Fischbach, their papers were presented by Messrs. R. S. Lehman, P. C. Rosenbloom, and D. M. Young, Jr., respectively.

C. B. Tompkins: *Application of automatic digital computers to problems with discrete variables.*

R. H. Bruck: *Computational aspects of certain combinatorial problems.*

Olga Taussky: *Some computational problems in algebraic number theory.*

Emma Lehmer: *Number theory on the SWAC.*

H. W. Wielandt: *Error bounds for eigenvalues of symmetric integral equations.*

Wallace Givens: *A method of error analysis.*

D. M. Young, Jr.: *On the solution of linear systems by iteration.*

M. R. Hestenes: *The conjugate gradient method for linear systems.*

Stefan Bergman: *Some methods for solutions of boundary value problems of linear partial differential equations.*

T. S. Motzkin: *The assignment problem.*

Richard Bellman: *On some computational problems in the theory of dynamic programming.*

Harold W. Kuhn: *The traveling salesman problem.*

S. E. Warschawski: *Recent results in numerical methods of conformal mapping.*

A. H. Taub: *Numerical solution of the equations of pseudo-stationary flow.*

J. W. Fischbach: *Some applications of gradient methods.*

S. P. Frankel: *Stability problems in partial differential equations.*

J. L. Walsh: *Best-approximation polynomials of given degree.*

Arthur Sard: *Function spaces and approximation.*

Cecil Hastings, Jr.: *Approximations in numerical analysis.*

W. R. Wasow: *On the asymptotic transformation of certain probability distributions.*

P. C. Rosenbloom: *The method of steepest descent.*

J. W. GREEN,
Associate Secretary

D. H. LEHMER
Chairman, Arrangements Committee

THE SUMMER MEETING IN KINGSTON

The fifty-eighth Summer Meeting and the thirty-fourth Colloquium of the American Mathematical Society were held at Queen's University and the Royal Military College, Kingston, Ontario, Canada, from August 31 to September 5, 1953 in conjunction with meetings of the Canadian Mathematical Congress, the Mathematical Association of America, the Institute of Mathematical Statistics, and the Economic Society.

Over 700 people registered for the meeting, including the following 377 members of the Society:

C. R. Adams, R. B. Adams, J. G. Adshead, M. I. Aissen, B. E. Allen, E. B. Allen, C. B. Allendoerfer, J. P. van Alstyne, S. A. Amitsur, A. G. Anderson, R. D. Anderson, R. G. Archibald, Richard Arens, E. L. Arnoff, H. E. Arnold, K. J. Arnold, Nachman Aronszajn, M. G. Arsove, M. C. Ayer, I. A. Barnett, C. F. Barr, J. H. Barrett, L. K. Barrett, R. C. F. Bartels, R. G. Bartle, P. T. Bateman, W. R. Baum, Samuel Beatty, E. G. Begle, Theodore Bennett, R. H. Bing, D. W. Blackett, David Blackwell, J. R. Blum, M. I. Blyth, Salomon Bochner, H. F. Bohnenblust W. M. Boothby, R. C. Bose, Evelyn Boyd, C. B. Boyer, A. T. Brauer, G. U. Brauer, F. R. Britton, F. E. Browder, F. H. Brownell, G. S. Bruton, F. J. H. Burkett, G. H. Butcher, James William Butler, P. L. Butzer, W. R. Callahan, E. A. Cameron, J. W. Campbell, L. L. Campbell, K. H. Carlson, A. B. Carson, Henri Cartan, C. R. Cassity, Lamberto Cesari, Abraham Charnes, Randolph Church, R. V. Churchill, Paul Civin, C. J. Clark, H. M. Clark, A. B. Clarke, H. E. Clarkson, D. E. Coffey, L. W. Cohen, Harvey Cohn, R. H. Cole, A. J. Coleman, J. B. Coleman, Esther Comegys, R. M. Conkling, T. F. Cope, A. H. Copeland, Sr., H. S. M. Coxeter, J. B. Crabtree, C. C. Craig, H. F. Cullen, A. B. Cunningham, H. B. Curry, A. E. Danese, J. M. Danskin, G. B. Dantzig, R. A. Dean, R. Y. Dean, J. C. E. Dekker, D. B. DeLury, Douglas Derry, W. F. Donoghue, Jr., C. H. Dowker, A. C. Downing, D. W. Dunn, W. L. Duren, Jr., W. F. Eberlein, H. W. Ellis, M. P. Emerson, H. P. Evans, R. L. Evans, Walter Feit, H. H. Ferns, J. V. Finch, D. T. Finkbeiner, Irwin Fischer, E. E. Floyd, G. E. Forsythe, J. S. Frame, Evelyn Frank, H. D. Friedman, A. H. Frink, Orrin Frink, R. M. Frisch, R. E. Fullerton, I. N. Gál, I. S. Gál, H. L. Garabedian, A. O. Garder, Boris Garfinkel, H. M. Gehman, Irving Gerst, Murray Gerstenhaber, K. S. Ghent, Leonard Gillman, Seymour Ginsburg, Sidney Glusman, Herbert Goertzel, Casper Goffman, Michael Goldberg, Harry Gonshor, S. H. Gould, R. L. Graves, J. W. Green, Harold Greenspan, J. A. Greenwood, F. L. Griffin, V. G. Grove, Simon Gruenzweig, Irwin Gutman, P. R. Halmos, M. E. Hamstrom, J. F. Hannan, B. I. Hart, G. E. Hay, Henry Helson, Melvin Henriksen, J. G. Herriot, I. N. Herstein, M. J. Herzberger, Edwin Hewitt, T. H. Hildebrandt, J. G. Hocking, R. V. Hogg, F. E. Hohn, D. L. Holl, M. P. Hollcroft, T. R. Hollcroft, W. A. Hurwitz, W. R. Hutcherson, Jack Indritz, S. L. Isaacson, J. R. Isbell, Arno Jaeger, T. J. Jaramillo, R. L. Jeffery, W. E. Jenner, Meyer Jerison, A. E. Johns, L. W. Johnson, R. E. Johnson, F. E. Johnston, B. W. Jones, P. S. Jones, G. K. Kalisch, L. H. Kanter, Leo Katz, D. E. Kearney, M. E. Kellar, J. B. Kelly, J. R. F. Kent, D. E. Kibbey, T. C. Koopmans, C. F. Kossack, H. L. Krall, Max Kramer, Saul Kravetz, Solomon Kullback, O. E. Lancaster, C. E. Langenhop, E. H. Larguier, J. W. Lawson, C. Y. Lee, H. L. Lee, A. B. Lehman,

F. C. Leone, Benjamin Lepson, C. A. Lester, W. J. LeVeque, D. J. Lewis, F. A. Lewis, M. M. Lipschutz, Mark Lotkin, R. C. Lyndon, H. T. McAdams, N. H. McCoy, S. W. McCuskey, W. H. McEwen, J. E. McLaughlin, E. J. McShane, R. W. MacDowell, G. W. Mackey, H. M. MacNeille, M. S. Macphail, W. G. Madow, G. R. Magee, Lawrence Markus, Imanuel Marx, K. O. May, Paul Meier, B. E. Meserve, E. A. Michael, Carman E. Miller, Norman Miller, H. D. Mills, J. M. Mitchell, Don Mittleman, E. B. Mode, E. E. Moise, H. F. Montague, M. D. Montgomery, J. C. Moore, J. T. Moore, T. W. Moore, K. A. Morgan, D. C. Morrow, T. A. Mossman, S. T. C. Moy, C. W. Munshower, W. L. Murdock, Zeev Nehari, August Newlander, Jr., D. J. Newman, C. V. Newsom, A. V. Newton, Jerzy Neyman, C. O. Oakley, R. E. O'Connor, Katsumi Okashimo, E. G. Olds, E. M. Olson, J. C. Oxtoby, F. D. Parker, Emanuel Parzen, S. E. Pence, F. W. Perkins, G. M. Petersen, H. P. Pettit, C. G. Phipps, George Piranian, Everett Pitcher, J. C. Polley, Pasquale Porcelli, D. H. Potts, I. R. Pounder, J. H. Powell, G. B. Price, L. E. Pursell, Tibor Radó, J. F. Randolph, R. B. Rasmusen, L. T. Ratner, G. E. Raynor, H. W. Reddick, R. M. Redheffer, P. K. Rees, R. F. Reeves, Eric Reissner, H. G. Rice, J. D. Riley, E. K. Ritter, Herbert Robbins, M. S. Robertson, G. de B. Robinson, Louis Robinson, Jack C. Rogers, P. G. Rooney, Arthur Rosenthal, Edward Rosenthall, M. F. Rosskopf, S. G. Roth, E. H. Rothe, Herman Rubin, Phyllis Rubin, M. F. Ruchte, J. P. Russell, Arthur Saastad, Charles Saltzer, H. E. Salzer, Hans Samelson, D. E. Sanderson, Leo Sario, H. M. Schaerf, R. D. Schafer, J. A. Schatz, F. J. Scheid, Peter Scherk, E. C. Schlesinger, E. R. Schneckenburger, Lowell Schoenfeld, B. L. Schwartz, W. R. Scott, D. H. Shaftman, Seymour Sherman, A. L. Shields, Edward Silverman, Annette Sinclair, Aubrey H. Smith, F. C. Smith, K. T. Smith, W. K. Smith, D. O Snow, W. S. Snyder, D. E. Spencer, G. L. Spencer, V. E. Spencer, E. R. Stabler, M. E. Stark, F. H. Steen, C. F. Stephens, B. M. Stewart, A. C. Sugar, D. B. Sumner, Irving Sussman, R. L. Swain, William Clare Taylor, G. H. M. Thomas, G. L. Thompson, D. L. Thomsen, Jr., R. M. Thrall, H. S. Thurston, H. E. Tinnappel, M. M. Torrey, E. A. Trabant, A. W. Tucker, J. W. Tukey, H. L. Turrittin, J. L. Ullman, H. E. Vansant, J. E. Vollmer, M. E. G. Waddell, T. L. Wade, R. W. Wagner, E. B. Walters, S. L. Ward, J. F. Wardwell, W. G. Warnock, J. V. Wehausen, C. P. Wells, J. G. Wendel, F. J. Weyl, G. W. Whitehead, K. B. Whitehead, A. L. Whiteman, G. T. Whyburn, L. S. Whyburn, W. L. G. Williams, L. M. Winer, Clement Winston, F. B. Wright, F. M. Wright, Hidehiko Yamabe, L. C. Young, Arthur Zeichner, A. D. Ziebur, Antoni Zygmund.

The Colloquium Lectures, *On the existence and properties of certain singular integrals*, were delivered by Professor Antoni Zygmund of the University of Chicago. The meeting opened with the first lecture at which Professor G. T. Whyburn, President of the Society, presided. Professors Tibor Radó, R. L. Jeffery, and Arthur Rosenthal presided at the remaining lectures.

Professor Henri Cartan of the University of Paris presented two lectures on *Recent results in the theory of analytical functions of several variables* under the joint auspices of the Canadian Mathematical Congress and the Society, Professors G. T. Whyburn and J. S. Frame presiding.

By invitation of the Committee to Select Hour Speakers for Summer and Annual Meetings, Professor Salamon Bochner of Princeton

University addressed the Society on *Fourier transforms in the theory of probability*. Professor T. H. Hildebrandt presided.

There were fourteen sessions for contributed papers presided over by Professors Paul Civin, H. B. Curry, Dr. G. E. Forsythe, Professors Murray Gerstenhaber, T. R. Hollcroft, W. A. Hurwitz, F. W. Perkins, George Piranian, Everett Pitcher, G. B. Price, Eric Reissner, Dr. Seymour Sherman, Professors A. L. Whiteman, L. C. Young.

The Council met on Tuesday evening, September 1.

The Secretary announced the election of the following thirty-five persons to ordinary membership in the Society:

Dr. Julian Adem, University of Mexico;
Miss Mary Ruth Arnette, Oak Ridge National Laboratory;
Mr. Robert Waller Bagley, University of Florida;
Dr. George L. Baldwin, Bell Telephone Laboratories, Murray Hill, New Jersey;
Mr. Walter Brunner, Redstone Arsenal, Huntsville, Alabama;
Miss Virginia Clyde Carlock, Oak Ridge National Laboratory;
Professor George Irvin Cohn, Illinois Institute of Technology;
Professor Wilfred James Combellack, Colby College;
Mr. Robert Lloyd Davis, University of Michigan;
Miss Irene L. Doto, Temple University;
Mr. Norbert A. Eisen, Mid Continent Petroleum Corporation, Tulsa, Oklahoma;
Mr. Frank August Engel, Jr., Westinghouse Atomic Power Division, Pittsburgh, Pennsylvania;
Professor Ahmed Cemal Eringen, Illinois Institute of Technology;
Dr. A. E. Foster, Newark College of Engineering;
Mr. James Phillip Frink, Brown University;
Mr. Dean Gillette, University of California, Berkeley;
Mr. Christopher Gregory, University of Hawaii;
Mr. Wayne Hewitt Jones, Department of Defense, Washington 25, D. C.;
Professor Hyman Kaufman, McGill University;
Dr. Kazar Kazarian, Franklin Institute Laboratories, Philadelphia, Pennsylvania;
Professor John William Keesee, University of Arkansas;
Mr. Robert Roy Korfhage, United Aircraft Corporation, East Hartford, Connecticut;
Mr. Karl Kozarsky, U. S. Naval Proving Ground, Dahlgren, Virginia;
Mr. Francis Patrick Larkin, Fordham College;
Mrs. Norma Lee (Ashby) Lindemann, University of Notre Dame;
Dr. John Edward Maxfield, U. S. Naval Ordnance Test Station, China Lake, California;
Mr. Kenneth Roy Mount, University of Illinois;
Mr. Richard Sheldon Palais, Harvard University;
Mr. Carl Henry Pollmar, Engineering Research Institute, University of Michigan;
Dr. Judah Ben Rosen, Forrestal Research Center, Princeton University;
Dr. Thomas Lorie Saaty, Yale University;
Mr. Gerald Sheldon Silberman, University of California, Berkeley;
Mr. Frederick C. Warner, University of Buffalo;
Mr. George Kilbourne White, United Aircraft Corporation, East Hartford, Connecticut;
Professor Kuo-Tai Yen, Rensselaer Polytechnic Institute.

It was reported that the following persons had been elected as nominees of the institutions indicated:

Brooklyn College: Mr. Kenneth Norman Geller, Mr. Allan Joseph Goldman, and Mr. Joel Lebowitz.

Harvard University: Mr. Harry Gonshor.

Stanford University: Mr. Paul G. Loewner, Mr. Edward Blake McLeod, Jr., Mr. Grove Crawford Nooney, Mr. Louis Anthony Schmittroth, Mr. Donald Platte Squier, and Mr. Martin Judy Vitousek.

University of Toronto: Rev. John Christian Egsgard and Mrs. Sudha Jain.

University of Utah: Mr. Harvey Junior Fletcher.

The Secretary announced that the following persons had been admitted to the Society in accordance with reciprocity agreements with the following organizations: Société Mathématique de France: Mr. Aimé Hennequin, Faculté des Sciences, Paris; Indian Mathematical Society: Professor H. G. S. Sharma, Siddaganga Sanscrit College, Dr. Tallamraju Venkatarayudu, Polytechnic Institute of Brooklyn; Svenska Matematikersamfundet: Dr. Gustav Olof Thorin, Valand, Ömsesidigt Livförakringsbolag, Stockholm, Sweden; Wiskundig Genootschap: Professor Dwinger, University of Indonesia.

The following appointments of representatives of the Society were reported: at the inauguration of John Thomas Wahlquist as President of San Jose State College on May 1, 1953: Professor H. M. Bacon; at the inauguration of Jay Frederick Wesley Pearson as President of the University of Miami on May 7, 1953: Mr. H. Raymond Wright; at the 50th anniversary of the Chicago Law School on May 8, 1953: Professor E. J. Moulton; at the dedication of two buildings on the campus of Mary Washington College of the University of Virginia on May 8–9, 1953: Professor E. E. Floyd; at the inauguration of Oliver S. Willham as President of Oklahoma Agricultural and Mechanical College on May 8–9, 1953: Professor O. H. Hamilton; at the inauguration of G. Brooks Earnest as President of Fenn College on May 9, 1953: Professor J. R. Mussleman; at the inauguration of Malcolm A. Love as President of San Diego State College on May 10, 1953: Mr. L. M. Klauber.

The following additional appointments by the President were reported: A. H. Taub appointed as Chairman of the Editorial Committee for Applied Mathematics Proceedings for the period July 1, 1953–June 30, 1954; A. E. Heins appointed a member of the Editorial Committee for Applied Mathematics Symposium Proceedings for a period of three years beginning July 1, 1953 (Committee now consists of A. H. Taub (Chairman), R. V. Churchill, and A. E. Heins); Hassler Whitney as Chairman of the Organizing Committee for Summer Institutes for the period July 1, 1953–June 30, 1954; S. S. Chern and F. B. Jones as members of the Organizing Committee for Summer Institutes for a period of three years beginning July 1, 1953 (Committee now consists of Hassler Whitney (Chairman), A. A. Albert, S. Bochner, S. S. Chern, F. B. Jones, and H. P. Robertson); R. P. Boas reappointed Chairman of the Committee on Translations from Russian and other Foreign Languages for the period July 1, 1953–June 30, 1954; J. L. Doob re-appointed as a member of the Committee on Translations from Russian and other Foreign Languages for a period of three years beginning July 1, 1953 (Committee now consists of R. P. Boas (Chairman), R. E. Bellman, J. L. Doob, Irving Kaplansky, and Hans Samelson); M. H. Martin as the Society's representative on the Program Committee in charge of arranging the two conferences co-sponsored by the Committee on Training and Research in Applied Mathematics of the National Research Council and the American Mathematical Society; B. P. Gill and C. H. W. Sedgewick as auditors for 1953; C. S. Pettis to replace T. F. Jordan as Chairman of the Arrangements Committee for the meeting to be held at Wofford College, Spartanburg, South Carolina, on November 27–28, 1953 (Committee now consists of C. S. Pettis (Chairman), John Hill, G. May, E. H. Shuler, and W. M. Whyburn); as an arrangements committee for the Annual Meeting to be held at The Johns Hopkins University on December 28–31, 1953: D. C. Lewis (Chairman), W. L. Chow, A. H. Clifford, L. W. Cohen, H. M. Gehman, E. K. Haviland, W. K. Morrill, and Marian M. Torrey; as a committee to advise with the Executive Director on an experimental project to produce the Mathematical Reviews by the photo offset method: William Feller (Chairman), M. R. Hestenes, G. B. Price; as a committee to consider participation on a National Roster of Mathematicians: E. G. Begle (Chairman), D. H. Lehmer, and S. S. Wilks; as a committee to survey the Society's program of periodical publications: G. A. Hedlund (Chairman), S. B. Myers, and J. C. Oxtoby.

The following dates of meetings have been approved by the Council: November 25–26, 1955 at the Milwaukee Extension of the University of Wisconsin. The Council voted to accept an invitation to hold the 1955 Summer Meeting at the University of Michigan, Ann Arbor, Michigan and the 1955 Annual Meeting at the Rice Institute, Houston, Texas.

Invitations to give addresses in 1953 were announced: Paul Erdős at the October meeting in New York City; B. J. Pettis and A. S. Householder at the November meeting in Spartanburg, South

Carolina; P. C. Rosenbloom at the November meeting in Evanston; M. M. Schiffer at the November meeting in Pasadena, and D. V. Widder and A. D. Wallace at the Annual Meeting in Baltimore.

The Secretary reported that the Trustees had changed the fiscal year, which will now run from June 1 to May 31.

The Secretary reported that Professor E. J. McShane and R. L. Wilder had been appointed as representatives of the Policy Committee on the United States National Committee for Mathematics of the National Research Council for a period of four years beginning July 1, 1953 and that Professor J. B. Rosser had been appointed by the Policy Committee to represent mathematics on the Committee to evaluate the present functions and operations of the National Bureau of Standards in relation to the present national needs.

The Council voted to approve a contract with the Office of Scientific Research of the Air Research and Development Command under which the Society will administer a panel of reviewers. Members of this panel will be appointed on nomination by the Office of Scientific Research and approval by the Society. Lists of suggested names for this panel will be furnished by the Editorial Committee for the Mathematical Reviews on request by the Office of Scientific Research. Research proposals submitted to the Office of Scientific Research will be reviewed by individual members of the panel selected by the Office of Scientific Research. Reviewers' reports will be sent directly to the Office of Scientific Research. Payments to the reviewers will be made by the Society.

The Executive Director reported increasing difficulties in distribution of our periodicals because of changes in postal regulations and also reported difficulties resulting from late arrival at Headquarters of change of address notices.

The Council adopted the following resolution: "The Society associates itself with Columbia University in supporting the theme of the University's Bicentennial Celebration—Man's Right to Knowledge and the Free Use Thereof, and joins with the University in inviting John von Neumann to address a special session of its April, 1954, meeting in New York in honor of the occasion."

The Council elected Professor H. J. Ettlinger and C. C. MacDuffee as representatives of the Society on the National Research Council for the period July 1, 1954–June 30, 1957 and elected Professor G. A. Hedlund and E. J. McShane as representatives of the Society on the Council of AAAS for 1954–55.

The Council voted to invite Professor Enrico Fermi to deliver the Gibbs Lecture in 1954.

A Business Meeting was held on Wednesday September 6 at 11:00 A.M., President Whyburn presiding.

The Secretary reported on the affairs of the Society.

Two amendments to the by-laws of the Society were presented with the recommendation of the Council. These amendments would:

1. empower the Council, under certain conditions, to speak in the name of the Society with respect to matters affecting the status, dignity, and effective position of mathematics or mathematicians.
2. require a referendum on such Council actions when requested by 200 or more members of the Society.
3. require a business meeting at the Summer as well as the Annual Meeting.
4. allow final action to be taken at a business meeting only when business is accepted by unanimous consent or notified to the full membership of the Society in the call for the meeting.

It was moved by Professor Radó, and seconded, "To defer final action on these amendments until the text of the amendments, together with a summary of the arguments which had been advanced for and against them, had been sent to the entire membership of the Society, and to invite the membership to express their opinions." It was moved by Professor R. L. Swain, and seconded to amend Professor Radó's motion by adding the phrase "the opinions received to be presented to the Council for their consideration and possible revision of the proposed amendments." Professor Swain's amendment was approved by the members present and the amended motion was carried.

The Committee on Arrangements provided, for the entertainment of those attending the meeting, an enjoyable program including a reception on Monday evening, a tour of the city and Old Fort Henry on Tuesday afternoon, an excursion among the Thousand Islands on Wednesday afternoon, and a theatre party on Thursday evening at which members were guests of the Canadian Mathematical Congress.

The audience at the theatre were constituted a Committee of the Whole, representing the several societies attending the meeting, to receive a resolution thanking Queen's University, the Royal Military College, the Canadian Mathematical Congress and all responsible individuals for their generous and effective contributions to the success of the meeting. The resolution was presented by Professor J. W. Tukey and passed by a unanimous rising vote.

The abstracts of papers presented at the meeting follow. Those with "*t*" after the abstract number were presented by title. Where a paper, presented in person, has joint authorship, (p) follows the name of the author who presented it. Professor Noonan was introduced by Dean W. H. McEwen, Dr. Schwartz by Professor L. W. Cohen, Mr. Snyder by Professor Albert Wilansky, Professor Reza by Professor Norman Levinson, Dr. Gilmore by Professor E. W. Beth, Dr. Wang by Mr. Joel Pitcairn, and Dr. Feidner by Professor Edith R. Schneckenburger.

ALGEBRA AND THEORY OF NUMBERS

557*t*. A. A. Albert: *The structure of right alternative algebras.*

If $\mathfrak{A}$ is a right alternative algebra, the attached commutative algebra $\mathfrak{A}^{(+)}$ is a special Jordan algebra. The author considers the case where the base field $\mathfrak{F}$ is algebraically closed and of characteristic not two and defines a natural "primitive" trace function $\delta(x)$ for $\mathfrak{A}^{(+)}$ with the usual properties as regards the radical. He then shows that $\delta(xy)=\delta(yx)$ for every x and y of $\mathfrak{A}$. The usual trace properties then hold for $\mathfrak{A}$ and there results the theorem which states that a semisimple right alternative algegra over any field of characteristic not two is alternative. (Received June 3, 1953.)

558. S. A. Amitsur: *Noncommutative cyclic fields.*

A cyclic extension F of order n of a sfield C is a sfield $F\supset C$ which possesses a cyclic group of automorphisms of order n leaving C invariant *and* which is a right C-module of dimension n. With this definition, the theory of commutative cyclic fields can be extended to the noncommutative case. Let $C[t;\,D]$ $(C[t;\,\rho])$ denote the ring of all noncommutative polynomials in t with coefficients of C defined by the relation: $ct=tc+cD$ $(ct=tc+c^{\rho})$, $c\in C$, where $D\colon c\rightarrow cD$ $(\rho\colon c\rightarrow c^{\rho})$ is a derivation (an automorphism) of C. It is shown that all cyclic extensions of degree p of a sfield C of characteristic $p\neq 0$ are isomorphic to quotients $C[t;D]/(t^p-t-a)C[t;D]$ for some derivation D of C satisfying among others the condition that D^p-D is the inner derivation: $c\rightarrow ca-ac$ of C. The automorphism of the extension is the one induced by the mapping; $t\rightarrow t+1$. Inductive results are obtained for extensions of degree p^n. For the case of characteristic zero and sfields whose centre contain a primitive nth root of unit ζ_n, the author obtains the result that every cyclic extension of degree n with a centre containing ζ_n is isomorphic to the quotient $C[t;\,\rho]/(t^n-a)C[t;\,\rho]$ for some automorphism ρ of C satisfying among others the condition that ρ^n is the inner automorphism: $c\rightarrow aca^{-1}$ of C. These results are used to construct new examples of noncommutative fields containing a nontrivial group of outer automorphisms. (Received June 22, 1953.)

559*t*. H. W. Becker: *The Euler, Lebesgue, and Kraitchik transforms of rational cuboids.* Preliminary report.

See Dickson's *History* II, pp. 497–501, and M. Kraitchik, Scripta Math. vol. 11 (1945) p. 325. A Diophantine solution of $x^2+y^2=u^2$, $y^2+z^2=v^2$, $x^2+z^2=w^2$ depends on $x,\ y,\ z,\ u,\ v,\ w=eg(g^2-h^2),\ 2efgh,\ gh(e^2-f^2),\ ef(g^2+h^2),\ gh(e^2+f^2),\ \{(e^2g^2+f^2h^2)\cdot(e^2h^2+f^2g^2)-4e^2f^2g^2h^2\}^{1/2}$. Under the Euler transform E: $e, f, g, h\rightarrow e\pm f, g\pm h$; and $x, y, z, u, v, w\rightarrow yz, zx, xy, uz, vx, wy$. Under the Lebesgue transform L: $e, f, g, h\rightarrow e, f, xu, wy$; *and* $x,\ y,\ z,\ u,\ v,\ w\rightarrow x^4-y^2z^2=u^2x^2-v^2y^2=w^2x^2-u^2z^2,\ 2uxwy,\ 2uxvz,\ u^2x^2+w^2y^2,\ 2uxwv,\ w^2x^2+u^2z^2=X,\ Y,\ Z,\ U,\ V,\ W$. There are analogous transforms L' and L'', under permutations $(xz)(uv)$ and $(xy)(vw)$ on L. Under the Kraitchik transform $EL''=K''$: $e, f, g, h\rightarrow vx, uy, uz, vy$; and $x,\ y,\ z,\ u,\ v,\ w\rightarrow z(y^4-x^2z^2),\ 2uvxyz,\ x(y^4-x^2z^2),\ z(v^2x^2+u^2y^2),\ x(u^2z^2+v^2y^2),\ w(y^4-x^2z^2)=Y''z,\ X''z=Z''x,\ Y''x,\ U''z,$

$V''x$, $Y''w$. Similarly, under $EL=K$ and $EL'=K'$: $x, y, z, u, v, w \rightarrow Yz=Zy$, Xz, Xy, Uz, Xv, Wy, and $Z'y$, $Z'x$, $X'y=Y'x$, $Z'u$, $V'x$, $W'y$. After cancellations, $E^2=I$, $LE=L$, $KE=K$, etc., but $EK=L$ though $EL=K$, $LK=L^2$ though $KL=K^2$, etc. The classic Saunderson solution is $e, f, g, h=r^2+s^2$, $2(r^2-s^2)$, r^2+s^2, $4rs$. Another is due to Rignaux "avec l'aide de M. Crussol," L'Int. des Math. vol. 25 (1919) p. 127: $e, f, g, h =2(r^2-rs-s^2)$, r^2+s^2, $r(r+2s)$, $s(2r-s)$, wherein $r, s \rightarrow r \pm s$ makes $e, f, g, h \rightarrow e \pm f$, $g \pm h$, so operating on this solution $E \subseteq I$ the identity. Rignaux forms cover nos 4, 15, 16, 38, 46, 49 of Kraitchik's table of 50 maverick empirical solutions; the other 44 should suggest new formulations. Do there exist: (a) Absolute mavericks? (b) A finite, or infinite, number of 2 parameter solutions of $w=\omega(e, f, g, h)$? (Received July 20, 1953.)

560t. Volodymyr Bohun-Chudyniv: *On a general method for constructing orthogonal square matrices of every order with differing integers.*

In this author's last two papers: (1) *On orthogonal and non-orthogonal closed systems of K-nions and their application*, (2) *On a method of determining orthogonal square matrices with differenting integers* (also complex numbers, quaternions, octonions, sedecimions, etc.) he considered methods constructed by the author for determining orthogonal square matrices of 2^k, 2^k-1; $(2^{1+\lambda}-1)2^\beta$ ($k \geqq 2$; $\lambda \geqq 1$, $\beta \geqq 1$; $\lambda \geqq 0$, $\beta \geqq 2$) orders. (Bull. Amer. Math. Soc. Abstracts 58-6-621, 59-3-232.) In the present paper the author determines: (A) The general method of constructing orthogonal square matrices of *every order* composed of differing integers, complex numbers, etc., with the help of expressions (I) constructed by this author: (I) $2x_\alpha x_1$, $2x_\alpha x_2, \cdots, 2x_\alpha^2-\sum_{i=1}^{i=n} x_i^2$, $2x_\alpha x_{\alpha+1}, \cdots, 2x_\alpha x_n$ ($\alpha=1, 2, \cdots, n$; $n \geqq 2$). Expressions (I) can also be obtained from the author's orthogonal schemes of K-nions of $2^{k+1}=n+\gamma$ ($2^{k+1}>\gamma \geqq 0$) rank as a separate case. The matrices of this type can be made to take on a symmetrical form. (B) The method of constructing orthogonal square matrices with the help of expressions (I) and two lemmas pointed out in the mentioned paper (2) of this author. The largest number of differing elements in type (A) matrices can equal $n(n+1)/2$ and in type (B) matrices can equal $n(n+2)/2$ (n is the order of the matrix). (Received July 14, 1953.)

561t. A. T. Brauer: *A new class of Hadamard determinants.*

Recently O. Perron [Math. Zeit. vol. 56 (1952) pp. 122–130] raised the following question. Let m be a composite number of form $4n-1$. Do there exist matrices with $4n-1$ rows and $2n$ columns whose elements are the numbers of a complete system of residues (mod m) and which have the following two properties: I. Each row has exactly n elements in common with each other row. II. The corresponding elements of each pair of two rows have the same difference (mod m)? Perron proved that such a matrix exists for $m=15$. Using a theorem of the author on the distribution of the Jacobian symbols [Bull. Amer. Math. Soc. Abstract 59-1-37] it is shown in this paper that such matrices exist for $m=p(p+2)$ if p and $p+2$ are twin primes. It follows from this result that it is possible to construct Hadamard determinants of order $(p+1)^2$ if p and $p+2$ are twin primes. (Received July 14, 1953.)

562. A. T. Brauer: *On the location of the characteristic roots of a matrix.*

The main result of the paper is the following theorem. Let $A=(a_{rs})$ be a square matrix of order n. Let D_{rs} be the principal two-rowed minor of the characteristic determinant of A which contains elements of the rth and sth rows. If Q_{rs} denotes the sum $\sum_\nu \{|a_{r\nu}a_{\nu s}|+|a_{s\nu}a_{\nu r}|+|a_{r\nu}a_{s\nu}|\}+\sum_{\nu,\mu}|a_{r\nu}a_{s\mu}+s_{\mu\nu}a_{r\mu}|$ where ν and μ run from 1 to n with $\nu<\mu$, and where $\nu\neq r, s$ and $\mu\neq r, s$, then each characteristic root of A lies in the interior or on the boundary of at least one of the $n(n-1)/2$ ovals of Cassini $|D_{rs}|\leq Q_{rs}$ $(r, s=1, 2, \cdots, n; r\neq s)$. This theorem improves former results of the author [Duke Math. J. vol. 13 (1946) pp. 387–395; vol. 14 (1947) pp. 21–26; vol. 15 (1948) pp. 871–877; vol. 19 (1952) pp. 75–91; vol. 19 (1952) pp. 553–562]. (Received July 14, 1953.)

563*t*. G. C. Byers: *Class number relations for quadratic forms over* $GF[q, x]$.

Analogues of certain of Kronecker's classic class number relations are obtained for quadratic forms with coefficients from $GF[q, x]$. This is done by establishing a relation between definite binary quadratic forms and associated quaternary bilinear forms. Defining $h(A)$, $A\in GF[q, x]$, to be the number of classes of quadratic forms of discriminant A, we quote a typical result. Given $\nabla\in GF[q, x]$ of degree $2m+1$, then $\sum_R h(R^2-\nabla)=2\sum_J q^j$ $(j=\deg J)$, where the summation on the left is over all $R\in GF[q, x]$, including $R=0$, of degree not exceeding m, while the summation on the right is over those primary $J\in GF[q, x]$ of degree greater than m dividing ∇. Similar results are obtained for ∇ of even degree. (Received May 29, 1953.)

564*t*. Leonard Carlitz: *Pairs of quadratic equations in a finite field.*

If $Q_1(\xi_1, \cdots, \xi_r)$, $Q_2(\xi_1, \cdots, \xi_r)$ denote quadratic forms with coefficients in $GF(q)$, consider the number of solutions of the system $Q_1=\alpha$, $Q_2=\beta$. If the Q's satisfy certain conditions, explicit results are obtained. In some cases the number of solutions is exhibited in terms of Jacobsthal sums. (It may be noted that the case $q=2^n$ is included.) (Received May 13, 1953.)

565*t*. Leonard Carlitz: *Representations by quadratic forms in a finite field.*

Let A be a symmetric matrix with elements in $GF(q)$, q odd, of order m and nonsingular; let B be symmetric of order t and rank r. The principal result of the paper is the determination of $N_r(A, B)$, the number of $m\times t$ matrices X such that $X'AX=B$. (Received May 13, 1953.)

566*t*. Leonard Carlitz: *The number of solutions of some special equations in a finite field.*

Explicit formulas for the number of solutions of certain special equations $f(\xi)=\alpha$ are obtained. In particular such results are obtained in the case f equal to a Pfaffian or a factorable polynomial as well as various combinations of these and other polynomials. In most of the cases discussed the number of solutions satisfies $N_f(\alpha)=N_f(1)$ for all $\alpha\neq0$. (Received May 13, 1953.)

567*t*. F. Marion Clarke: *An extension of the n-step gradient method.*

Given n linear equations in n unknowns represented by the vector equation $Dx+c=0$, the matrix operator D is called G-Hermite if (i) the elements of D belong to a finite cyclic extension K of a base field F and (ii) there exists a unitary matrix R whose elements are relative automorphisms of K over F such that "transpose $RD=D$." It is shown that when D is nonsingular and $Rx\cdot x>0$, the Stiefel n-step gradient method (Zamp vol. 3 (1952) pp. 1–33) can be generalized to yield the solution in K of the given system of equations. The extension uses G-Hermite forms and the lemma: if the vectors $x_1, x_2, \cdots, x_n$ with coefficients in K have the property that $Rx_i \cdot x_j = 0$ for all $i \neq j$ and $Rx_i \cdot x_i \neq 0$, then $x_1, x_2, \cdots, x_n$ are linearly independent over F. It is shown that the method applied with a G-Hermite matrix includes the familiar Hermite matrix as a special case. (Received July 21, 1953.)

568*t*. C. W. Curtis: *On the structure of non-semisimple algebras.*

The following extension of the Wedderburn-Malcev theorem for associative algebras is proved. Let A be an algebra over a field K with Jacobson radical N, such that A/N is finite-dimensional and separable over K. If $\bigcap_{i=1}^{\infty} N^i = (0)$, and if A is complete in the natural topology determined by the powers of N, then there exists a subalgebra B of A with the following properties: (i) $A = B + N$ (vector space direct sum); and (ii) if B' is a separable subalgebra of A, then there exists an element $z \in N$ such that if z' is the quasi-inverse of z, then the mapping $b' \rightarrow b' - zb' - b'z' + zb'z'$ is an isomorphism of B' into B. Consider now a complete semi-local ring R contained in the center of a ring S. A sufficient condition is derived in order that S be complete in the topology determined by the powers of its radical. These results yield a theorem on the structure of a ring S which is a finitely generated module over a complete local ring contained in its center. (Received June 15, 1953.)

569*t*. R. P. Dilworth: *Proof of a conjecture on finite modular lattices.*

It is proved that in a finite modular lattice the number of meet irreducibles is equal to the number of join irreducibles. (Received July 20, 1953.)

570*t*. D. O. Ellis: *Notes on the foundations of lattice theory.* II.

Let L be a lattice. A mapping $f: L \otimes L \rightarrow S$, where S is an arbitrary set, is called a c-function on L provided $f(a, b) = f(a \wedge b, a \vee b)$. The notion of c-function is a unifying concept in lattice theory. Relations of c-functions to valuations, distance spaces, lattice-ordered algebras, and algebraic identities are discussed. Examples of results are: 1. *A lattice L is distributive if and only if there is a c-function $f(a, b)$ on L which is biuniform in b for fixed a.* 2. *A lattice L is modular if and only if there is a c-function $f(a, b)$ on L so that $f(a, b) = f(a, c)$ implies $b = c$ or $b \| c$ ("b and c are incomparable").* The methods are lattice-theoretic and quite elementary. (Received July 3, 1953.)

571. Irwin Fischer: *Singular points on the moduli-variety for hyperelliptic curves.*

Let k be any algebraically closed ground field. It is shown that one can construct, in a natural manner, an algebraic variety M which, except for an algebraic subvariety, is in one-one correspondence with the birational classes of hyperelliptic curves of genus g, defined over k. The question of singular points on M is then investigated. If g is greater than 2, a point P is a singular point of M/k if and only if the function field of the birational class of hyperelliptic curves which P represents possesses at least one nonidentical k-automorphism. If $g = 2$, a singular point on M is exhibited. (Received July 17, 1953.)

572. J. S. Frame(p), G. de B. Robinson, and R. M. Thrall: *The hook graphs of the symmetric group.*

To each irreducible representation $[\lambda]$ of the symmetric group S_n corresponds a Young diagram having λ_i nodes in the ith row and λ_j' nodes in the jth column. The

ij-node is the corner of a right hook of length $h_{ij}=1+(\lambda_i-j)+\lambda'_j-i$. If $h_{ij}=kq$ and $\lambda_i-i+1\equiv j-\lambda'_j\equiv r(\text{mod } q)$, the ij-node is called a q-node of residue r or a (q, r) node. The hook graph of $[\lambda]$ is an array $H[\lambda]$ obtained by replacing each node of $[\lambda]$ by its hook number h_{ij}. The hook product H_λ is the product of all the h_{ij}. The degree f_λ of $[\lambda]$ is found to be $n!/H_\lambda$. For any positive integer q, a q-quotient graph $[\lambda]_q$ can be constructed very simply from $[\lambda]$ by deleting all nodes except the q-nodes. Those of residue r form one of the disjoint constituents of $[\lambda]_q$. The hook numbers of $[\lambda]_q$ are h_{ij}/q. The q-quotient graph is closely related to the star diagram used by Robinson, Staal, Nakayama and Osima, and Nagao. Simple proofs of the theorems of Staal [*Star diagrams and the symmetric group*, Canadian Journal of Mathematics vol. 2 (1950) pp. 79–92] and other important properties of $[\lambda]$ follow immediately from the use of the hook graphs. (Received July 20, 1953.)

573*t*. B. R. Gelbaum: *Symmetric games.*

(This work was supported by the Office of Naval Research.) The characteristic function of any symmetric zero-sum n-person game can be defined by giving its values v_q on sets with q elements. This paper discusses properties of solutions based on the "most efficient coalitions" in such games. These coalitions are defined by setting $a=\max\{v_q/q \text{ for } q>0\}$ and $p=\min\{q \text{ with } v_q=qa\}$. The coalitions with p members are said to be most efficient. If one sets $n=pm+r$ with $0\leqq r<p$, it is shown that: (1) $r>0$. (2) $v_q=qa$ if and only if p divides $q\neq 0$. (3) $v_k=-k$ implies $k\leqq r$. (4) $1/(n-1)\leqq a\leqq r/pm$. The main result is to give solutions for those games in which $v_r=-r$ and $v_{r+1}\geqq a[r-(n-p+1)/(p-1)]$. For these games, the permutations of the imputations $\alpha^0=(\alpha^0_1, \cdots, \alpha^0_n)$ and $\alpha^1=(\alpha^1_1, \cdots, \alpha^1_n)$ with $\alpha^0_i=a$ for $1\leqq i\leqq pm$, $\alpha^0_i=-1$ for $pm+1\leqq i\leqq n$, $\alpha^1_i=a$ for $1\leqq i\leqq n-p+1$, $\alpha^1_i=-((n-p+1)/(p-1))a$ for $n-p+2\leqq i\leqq n$ form a solution. If $2(p-r)-2\leqq p\leqq 2(p-r)-1$ then the condition on v_{r+1} can be realized; however there exist cases for which it cannot be realized. This method provides symmetric zero-sum n-person games with finite symmetric solutions for a large class of integers n (in particular, for all odd integers). (Received July 20, 1953.)

574*t*. T. R. Hollcroft: *On sums of powers of n consecutive integers.*

Let $S_p=1^p+2^p+3^p+\cdots+n^p$, in which n and p are any finite positive integers. The well known relation, $S_3=S_1^2$, led the author to search for a more general relation (Theorem I below) of which this would be a special case. After obtaining general formulas for S_p, it is shown that for p odd, S_p can be expressed as an algebraic polynomial in $N=n(n+1)$, and for p even, S_p can be expressed as the product of $(2n+1)$ and an algebraic polynomial in N. Since $N=2S_1$ and $N(2n+1)=6S_2$, the following theorems result: Theorem I. For p odd, S_p can be expressed as an algebraic polynomial in S_1. Theorem II. For p even, S_p can be expressed as the product of S_2 and an algebraic polynomial in S_1. Examples: $S_{11}=S_1^2(16S_1^4-32S_1^3+34S_1^2-20S_1+5)/3$; $S_{10}=S_2(48S_1^4-80S_1^3+68S_1^2-30S_1+5)/11$. Similar theorems are found for sums of powers of consecutive integers beginning with any positive integer and for sums of powers of consecutive even or odd positive integers. (Received August 31, 1953.)

575. R. E. Johnson: *Semi-prime rings.*

A ring R is semi-prime if the intersection of the prime ideals of R is zero. Prime right ideals may be defined in a semi-prime ring R much as they are in a prime ring (*Prime rings*, Duke Math. J. vol. 18 (1951) pp. 799–809). The algebra $\mathfrak{P}$ of all prime right ideals of R forms a *structure* of R in that: (1) $0, R\in\mathfrak{P}$; (2) $(I\cap I')^*=I^*\cap I'^*$; (3) $(I:a)^*=(I^*:a)$; where $a\in R$, I, I' are right ideals of R, and I^* is the least element of $\mathfrak{P}$ containing I. In general, any subalgebra of $\mathfrak{P}$ satisfying (1)–(3) is called a structure of R. If the structure $\mathfrak{R}$ of R has minimal nonzero elements (atoms), $\mathfrak{R}$ is called an a-structure. Associated with each semi-prime ring R is a universal extension ring $N(R)$ containing R as an ideal and having the property that the annihilator of R in $N(R)$ is zero. The main theorem of the paper is that R is a semi-prime ring with a-structure if and only if $B \oplus B' \subseteq R \subseteq N(B) \oplus N(B')$, where B is a direct sum of prime rings each having an a-structure and B' is a semi-prime ring having a nonatomic structure. (Received June 3, 1953.)

576. G. K. Kalisch: *m-quota games and m-quota solutions.*

(This work was supported by the Office of Naval Research.) An m-quota n-person game $Q_{n;m}$ (for $Q_{n;2}$ see L. Shapley, *Contributions to the theory of games*, II) has the property that there exist numbers ω_i $(i=1, \cdots, n)$ such that (1) $\sum_{k=1}^{m} \omega_{i_k} = v((i_1, \cdots, i_m))$ (see v. Neumann and Morgenstern, *Theory of games* $\cdots$, for notations), (2) $\Omega = \sum_{i=1}^{n} \omega_i = 0$, (3) $\omega_i \geqq -1$. Let there be defined integers b_i^j $(i=1, \cdots, n; j=2, \cdots, m)$ such that the sets $(i, b_i^2, \cdots, b_i^m)$ each contain m distinct elements. Define the sets of imputations $\lambda_i = [\{\omega_1, \cdots, \omega_i - \sum_2^m x_k, \cdots, \omega_{b_i^2} + x_2, \cdots, \omega_{b_i^m} + x_m, \cdots, \omega_n\}]$. Then for suitable b_i^j, and v, $\Lambda = \bigcup_{i=1}^{n} \lambda_i$ is a solution of $Q_{n;m}$. It is possible to weaken (1)–(3) in various ways and still have a solution of the type above. (Received July 20, 1953.)

577*t*. J. Lambek and Leo Moser: *Elementary methods in the theory of distribution of primes.*

Let $h(a) = \sum_{r \leq a} 1/r$ and $l_k(a) = h(ka) - h(k)$. Using finite manipulations with rationals only the following are proved: 1. For $k \geqq n \geqq 2$, $.6 < \pi(n)/l_k(n) < 2$. This improves results of T. S. Broderick [J. London Math. Soc. vol. 14 (1939) pp. 303–310] and F. Behrend [Proc. Roy. Soc. New South Wales vol. 25 (1942) pp. 169–174]. 2. $\left| \sum_{p \leq a} l_k(p) + \sum_{p\, q \leq a} l_k(p) l_k(q) - 2a l_k(a) \right| < 61a + 18a^2/k$. This is an analogue of the fundamental lemma of A. Selberg [Ann. of Math. vol. 50 (1949) pp. 305–313]. 3. For each $n > 1$, there is a number of the form p or $p \cdot q$ $(p \neq q)$ between n and $n(1+100/h(n))$. (Received July 16, 1953.)

578*t*. J. Lambek and Leo Moser: *Inverse and complementary functions.*

Two arithmetic functions f and g are defined to be inverse if for every pair of positive integers m, n, either $f(m) < n$ or $g(n) < m$ but not both. It is shown that two nondecreasing functions f and g are inverse if and only if the sequences $n + f(n)$ and $n + g(n)$ are complementary. Thus it turns out that the sequences $[n + \log n]$ and $[n + e^n]$ are complementary. Also, for irrational θ, the sequences $[n(1+\theta)]$ and $[n(1+1/\theta)]$ are complementary. This is a result of S. Beatty. Methods for finding the nth term of the complement of a given sequence are developed. Thus the nth positive integer which is not a kth power is $n + [(n + [n^{1/k}])^{1/k}]$. The nth nonprime is the limit of the sequence: n, $n + \pi(n)$, $n + \pi(n + \pi(n))$ $\cdots$. A method due to V. Brün [Norsk. Mat. Tidsskr. vol. 13 (1931) pp. 73–79] for computing the nth prime is generalized. (Received July 16, 1953.)

579*t*. J. Lambek and Leo Moser: *On the distribution of pythagorean triangles.*

Estimates are obtained for the number of relatively-prime lattice points in certain regions which may extend to infinity. These are used to extend some results of D. N. Lehmer [Amer. J. Math. vol. 22 (1900) pp. 293–335] and D. H. Lehmer [Bull. Amer. Math. Soc. vol. 54 (1948) pp. 1185–1190] on the number of primitive pythagorean triangles satisfying various conditions. Thus, for example, it is shown that the number of primitive pythagorean triangles with area not exceeding n is $cn^{1/2}+O(n^{1/3})$ where $c=\Gamma(1/4)^2 2^{-1/2}\pi^{-5/2}=.531340\cdots$. This agrees well with recently completed tables of F. L. Miksa. These tables list all primitive pythagorean triangles with area less than 10^9. (Received July 16, 1953.)

580. W. J. LeVeque: *The distribution of values of multiplicative functions.*

Let $r_2(m)$ be the number of representations of m as a sum of two squares, and let $\tau(m)$ be the number of divisors of m. Let $R_k(x)$ and $T_k(x)$ be the number of $m\leq x$ such that $r_2(m)=k$, and $\tau(m)=k$, respectively. Complicated asymptotic estimates are obtained for $R_k(x)$ and $T_k(x)$; that for $R_k(x)$ disproves a conjecture of P. Levy. (Received July 15, 1953.)

581. D. J. Lewis: *Polynomial functions over the residue rings* $\mathfrak{O}/\mathfrak{p}^m$.

Let $\mathfrak{O}$ be any ring of $\mathfrak{p}$-adic integers, $\mathfrak{p}$ the prime ideal of $\mathfrak{O}$, and π a prime in $\mathfrak{O}$, hence $\mathfrak{p}=\pi\mathfrak{O}$. Suppose $\mathfrak{O}/\mathfrak{p}\cong GF(q)$. Let $\mathcal{B}_m=\{f$ in $\mathfrak{O}[x]$ such that $f:\mathfrak{O}\rightarrow\mathfrak{p}^m\}$. It is well known that $\mathcal{B}_1=(x^q-x,\pi)$ and this has been most useful in considering functions over finite fields. It is shown that there exist polynomials λ_m in $\mathfrak{O}[x]$ such that $\mathcal{B}_m=(\lambda_m,\pi\mathcal{B}_{m-1})$. A similar result is obtained for the case of several variables. These facts are used in considering functions over the residue ring $\mathfrak{O}/\mathfrak{p}^m$ and some results analogous to those for finite fields are obtained. (Received July 9, 1953.)

582*t*. L. J. Mordell: *On intervals containing an affinely equivalent set of n integers* mod *k*.

L. Redei, improving an old result of Thue, has recently proved that if k is a prime, there exist integers a, b with $(a,k)=1$ such that a set of integers (y) defined by $y_r\equiv ax_r+b \pmod{k}$ $(r=1,2,\cdots,n)$ lies in an interval of width L where $nL^{n-1}\leq 2^{n-1}k^{n-2}$. The paper contains a simpler proof and also other estimates for L found by using Dirichlet's pigeonhole result. (Received July 9, 1953.)

583*t*. D. C. Murdoch: *On the automorphism group and endomorphism ring of an Abelian group.*

The quasi-regular elements of any ring R form a group $\mathfrak{Q}^*$ under the operation $x*y=x+y+xy$. Every ideal B_1 of R intersects $\mathfrak{Q}^*$ in a normal subgroup $\mathfrak{H}(B_1)$ of $\mathfrak{Q}^*$, while every subgroup $\mathfrak{H}_1$ of $\mathfrak{Q}^*$ generates an ideal $B(\mathfrak{H}_1)$ in R. The normal subgroup $\mathfrak{H}[B(\mathfrak{H}_1)]$ is called the closure of $\mathfrak{H}_1$, and the ideal $B[\mathfrak{H}(B_1)]$ the closure of B_1. A subgroup (ideal) which is equal to its closure is closed. The closed subgroups of $\mathfrak{Q}^*$ and the closed ideals of R form isomorphic lattices under $\mathfrak{H}\leftrightarrow B(\mathfrak{H})$. If R is the endomorphism ring of an Abelian group $\mathfrak{G}$, $\mathfrak{Q}^*$ is isomorphic to $\mathfrak{A}$, the automorphism group of $\mathfrak{G}$, and the concept of closed subgroup is extended to $\mathfrak{A}$ via this isomorphism. If $\mathfrak{A}_1$ is a subgroup of $\mathfrak{A}$, $\mathfrak{G}_0(\mathfrak{A}_1)=\mathfrak{G}$, and $\mathfrak{G}_i(\mathfrak{A}_1)$ is the group generated by all elements $g^{-1}g^s$, $g\in\mathfrak{G}_{i-1}(\mathfrak{A}_1)$ and $s\in\mathfrak{A}_1$ $(i=1,2,3,\cdots)$, then $\mathfrak{G}$ is $\mathfrak{A}_1$-nilpotent if $\mathfrak{G}_r(\mathfrak{A}_1)=1$ for some integer r. If the minimum condition for left ideals holds in R, then $\mathfrak{A}$ contains a unique maximal closed subgroup $\mathfrak{A}_N$ such that $\mathfrak{G}$ is $\mathfrak{A}_N$-nilpotent. $\mathfrak{A}_N\cong\mathfrak{H}(N)$ where

N is the radical of R and $\mathfrak{A}/\mathfrak{A}_N$ is isomorphic to the direct product of the quasi-regular groups of the simple ideals of R/N. (Received July 20, 1953.)

584. Bernard Noonan: *The construction of faithful representations of Lie algebras of characteristic zero.*

This paper gives an explicit procedure for the construction of a faithful representation S of a Lie algebra L, over an algebraically closed field of characteristic zero, from a given faithful representation Q of an ideal T of L, where T is in the radical R of L and where Q induces a nil representation on T_1, the intersection of the ideal LoL (o meaning Lie multiplication) and the radical R, and where, furthermore, Q is invariant under V ($L = V \dotplus R$ by the theorem of Levi). Q is defined to be invariant under V if, for any t in T, and v in V, there exists a matrix $C(v)$ such that $C(v)oQ(t) = Q(vot)$. The representation S so constructed possesses great generality in that one arbitrary eigenvalue can be assigned to the representation of each basis element of the radical R not in T. The theorem of Ado is proved as a corollary, using the Birkhoff representation of a nilpotent Lie algebra. (Received July 17, 1953.)

585t. Everett Pitcher: *A class of group extensions.*

A system (H, ϕ, ι) is called an *α-extension* of $\alpha: N \rightarrow E$ if N, E, H are topological abelian groups, α is a homomorphism, and (i) $\phi: H \rightarrow E$ is an open homomorphism onto, (ii) $\iota: N \rightarrow H$ is an isomorphism (into), (iii) $\phi\iota = \alpha$, (iv) $\phi^{-1}\alpha N = \iota N$. This reduces to the usual definition of abelian extension when α is trivial. (With an additional hypothesis, non-abelian extensions could be considered.) The existence of α-extensions may be regarded as the question of existence of extensions of E by $\alpha^{-1}0$ which, through N, are partially prescribed. It may be regarded as a problem in inverting the order of homomorphism onto and isomorphism into, with $\alpha: N \rightarrow \alpha N$, $\alpha N \subset E$ replaced by $\iota: N \rightarrow H$, $\phi: H \rightarrow E$. Two α-extensions (H_s, ϕ_s, ι_s), $s = 1, 2$, are *α-equivalent* if there is an isomorphism $\omega: H_1 \rightarrow H_2$ which is onto and for which $\phi_2\omega = \phi_1$ and $\omega\iota_1 = \iota_2$. *Theorem*: (a) *If E is discrete or if N is compact and E is locally compact, there exist α-extensions.* (b) *The set of α-extensions can be put into one-to-one correspondence with the set of extensions of $E/\alpha N$ by $\alpha^{-1}0$.* (Received July 20, 1953.)

586t. Irving Reiner: *Symplectic modular complements.*

An integral $(j+k) \times 2n$ array ($j \leqq n$, $k \leqq n$) with first j rows $(A\ B)$ and next k-rows $(C\ D)$ is called a normal (j, k) array if AB' and CD' are symmetric, and $AD' - BC' = (I\ 0)$ or $(I\ 0)'$ (depending on whether $j \leq k$ or $j \geq k$). The symplectic modular group Γ_{2n} is known (see C. L. Siegel, Math. Ann. vol. 116 (1939) pp. 617–657) to consist of all normal (n, n) arrays. In the present paper it is shown that a given normal (j, k) array can always be completed to an element of Γ_{2n} by placing $(n-j)$ rows after the first j rows, and $(n-k)$ rows after the last k rows. Further, a parmetrization of the general completion of a given normal array is found. This generalizes Siegel's result (loc. cit.) for the special case $j = n$, $k = 0$. (Received July 13, 1953.)

587. M. F. Ruchte: *On measure density: Addition of sets.*

The definitions and notations used are those given by R. C. Buck [Amer. J. Math. vol. 68 (1946) pp. 560–580]. In this paper the sum of two sets of positive integers is studied with respect to measure density. Principal results are: (1) $\underline{\mu}(A) + \bar{\mu}(B) > 1$ implies that $A + B \doteq I$ and this result is, in a sense, best possible. (2) $\underline{\mu}(A+B) \geqq \bar{\mu}(A) + \underline{\mu}(B)/2$ or $A + B = I$ if $1 \in B$. (3) $\mu(A \breve{+} B) \geqq \underline{\mu}(A) + \underline{\mu}(B)$ or $A \breve{+} B \doteq I$, where $A \breve{+} B = A \cup B \cup (A+B)$, providing $\underline{\mu}(A \cap B \cap H) = 0$ for all $H = [an]$. The method of the proofs is a reduction to the case of finite groups. Definitions of measure density are given for abelian groups, with translates of subgroups of finite index replacing progressions. Using these definitions, the above results are proved for the special case of the Gaussian integers. (Received July 13, 1953.)

588. W. R. Scott: *The number of subgroups of given index in nondenumerable Abelian groups.*

Let G be an Abelian group of order $A>\aleph_0$ and let $\aleph_0\leq B\leq A$. Then G has 2^A subgroups of index B and order A, and the intersection of all such subgroups is the 0 element. Furthermore, there exist 2^A subgroups H_α of index B and order A such that all the factor groups G/H_α are isomorphic to the same direct sum $\sum C_\beta$ where either (i) All C_β are cyclic of prime order, or (ii) all C_β are p^∞ groups where p need not be fixed. The first theorem is an improvement of a result, due to the author (Amer. J. Math. vol. 74 (1952) p. 195), that a nondenumerable Abelian group of order A has 2^A subgroups of order A, and that the intersection of these subgroups is 0. (Received June 22, 1953.)

589. Irving Sussman (p) and A. L. Foster: *Theory of associate rings.* Preliminary report.

The lattice theory of Boolean rings is a special case of a more general lattice theory holding in a wider classification called associate rings, and which contain as subclasses regular rings without nilpotents, p-rings, and algebraic rings in which $a^{n(a)}=a$. Associate rings are subdirect sums of domains of integrity (not necessarily commutative), with identity, with the sole governing restriction that with each element the projected (associated) idempotent is also in the ring. An abstract characterization, independent of the subdirect structure, is given for the commutative case. Via a relation called compatibility (a is compatible with b if $a^2b=ab^2$) and a generalized partial ordering (a is enclosed in b if $ab=a^2$), the ring breaks up into maximal distributive lattices with identities. Under appropriate operative definitions these sets become Boolean rings, all of which are then shown to be isomorphic. A decomposition of the ring into disjoint multiplicative-domains, each determined by an idempotent element, is related to the maximal lattices, and those local multiplicative-domains which are also additively closed and form principal ideals of the ring are identified and related to the component domains of the initial subdirect sum. (Received July 17, 1953.)

590. G. L. Thompson: *Projective quotients in modular lattices.*

In this paper a study is made of quotients of arbitrary (finite) dimension in a modular lattice L in which bounded chains are finite. Given two quotients which are projective in L, a generalization of the first normal form for a projectivity, established by R. M. Thrall [Proc. Amer. Math. Soc. vol. 2 (1951) pp. 146–152, §3], is proved. In this theorem a seven element sub-lattice $\{r; s, t, u; w, v, x\}$, such that $\{r; s, t, u; w\}$ is a projective root and x/v and w/t are transposes, called a *lower root*, and its dual called an *upper root*, play the role of the prime projective root in Thrall's theorem. A *root* is the lattice generated by a lower or an upper root and a maximal chain between s and x, and is characterized as a finite lattice by means of structure numbers. A more complete normal form for a projectivity is given in terms of roots and their structure numbers. [The work on this paper was supported by ONR Contract N8-ONR 71400 at the University of Michigan.] (Received July 9, 1953.)

591. H. S. Thurston: *On matric solutions of a cyclic equation.* Preliminary report.

If $f(x)=0$ is a cyclic equation over the rational field, then its companion matrix A_1 together with $A_2=\theta(A_1)$, $A_3=\theta(A_2)$, $\cdots$, $A_n=\theta(A_{n-1})$ form a conjugate set of rational matric solutions of $f(x)=0$. There exists a rational matrix T which transforms the set A_i into a set B_i $(B_i=T^{-1}A_iT)$ such that $B_{i+1}=U^{-1}B_iU$, where U is the matrix obtained by cyclicly permuting the columns of the identity matrix. Thus every cyclic equation over the rational field possesses a conjugate set of rational matrices B_i where B_{i+1} is obtained from B_i by cyclicly permuting rows and columns. (Received July 17, 1953.)

592*t*. A. D. Wallace: *Cohomology, codimension, and mobs.*

To obtain more precise results, not involving a favored coefficient group, it is necessary to use codimension (Haskell Cohen, Tulane Dissertation, 1952). For reasons of brevity we state our results in terms of the cruder covering dimension and employ integral cohomology groups. Let M be a compact connected mob with dim $M\leqq n$. Let A be a closed subset of M such that $\{x \mid xA\cap A\neq\square\}$ is a proper subset of M. If M has a left unit, then $H^n(A)=0$. This result is based on some ideas of Deane Montgomery. It follows that, if M has a unit and if B is a closed subgroup of M with $H^n(B)\neq 0$, then B coincides with the minimal closed ideal K of M. The proof employs Cohen's extension of the Hurewicz theorem to the effect that if X is a compact Hausdorff space of dimension n, then dim $(X\times I)=n+1$, I being the closed unit interval. It is a corollary that if e is an idempotent of M, if $H(e)$ is the maximal subgroup of M including e, and if $H^n(H(e))\neq 0$, then $e\in K$. In the corollary we need not assume that M has a unit. If M does not have a unit, then $H^n(M)\neq 0$ implies $H^n(H(e))\neq 0$ for each idempotent $e\in K$. (Received January 26, 1953.)

593*t*. F. B. Wright: *Ideals in rings of operators.*

Let A be an AW^* algebra, and let L be the lattice of projections in A. Define a p-ideal P of L to be a lattice ideal of A with the property that if $e\in P$, $f\in L$, and $e\sim f$, then $f\in P$. Then if I is a closed two-sided ideal of A, $I\cap L$ is a p-ideal of L. If P is a p-ideal of L, then the right ideal of A generated by P is a two-sided ideal, and the set of projections in the closure of this ideal is precisely P. Hence the closed two-sided ideals of A are in one-one correspondence with the p-ideals of L. This is a noncommutative version of Stone's characterization of the closed ideals in $C(X)$. Many results, both known and new, are consequences. Examples: (1) a finite AW^* algebra is strongly semi-simple, in the sense of Segal; (2) there is a one-one correspondence between the maximal two-sided ideals of a finite AW^* algebra and the maximal ideals of its center; (3) the closed two-sided ideals of an AW^* factor form a chain. In particular, in the algebra of all bounded operators on a Hilbert space, the closed two-sided ideals may be identified explicitly. (Received July 17, 1953.)

ANALYSIS

594*t*. H. A. Antosiewicz and Philip Davis: *Some implications of Liapunov's conditions for stability.*

The two fundamental criteria of Liapunov concerning the stability of solutions of systems of differential equations $dx/dt=p(x, t)$, which are based upon the existence of a Liapunov function, are examined. Together with results of I. G. Malkin, K.

Persidskii, and others they are shown to imply exponential dampedness of all solutions of linear as well as certain nonlinear systems. In particular, it is shown that a necessary and sufficient condition for all solutions of $dx/dt = A(t)x + b(t)$ to be bounded for every bounded vector $b(t)$ is that all solutions of $dy/dt = A(t)y$ are $O(e^{-kt})$ for some $k > 0$ as $t \to \infty$. (Received July 20, 1953.)

595. Richard Arens: *Some results of analytic functions generalized.*

Let Y be a compact Hausdorff space and let A be a separating subalgebra with unit of $C(Y$, complex values) with the usual norm. According to Šilov, there is a unique minimal closed subset X of Y such that every f in A attains its maximum modulus on X. The first proposition regarding X is this: for each point y of Y there is at least one Baire measure m_y defined on X such that $f(y) = \int f(x) m_y(dx)$ (integration over X) for every f in A. The measure m_y is said to represent y. When the measure representing m_y is unique, the one-parameter family of measures is a generalization of Poisson's kernel. The second proposition is this: with respect to an arbitrary f_0 in A, the family of measures m_y may be so selected that $\log |f_0(y)| \leqq \int \log |f_0(x)| m_y(dx)$ for that f_0. Finally, if for any point y and every representing measure m_y, $\int \log |f(x)| m_y(dx)$ diverges to $-\infty$, then f is a topological zero-divisor. (Received July 7, 1953.)

596. Nachman Aronszajn: *Boundary values of harmonic functions with finite Dirichlet integral.*

A complete characterization of these boundary values for an n-dimensional domain D with smooth boundary B (of class C^1) is given. The characterization is obtained by properties of developments in specific orthonormal systems in local coordinates on B. In terms of chosen local coordinates on B, a quadratic norm is defined for functions on B, stronger than the L^2 norm and essentially equivalent to the Dirichlet norm of the corresponding harmonic function. For hypersurfaces S_n in D converging to B in a "regular" manner (which implies in particular that the same systems of local coordinates can be used for the S_n and for B) it is proved: if h is a harmonic function in D with a finite Dirichlet integral, the values of h on S_n converge strongly in the new norm to the boundary values of h. (Received July 17, 1953.)

597. M. G. Arsove: *A generalization of the Dirichlet integral.*

In Euclidean n-space let Ω be a bounded Dirichlet region, S the set of all subharmonic functions bounded on Ω and taking on continuous values on the boundary $\partial\Omega$, and A the set of all differences of functions in S. Under pointwise multiplication the elements w of A form an algebra. If Ω^* is any region with closure in Ω, then $M_{\Omega^*}(w^2)$, the total mass of w^2 on Ω^*, and $\int_{\Omega^*} w\,dm$, where m is the mass distribufor w, both exist finitely. For $\{\Omega_k\}$ a monotone exhaustion of Ω let $D(w) = \lim_{k\to\infty} [2\int_{\Omega_k} w\,dm - M_{\Omega_k}(w^2)]$. This exists $\geqq 0$ (in the extended sense) independently of the exhaustion and is proportional to the classical Dirichlet integral for functions in $C''(\Omega)$. Let A_ϕ denote the class of all functions in A coinciding on $\partial\Omega$ with a prescribed continuous function ϕ. If $D(w)$ is finite for some $w \in A_\phi$, then D attains a minimum over A_ϕ, the minimizing function being the solution of the Dirichlet problem on Ω for the boundary function ϕ. (Received July 20, 1953.)

598t. Frederick Bagemihl and Wladimir Seidel: *A general principle involving Baire category, with applications to function theory and other fields.*

Let S be a nonempty complete metric space, $P_n(z)$ an at most enumerable class of properties meaningful for every $z \in S$, and $Q(z) = (P_1 \vee P_2 \vee \cdots)'$. Suppose that, for every P_n, if X is a dense subset of a nonempty open set $G \subseteq S$, and if every $x \in X$ has property P_n, then every $z \in G$ has property P_n, and every nonempty open subset of S has an element with property P_n'. Then there exists a residual set $R \subseteq S$, every element of which has property Q. This principle leads to a topological analogue of Kronecker's approximation theorem, and results on Hausdorff spaces and functions $f(z)$ meromorphic in $|z| < 1$, including the following, some of which contain answers to questions raised by P. Erdös: If the cluster set of $f(z)$ at every point of $|z| = 1$ is the whole plane, then that of $f(z)$ on each of a residual set of radii is the whole plane. The conclusion remains valid if, e.g., (1) the set of Fatou points of $f(z)$ on every subarc of $|z| = 1$ is of measure less than the length of that subarc; or, on any radius terminating in a point of a set on $|z| = 1$, of positive measure on every subarc, (2) $f(z)$ has no limit, or (3) the cluster set of (a nonconstant) $f(z)$ contains a fixed constant. If E is a subset of $|z| = 1$, meas $E > 0$, and for every $e^{i\theta} \in E$, the cluster set of $f(z)$ on every one of a residual set of segments terminating in $e^{i\theta}$ contains a fixed constant c but not the whole plane, then $f(z) \equiv c$. (Received July 20, 1953.)

599. Evelyn Boyd: *Regularity preserving factor sequences for Laguerre series.*

With the aid of an expression derived by Szegö describing the asymptotic behavior of Laguerre polynomials, it can be shown that the series $\sum_{n=0}^{\infty} f_n L_n^{(\alpha)}(z)$, where $\alpha > -1$, the f_n are arbitrary constants, and the $L_n^{(\alpha)}(z)$ are Laguerre polynomials of the complex variable z, converges in the interior of a parabola Re $[(-z)^{1/2}] < \tau$ and possibly on the boundary of this domain. The series defines a function which is analytic at least in the domain of convergence. A sequence $\{a_n\}$ is said to be a regularity preserving factor sequence (r.p.f.s.) if the series $\sum_{n=0}^{\infty} a_n f_n L_n^{(\alpha)}(z)$ converges at least at every point in the domain of convergence of the series $\sum_{n=0}^{\infty} f_n L_n^{(\alpha)}(z)$ and if it is possible to continue $F(z)$ analytically along any finite path along which $f(z)$ can be continued analytically. If $g(\omega)$ is defined for $\omega = n$ $(n = 0, 1, 2, \cdots)$ and is analytic in some neighborhood of the point at infinity, then $\{g(n)\}$ is a r.p.f.s. for the series $\sum_{n=0}^{\infty} f_n L_n^{(\alpha)}(z)$. Likewise, if $G(\omega)$ is an entire function of order $< 1/2$ and of minimal type if the order equals $1/2$, $\{G(n)\}$ is a r.p.f.s. for $\sum_{n=0}^{\infty} f_n L_n^{(\alpha)}(z)$. The proofs make use of properties of the differential operator $\delta_z = -zD^2 + (z - \alpha - 1)D$. (Received July 10, 1953.)

600. G. U. Brauer: *Some Abelian semi-groups of linear transformations of Hausdorff type.*

Various semi-groups of Toeplitz matrices of the form $A\mu A^{-1}$, where A is a normal matrix and μ is a diagonal matrix, are studied with regard to regularity conditions and convergence fields. It is shown that if m is any integer greater than 1 and C_m denotes the Cesàro matrix of order m, then no regular matrix $C_m \mu C_m^{-1}$ evaluates divergent sequences. (Received July 22, 1953.)

601. F. E. Browder: *Differentiability theorems of the Lichtenstein type for the general linear elliptic differential equation.*

Let K be a linear elliptic differential operator of order $2m$ with suitably differentiable coefficients on a domain D of Euclidean n-space. Let D_1 be a bounded subdomain of D whose closure is contained in D. If $e(x, z)$ is a fundamental solution for K on D, $\eta\in L^2(D)$, then $\phi(s)=\int e(x, s)\eta(x)dx$ is uniquely determined a.e. in D as a limit in square-mean. Generalizing results of Lichtenstein and Friedrichs for the Laplace equation, it is shown that ϕ is $2m$-times differentiable in a certain almost-everywhere sense and that the derivatives obtained satisfy the equation $K\phi=\eta$ a.e. in D. (Received July 20, 1953.)

602*t*. F. E. Browder: *The asymptotic distribution of the eigenvalues and eigenfunctions of the general self-adjoint elliptic differential operator*. II.

Let K be a self-adjoint linear elliptic differential operator of order $2m$ with variable coefficients on the bounded domain D of Euclidean n-space. The theorems on the asymptotic distribution of the eigenvalues and eigenfunctions of K on D for $2m>n$, presented by the author in a previous abstract, are established for general m and n. For $2m\leqq n$, mild smoothness conditions on the boundary of D are required in obtaining the eigenvalue distribution but not for the eigenfunctions. Analogous results are obtained for the general self-adjoint strongly elliptic systems of differential equations. (Received July 20, 1953.)

603*t*. A. P. Calderón and Antoni Zygmund: *On a problem of Mihlin*.

Let $x=(\xi_1, \cdots, \xi_n)$, $y=(\eta_1, \cdots, \eta_n)$, $\cdots$ be points of the n-dimensional vector space E^n with metric $|x|=(\xi_1^2+\cdots+\xi_n^2)^{1/2}$. Let $\Omega(x, \theta)$ be a measurable function of the point x and direction θ and let $K(x, y)=\Omega(x, \theta)\ |x-y|^{-n}$, where θ now denotes the direction from x to y, and let $\tilde{f}(x)=\int_{E^n} f(y)K(x, y)dy$, with $dy=d\eta_1, \cdots, d\eta_n$. The following theorem was proved by Mihlin for $n=2$ and stated as conjecture for $n>2$. Consider θ as a point of the unit sphere Σ with center at the origin and suppose that, for all x, 1° $\int_\Sigma \Omega(x, \theta)d\theta=0$, 2° $\int_\Sigma |\Omega(x, \theta)|^2 d\theta<C$, C a constant independent of x and $d\theta$ the $(n-1)$st dimensional measure on Σ. Then the singular integral defining $\tilde{f}(x)$ and taken in the principal value sense converges to a limit in the metric L^2. Moreover $\|\tilde{f}\|_2\leqq A\|f\|_2$. It can be shown that not only is the conjecture correct but that the conclusion holds if the exponent 2 in 2° is replaced by any $p>2(k-1)/k$, and the latter condition is best possible. (Received September 2, 1953.)

604. Lamberto Cesari: *Frechet surfaces of finite Lebesgue area in E_n*.

Let $S\subset E_n$ be any continuous surface $(n\geqq 3)$, T: $x=T(w)$, $w\in Q$, $x=(x_1, x_2, \cdots, x_n)$, any representation of S (Q simple Jordan region of the w-plane E_2), $L(S)$ the Lebesgue area of S, T_{ij} the plane mapping which is the projection of S on the plane $E_{ij}=[x_k=0, k\neq i, j, k=1, 2, \cdots, n]$, $E_{ij}\subset E_n$, $W(T_{ij})$ the total variation of the plane mapping T_{ij}, $N(p; T_{ij}, Q)$, $p\in E_{ij}$, the characteristic function of T_{ij} $(i\neq j, i, j=1, 2, \cdots, n)$, F_{ij} the set of all points $p\in E_{ij}$ such that the components of $T_{ij}^{-1}(p)$ are continua of constancy for T on Q, C the boundary curve of S, r_{ij} the $(n-2)$-dim. subspace of E_n defined by $[x_i=0, x_j=0]$, R the set $R=\sum r_{ij}\subset E_n$ where $\sum$ ranges over all $i\neq j$, $i, j=1, 2, \cdots, n$. Let G_{ij} be the set of all points $p\in F_{ij}$ where $N(p; T_{ij}, Q)=0$. The following lemma holds (generalization of the analogous lemma for $n=3$; Bull. Amer. Math. Soc. vol. 58 (1952) p. 178): If $CR=0$, if $(0, 0)\in G_{ij}$ and is a point of density for G_{ij} in E_{ij}, $i\neq j$, $i, j=1, 2, \cdots, n$, then C is null-homotop in E_n-R. The proof is the same as for $n=3$. From here it follows that for every surface S, we have $L(S)\leqq\sum W(T_{ij})$. This inequality holds also if Q is any finitely connected closed Jordan region, or any open set in E_2. (Received July 20, 1953.)

605t. R. B. Davis: *A boundary value problem for certain third-order linear composite partial differential equations.*

Using the methods of the author's previous papers, a boundary value problem existence theorem is proved for a certain class of third-order partial differential equations. Uniqueness is not proved, but rather one obtains the Fredholm alternatives. The boundary value problem consists in prescribing u around the boundary of the region, and Δu along a suitable curve. The hypothesis is that, besides having a special form, the differential equation has coefficients belonging to C^1, and the region is simply-connected, with boundary having continuous curvature. The special form is the $(\Delta u)_x+au_x+bu_y+cu=d$ form described in Proc. Amer. Math. Soc. vol. 3 (1952) pp. 751–756, but with the omission of the terms in u_{xx} and u_{yy}, so as to facilitate greatly the estimation of the kernel in the Fredholm integral equation obtained by the method of Levi. (Received July 14, 1953.)

606t. M. R. Demers: *On the isoperimetric inequality.* Preliminary report.

The isoperimetric inequality is shown to be valid for any subset A of the Euclidean plane having finite Lebesgue (outer) measure if the perimeter of A is taken to be the Favard 1-dimensional measure of the boundary of A. Moreover, if equality holds for A and if A has positive Lebesgue (outer) measure, then the interior of the closure of A is a circular disc. It follows immediately from the definitions and a theorem of Federer (Trans. Amer. Math. Soc. vol. 62 (1947) pp. 122 and 182) that the Favard measure may be replaced by Carathéodory linear measure or Gillespie linear measure. (Received July 16, 1953.)

607. W. F. Donoghue: *Invariant subspaces of unitary transformations.*

It is known from the Hellinger-Hahn multiplicity theory that the lattice of closed reducing subspaces of a unitary transformation in a separable Hilbert space is isomorphic to the lattice of sets measurable with respect to a certain Borel measure on the reals modulo the sets of measure zero. Using results due to Wiener, Kolmogoroff, Beurling and Karhunen, the lattice of closed invariant subspaces which are not reducing subspaces (one-sided invariant subspaces) is investigated. (1) The lattice of one-sided invariant subspaces is isomorphic to the lattice of such subspaces for the transformation $V=$ multiplication by $e^{i\theta}$ on the direct sum of L^2 $(0\leqq\theta<2\pi$, Lebesgue measure) with itself N times for some finite or infinite N. (2) If $N=1$, the lattice is isomorphic to the multiplicative group G of measurable functions on $0\leqq\theta<2\pi$ of absolute value 1 almost everywhere modulo the constants, identifying functions which coincide almost everywhere. If ϕ and ψ in G correspond to the subspaces $\mathfrak{M}$ and $\mathfrak{N}$ then $\mathfrak{M}\subseteq\mathfrak{N}$ if and only if $\phi=f\psi$ where f is the boundary value function of a function bounded and analytic in $|z|<1$. (3) Analogous results are obtained for continuous unitary representations of the reals. (Received July 17, 1953.)

608. W. F. Eberlein: *Generalized harmonic analysis.* I. Preliminary report.

A uniform spectral synthesis for weakly almost periodic functions is obtained under restrictions on the mean rate of growth at infinity. (Received July 20, 1953.)

609. H. W. Ellis (p) and Israel Halperin: *Function spaces determined by a levelling length function.*

Generalizing the Lebesgue spaces $L^p(B)$, the function spaces $L^\lambda(B)$ with elements which are functions $f(P)$ valued in a Banach space B and defined on a measure space S with $\|f\| = \lambda(|f(P)|)$ are introduced. Here λ is a length function, i.e. for arbitrary measurable, non-negative $u(P)$: $0 \leq \lambda(u) \leq \infty$, $\lambda(u) = 0$ if $u(P) = 0$ almost everywhere, $\lambda(u) \leq \lambda(u_1)$ if $u(P) \leq u_1(P)$ for all P, $\lambda(u+v) \leq \lambda(u) + \lambda(v)$, $\lambda(ku) = k\lambda(u)$ for $k > 0$, and $\lambda(\sup u_n) = \sup \lambda(u_n)$ if the u_n are nondecreasing in n. The conjugate length function is defined by $\mu(u) = \lambda^*(u) = \sup \int uv$ for all v with $\lambda(v) \leq 1$. A special class called levelling length functions (which include the L^p) are defined by the conditions: $\sup \lambda(u_e) = \lambda(u)$ when e varies over all sets of finite measure and $\lambda(u') \leq \lambda(u)$ if u' coincides with u except on a set of finite measure and on that set u' is constant, equal to the average of u on that set. For levelling length functions: (i) $L^\lambda(B)^* = L^\mu(B^*)$ under conditions less restrictive than Dieudonné's for L^p (Canadian Journal of Mathematics vol. 3 (1951) pp. 129–140) and (ii) $L^\lambda(V)$ is reflexive under conditions which generalize the corresponding theorem of Phillips for L^p (Amer. J. Math. vol. 65 (1943) pp. 108–136). (Received July 13, 1953.)

610. Paul Erdős, Fritz Herzog, and George Piranian (p): *On Taylor series of functions regular in Gaier discs.*

Gaier [Math. Zeit. vol. 56 (1952) pp. 326–334, see p. 327] has shown that if $f(z) = \sum a_n z^n$ is regular and bounded in a circular disc $|z+a| < 1+a$ $(a>0)$, then $a_n = O(n^{-1/2})$. The authors generalize this theorem by showing that if k is a real constant and $(1-z)^k f(z)$ is regular and bounded in such a disc, then the Taylor coefficients of $f(z)$ are $O(\phi(n, k))$, where $\phi(n, k) = n^{k-1}$, $\log n$, or $n^{(k-1)/2}$, according as $k > 1$, $k = 1$, or $k < 1$. The authors construct functions which show that for each k this bound is the best possible in the sense that the O cannot be replaced by o. If $k < 0$ then $\sum a_n z^n$ converges uniformly on the unit circle; and if $k < -1/2$ then $\sum |a_n| < \infty$. Fejer's work on the Taylor coefficients of the function $(1-z)^{-k} \exp [1/(z-1)]$ (see Perron, Arch. d. Math. u. Phys. (3) vol. 22 (1914) pp. 329–340) shows that the latter conclusion does not hold when $k \geq -1/2$ (Received July 17, 1953.)

611t. Robert Finn: *On a theorem of S. Bernstein.*

Elliptic partial differential equations which arise from variational problems $\delta \iint F(p^2+q^2)dxdy = 0$, $p = \phi_x$, $q = \phi_y$, are considered. Conditions on $F(\omega^2)$ are stated which insure that every solution $\phi(x, y)$ regular throughout the (x, y) plane is a linear function $\phi = ax+by+c$. This theorem was first stated as a property of minimal surfaces by S. Bernstein (Math. Zeit. vol. 26 (1927) pp. 551–558). The precise conditions obtained by the author are rather complicated, but it is sufficient that $F(\omega^2) = (1+\omega^2)^{1/2} + \lambda\epsilon(\omega^2)$ where $\epsilon = O(\omega^{-1})$, $\epsilon' = O(\omega^{-2})$, $\epsilon'' = O(\omega^{-3})$ as $\omega \to \infty$ and λ is a real number sufficiently small that $F_{pp}F_{qq} - F^2_{pq} > 0$. The proof follows in outline that given by T. Radó for minimal surfaces (Math. Zeit. vol. 26 (1927) pp. 559–565). It involves in addition an investigation of certain growth properties of quasi-conformal mappings and the use of the hodograph metric of Chaplygin. The author has knowledge of a similar theorem, proved by other methods, due to L. Bers (Proc. of Arden House Conference, 1952). The author's claim to priority is therefore limited to the methods employed. (Received July 6, 1953.)

612t. Charles Fox: *The inversion of convolution transforms by differential operators.*

This paper studies the type of kernel $K(x, v)$ for which the integral equation $f(x)=\int_0^\infty K(x, v)\phi(v)dv$ is inverted by means of differential operators. Let $E(u)=\prod_{n=1}^\infty (1+u^2a_n^{-2})$, where $\sum_{n=1}^\infty a_n^{-2}$ is convergent and let $K(x, v)=\int_0^\infty (k(xu)k(vu)/E(u))du$, where $k(x)$ is a generalized Fourier kernel (E. C. Titchmarsh, *The theory of the Fourier integral*, chap. 8). If $y=k(ux)$ satisfies a differential equation of the type $L(x, D)y=-u^2y$, where D denotes the differential operator d/dx and $L(x, D)$ is independent of u, then $\prod_{n=1}^\infty \{1-L(x, D)/a_n^2\}f(x) =\phi(x)$. Suitable conditions of convergence are required. An example is $k(x)=x^{1/2}J_\nu(x)$, where J_ν denotes the Bessel function of order ν. This result generalizes discoveries by Post, Hirschman Jr., and Widder in the theory of inversion of convolution transforms by means of differential operators. (Received June 29, 1953.)

613*t*. Evelyn Frank and Oskar Perron: *Remark on a certain class of continued fractions.*

Continued fractions of the form (A) $k_0\gamma_0+k_0(1-\gamma_0\bar{\gamma}_0)z/\bar{\gamma}_0z-1/k_1\gamma_1+k_1(1-\gamma_1\bar{\gamma}_1)z/\bar{\gamma}_1z-1/k_2\gamma_2+\cdots$, where the k_ν and γ_ν are arbitrary constants different from zero, $|\gamma_\nu|\neq 1$, were considered by Frank [Proc. Amer. Math. Soc. vol. 3 (1952) pp. 921–937]. To every such continued fraction there corresponds a power series $c_0+c_1z+c_2z^2+\cdots$ such that the Taylor's series for the approximants $A_{2\nu}(z)/B_{2\nu}(z)$, $A_{2\nu+1}(z)/B_{2\nu+1}(z)$, agree with the power series up to and including the terms involving z^ν and $z^{\nu-1}$, respectively. In this paper the function to which (A) converges is considered. It is found that in some cases (A) converges in a larger region than the series, so that (A) represents the analytic continuation. For example, in the case of the geometric series $1+z+z^2+\cdots$, depending on the values of the k_ν and γ_ν, (i) (A) may converge over the entire z-plane and may equal $1/(1-z)$, or may equal an entirely different function of z; (ii) (A) may converge in various regions to functions which are different. This is the case when $k_\nu=2$, $\gamma_\nu=(-1)^\nu/(\nu+2)$; for $|z|<1/2$, (A) is equal to $1/(1-z)$; for $|z|>1/2$, (A) is equal to $4-1/z$. (Received July 1, 1953.)

614. Bernard Friedman and Luna Mishoe (p): *A Sturm-Liouville expansion depending on a boundary condition.*

In the classical Sturm-Liouville theory the convergence of the expansion of $f(x)$ in terms of the eigenfunctions of a second-order differential equation does not depend on whether $f(x)$ satisfies the boundary conditions imposed on the differential equation, but depends solely on the local properties of $f(x)$. A case in which the convergence of the expansion to $f(x)$ depends on whether $f(x)$ satisfies a boundary condition is presented. Consider the eigenfunctions of the equation (*) $u''+qu+\lambda(u'+pu)=0$, where $u(0)=u(1)=0$. If $f(x)$ is of bounded variation and satisfies the boundary condition (†) $f(0+)+f(1-)\exp\{\int_0^1 p(t)dt\}=0$, then the eigenfunction expansion converges to $\{f(x+0)+f(x-0)\}/2$. If $f(x)$ is of bounded variation but does not satisfy the boundary condition (†), then the eigenfunction expansion converges to $\{f(x+0)+f(x-0)\}/2-[f(0+)+f(1-)\exp\{\int_0^1 p(t)dt\}]\exp\{\int_0^x p(t)dt\}$. This work was performed at Washington Square College of Arts and Science, New York University, and was supported in part by the Office of Scientific Research, H.Q., Air Research and Development Command, U.S.A.F., Baltimore, Md., under Contract No. AF-18(600)-367. (Received July 8, 1953.)

615*t*. R. E. Fullerton: *Properties of L cones in a linear space.*

A cone C in a topological linear space X which is the closure of the convex set determined by its extreme rays is an L cone if for every extreme ray r of C there exist $y\in X$, $y\notin C$, $z\in X$ such that $C\cap(y+C)=z+C$ and r is the ray determined by z and the vertex of C. It is shown that a necessary and sufficient condition that C be an L cone is that for any $y\in X$ the set $C\cap(y+C)$ is either a cone or a conoid with a one-dimensional finite interval as a vertex. A necessary and sufficient condition that a Banach space X whose unit sphere is determined by its set of extreme points be a space of continuous functions over a compact Hausdorff space is that the cone determined by the origin and a maximal convex subset of the surface of the unit sphere be an L cone which is not a C cone. (Received July 20, 1953.)

616*t*. P. R. Garabedian and M. M. Schiffer: *On estimation of electrostatic capacity.*

Let D be a domain in space containing the origin, let S_r be the sphere of radius r about the origin, let V_r be the volume of $D+S_r$, and let γ_r be the capacity constant, with respect to the origin, associated with $D+S_r$. By means of Hadamard's variational formula, it is shown that the dimensionless quantity $\gamma_r^3V_r$ is a nonincreasing function of r, whence it follows that the quantity has its least value for a sphere about the origin. Let Γ be the capacity, with respect to the point at infinity, of a lamina E spanned through a curve C. Suppose that the portion of E which was in the region $z>z_0$ has been replaced by corresponding sections of the plane $z=z_0$, while the rest of E, including the boundary curve C, remains in the region $z<z_0$. Then Hadamard's formula shows that Γ is a nondecreasing function of z_0. The analogous method in two dimensions yields the known theorem that the curve of minimum logarithmic capacity joining two given points is a line segment. Proofs depend upon estimating the the natural charge distribution, which appears in Hadamard's formula, by suitable application of the maximum principle and of comparison domains. (Received June 24, 1953.)

617. Seymour Ginsburg: *Order types of simply ordered sets of solutions to problem* P.

Let L denote the order type of the real numbers. Theorem: Let c be any order type, of power $2^{\aleph_0}$, which is $<L$. Let Y be any set of power $2^{\aleph_0}$, and H a simply ordered family of subsets of Y. Then problem P (see Trans. Amer. Math. Soc. vol. 74 (1953) p. 514), as applied to $a=c$ and $b=L$, and problem P, as applied to $a=0$ and $b=c$, each admits of a set, which is similar to H, of solutions. A consequence of the preceding is the following result: Let c be any order type, of power $2^{\aleph_0}$, which is $<L$. Then problem P, as applied to $a=0$ and $b=c$, and problem P, as applied to $a=c$ and $b=L$, each admits of a set, of order type L, of solutions. Finally, the following generalization of a result due to Sierpiński (*Types d'ordre des ensembles linéaires*, Fund. Math. vol. 37 (1950) pp. 253–264, Theorem 9) is shown: Let $\{a_m/m\in M\}$, where the power of M is $2^{\aleph_0}$, be a set of order types, each $a_m<L$. Then there exists an order type c such that $a_m<c<L$ for each m in M. (Received July 13, 1953.)

618. Casper Goffman: *On a theorem of Henry Blumberg.*

H. Blumberg proved (Trans. Amer. Math. Soc. vol. 24 (1922) pp. 113–128) the following remarkable, although apparently isolated, fact: Every real function $f(x)$ on $(0, 1)$ has an associated dense set D such that $f(x)$ is continuous on D relative to D. The purpose here is to show that the analogue for one-one transformations is not valid. (Received July 3, 1953.)

619*t*. J. W. Green: *Further remarks on approximately convex functions.*

Approximately convex functions have been studied by Hyers and Ulam (Proc. Amer. Math. Soc. vol. 3 (1952) pp. 821–828) and the author (Duke Math. J. vol. 19 (1952) pp. 499–504), and it was shown that such a function f can be approximated by a convex function to within $k_n\epsilon$ when k_n depends on the dimension n. The value of k_n found is the best possible for $n=1, 2, 3$. In the present work, more detailed investigation is made into the cases $n=2, 3$ of those functions for which the approximation is poorest possible. Typical result: If F is the lower convex envelope of f and $f(x_0)-F(x_0)=k_n\epsilon$, then $f-F<k_n\epsilon$ near x_0, and F is linear in precisely a triangle containing x_0. (Received July 16, 1953.)

620. J. W. Green: *Mean values of harmonic functions on homothetic curves.*

Let C be a smooth closed curve containing the origin and C_λ the homothetic transforms of C with respect to the origin and with scale factor λ. Suppose that, for each harmonic function U, the perimeter average of U over all C_λ is constant, or the area average of U over the interiors of the C_λ is constant. Then C is a circle. (Received August 21, 1953.)

621. J. G. Herriot: *The solution of Cauchy's problem for a third order linear hyperbolic differential equation by means of Riesz integrals.*

Cauchy's problem for the wave equation has been solved by means of a generalization of the Riemann-Liouville integral by M. Riesz (Acta Math. vol. 81 (1948) pp. 1–223). In the present paper the explicit solution of Cauchy's problem for the third order partial differential equation $\partial^3 u/(\partial x_1\partial x_2\partial x_3)=h(x_1, x_2, x_3)$ is given by means of a similar generalization of the Riemann-Liouville integral. The discussion is restricted to the case in which u and its first and second derivatives are given on the plane S whose equation is $x_1+x_2+x_3=0$. It is verified in detail that the solution given actually satisfies the differential equation and that it and its derivatives assume the proper values on S. The main tool used is the analytic continuation of the integrals which arise. (Received July 3, 1953.)

622. M. J. Herzberger: *Approximate methods in mathematics.*

The calculation with polynomials in which members of higher order are neglected is shown to be equivalent to the calculation with a special system of triangular matrices. It is shown how to apply this fact to obtain an approximate solution of nonlinear partial differential equations and to solve implicit equations. (Received June 10, 1953.)

623. Edwin Hewitt: *A maximum problem in harmonic analyses.*

Let G be a locally compact Abelian group, $\hat{G}$ its character group, and p a number such that $1<p<2$. Let $f\in\mathfrak{L}_p(G)$ and let $\hat{f}$ be the Fourier transform of f; $\hat{f}$ is a function in $\mathfrak{L}_{p'}(\hat{G})$ $(p'=p/(p-1))$ such that $\|\hat{f}\|_{p'}\leq\|f\|_p$ (Hausdorff's inequality). A subcharacter of G is any function on G which is the product of a complex number, a (continuous) character, and the characteristic function of a compact open subgroup. It is proved that if G is compact or discrete, then equality occurs in Hausdorff's inequality if and only if f is a subcharacter or a translate of a subcharacter. If G is an arbitrary locally compact Abelian group, and if $1\leq p\leq 4/3$, then equality occurs in Hausdorff's inequality if and only if f is a subcharacter or a translate of a subcharacter. If G is an arbitrary locally compact Abelian group, if f produces equality in Hausdorff's inequality, and if f is in $\mathfrak{L}_p(g)$ and $\mathfrak{L}_{p_1}(G)$, where $1<p\neq p_1<2$, then f is a subcharacter or a translate of a subcharacter. These theorems generalize a classical result of Hardy and Littlewood for Fourier series [Math. Ann. vol. 97 (1926) pp. 158–209]. (Received July 9, 1953.)

624. Jack Indritz: *Convergence of the Rayleigh-Ritz method to solutions of variational problems.* Preliminary report.

Let R be a plane bounded region whose boundary B is a finite number of regular arcs of class c^k, $k \geqq 3$, no two intersecting. Let a, b, c, f satisfy $a>0$, $b>0$, $c \geqq 0$; a, b, c bounded and integrable, f integrable on $R+B$. Assume that there exists a function u_1' minimizing the functional $I(u)=\iint_R(au_x^2+bu_y^2+cu^2+2fu)dxdy$ in the admissible set of functions u of class c^k on $R+B$ whose values on B are those of a given function $g(x, y)$ of class c^k on $R+B$. Here a function is of class c^k on $R+B$ if its partial derivatives of order $\leqq k$ exist in R with limits on B. Let $G(x, y)$ be of class c^k on $R+B$, vanish on B, positive in R, with positive normal derivative on B. The author obtains expansion theorems together with rates of convergence for admissible functions, and finds properties of the expansion analogous to those of Fourier series. An explicit solution is given for u' in terms of g, a, b, c, f, and functions $f_i(x, y)$ obtained by orthonormalizing the set Gx^iy^j where the square of the norm is $D(u)=\iint_R(au_x^2+bu_y^2+cu^2)dxdy$. (Received July 20, 1953.)

625. Meyer Jerison: *The set of all generalised limits of bounded sequences.*

A construction of Kakutani (Ann. of Math. vol. 42 (1941) p. 1019) is reformulated and extended so as to yield *all* generalized limits of bounded sequences. Let M denote the Banach space of bounded sequences, M^* its conjugate space, F the subset of M^* consisting of nontrivial multiplicative functionals, and T the transformation of M into itself which sends $x=(\xi_1, \xi_2, \cdots)$ into $Tx=(\xi_2, \xi_3, \cdots)$. Then every weak*-limit point of $\{n^{-1}\sum_{i=0}^{n-1} T^{*i}f\}_{n=1}^{\infty}$, for each $f \in F$, is a generalized limit, and every generalized limit is in the closed convex hull of those limit points. (Kakutani's construction amounts to considering the functional $f(x)=\xi_1$.) This provides an alternative approach to some of the results of G. G. Lorentz (Acta Math. vol. 80 (1948) pp. 167–190). (Received July 13, 1953.)

626t. R. V. Kadison: *On the orthogonalisation of operator representations.*

The problem of when a group representation by operators on a Hilbert space is similar to a unitary representation along with the problem of when a representation of a C^*-algebra is similar to a * representation is investigated. A concept of bounded local semi-simplicity of a group representation is introduced and it is shown that a group representation by operators on a Hilbert space is boundedly locally semi-simple if and only if it is similar to a unitary representation. A necessary and sufficient condition is given for a representation of a C^*-algebra to be similar to a * representation. (Received July 15, 1953.)

627t. Jacob Korevaar: *A numerical Tauberian theorem for Lambert series.*

The author has recently shown (*Best L_1 approximation and the remainder in Littlewood's theorem*, Nederl. Akad. Wetensch. Proc. Ser. A vol. 56 = Indagationes Math. vol. 15 (1953)) that the hypotheses $|na_n| < K_1$ $(n=1, 2, \cdots)$ and $|\sum a_nx^n - s| < K_2(1-x)^{\epsilon}$ $(\epsilon>0, 0<x<1)$, imply $|s_n - s| < C/\log(n+2)$, $n=0, 1, \cdots$. This estimate is best possible. In the present paper an analogous result is proved for Lambert series. Let $\mu(k)$ denote the Moebius function and set $\sum_{k \leq n} \mu(k)/k = g(n)$. Vinogradov has shown that there is a constant α $(1/2 < \alpha < 1)$ for which one can find $A > 0$, $\beta > 0$ such that $|g(n)| < A \exp\{-\beta (\log n)^{\alpha}\}$. From this result and the Tauberian theorem for power series mentioned above the following Tauberian theorem is derived. The hypotheses $|na_n| < K_1$ and $|(1-x)\sum a_n n x^n/(1-x^n) - s| < K_2(1-x)^{\epsilon}$ $(\epsilon > 0, 0 < x < 1)$ imply $|s_n - s| < C/\{\log(n+2)\}^{\alpha}$. If there exists a $\gamma > 0$ such that the Riemann zeta-function $\zeta(s)$ has no zeros in the half-plane $\operatorname{Re} s > 1 - \gamma$, then one may take $\alpha = 1$. It is shown by an example that the estimate certainly can not be improved beyond $|s_n - s| < C/\log(n+2)$. (Received July 17, 1953.)

628. Saul Kravetz: *On the geometry of Teichmueller manifolds and the structure of the mapping-class group*. I. Preliminary report.

Let ${}^t\overline{\mathfrak{R}}(g, k)$ be the *Teichmueller manifold* of conformal-equivalence classes of topologically-fixed compact Riemann surfaces of genus g with k distinguished points, with the metric induced by extremal quasi-conformal mappings, introduced by O. Teichmueller ([1] Abh. der Preuss. Akad. der Wiss., Math. Naturw. Klasse (1939) No. 22; [2] ibid. (1943) No. 4). Let ${}^t\overline{\mathfrak{F}}(g, k)$ be the mapping-class group associated with ${}^t\overline{\mathfrak{R}}(g, k)$. The author solves problems indicated by Teichmueller, proving the following results. ${}^t\overline{\mathfrak{R}}(g, k)$ is an open straight line space [1, §§113–120]. A necessary and sufficient condition that a certain mapping be conformal is obtained and is applied to proving that the complex lines of ${}^t\overline{\mathfrak{R}}(g, k)$ are hyperbolic planes, by the methods of the metric foundations of geometry [1, §121]. The fixed-point set in ${}^t\overline{\mathfrak{R}}(g, k)$ of a finite subgroup of ${}^t\overline{\mathfrak{F}}(g, k)$ is void or a linear subspace of ${}^t\overline{\mathfrak{R}}(g, k)$ homeomorphic to a complex Euclidean space (as indicated by W. Fenchel (Mat. Tidsskr. B. (1950) pp. 90–95), the topological part of this result and Smith's Fixed-Point Theorem solves the Hurwitz-Nielsen realization problem for solvable groups, after noting that it is equivalent to Teichmueller's fixed-point problem, but Fenchel does not mention Teichmueller's contribution [1, §149]). (Received May 11, 1953.)

629t. Saul Kravetz: *On the geometry of Teichmueller manifolds and the structure of the mapping-class group*. II. Preliminary report.

Given any topologically-fixed hyperelliptic surface of genus g, its sheet-transformation mapping induces an element of ${}^t\overline{\mathfrak{F}}(g, 0)$, which is called a ***hyperelliptic mapping-class***, and the latter leaves fixed a $(2g-1)$-dimensional linear subspace of ${}^t\overline{\mathfrak{R}}(g, 0)$, which is called a ***hyperelliptic subspace*** of ${}^t\overline{\mathfrak{R}}(g, 0)$. The author proves that there are an infinite number of disjoint hyperelliptic subspaces in ${}^t\overline{\mathfrak{R}}(g, 0)$, $g > 2$ [1, §§150–152]. If $\overline{\mathfrak{R}}(g, k)$ is the set of conformal-equivalence-classes of compact Riemann surfaces of genus g with k distinguished points, then $\overline{\mathfrak{R}}(g, k)$ is just ${}^t\overline{\mathfrak{R}}(g, k)$ modulo ${}^t\overline{\mathfrak{F}}(g, k)$. Let ${}^t\overline{\mathfrak{S}}(g, k)$ be the set of all points of ${}^t\overline{R}(g, k)$ which are left fixed by elements of ${}^t\overline{\mathfrak{F}}(g, k)$ which do not leave all the elements of ${}^t\overline{\mathfrak{R}}(g, k)$ fixed, and let $\overline{S}(g, k)$ be the set of corresponding points on $\overline{R}(g, k)$. It is shown that the mapping-class group of a compact topological surface of genus g with k distinguished points is a discrete topological group. Then ${}^t\overline{\mathfrak{F}}(g, k)$ is shown to be a properly discontinuous group on ${}^t\overline{\mathfrak{R}}(g, k)$. It is then shown that, in the induced topology, $\overline{\mathfrak{R}}(g, k)$ is a Hausdorff space while $\overline{\mathfrak{R}}(g, k) - \overline{\mathfrak{S}}(g, k)$ is a connected topological manifold. $\overline{\mathfrak{R}}(g, k)$ is called the ***singular moduli manifold***. (Received May 11, 1953.)

630. C. E. Langenhop: *On a generalisation of an inequality and some applications to the boundedness of solutions of differential equations*. Preliminary report.

If $w(t) \geqq 0$ and if for $t \geqq 0$ the inequality $u(t) \leqq v(t)+\int_0^t w(\tau)u(\tau)d\tau$ is true, then $u(t) \leqq v(0) \exp \int_0^t w(\tau)d\tau+\int_0^t v'(\tau)(\exp \int_\tau^t w(s)ds)d\tau$. This is a generalization of a well known result when $v(t)$ is a constant. The above inequality can easily be applied to prove such results as the following: let the constant matrix A be such that $\|e^{At}\| \leqq Ke^{-\sigma_0 t}$, $\sigma_0 > 0$, and let the vector function $f(x, t)$ be such that $\|f(x, t)\| \leqq \phi(t)\|x\|$ where $K\phi(t) \leqq \sigma_0 - \eta$, $\eta > 0$. Then if $p(t)$ is a bounded vector for $t \geqq 0$, all solutions of the equation $dx/dt = Ax+f(x, t)+p(t)$ are bounded as $t \to \infty$ and ultimately satisfy the condition $\|x\| \leqq 2K \max \|p\|/\eta$. (Received July 20, 1953.)

631*t*. R. B. Leipnik: *Von Neumann inverses and negligible sets for semi-gauged operators.*

The generalization of the Hahn-Banach Theorem to semi-gauges is used to obtain a simplified proof of the existence of von Neumann inverses for semi-gauged operators. (B is a von Neumann inverse of A in case $ABA = A$ and $BAB = B$.) The same method is applied to characterize negligible sets for inverses of centered semi-gauged operators. (Received July 16, 1953.)

632. Mark Lotkin: *The propagation of error for methods of integration.*

From both the theoretical as well as practical point of view it is of importance to be aware of the stability properties of numerical methods of integration, and the growth of error in the large. Proceeding along lines similar to those employed by H. Rademacher, H. Rutishauser, L. H. Thomas, and others there have been developed criteria for stability of a number of well known methods of integration of ordinary differential equations. The knowledge of the proper values of the pertinent characteristic equations permits the derivation of expressions for the propagation of error due to truncation and rounding. It is thus possible to obtain satisfactory estimates of the total error accumulated during an integration of a prescribed system of differential equations by means of a particular numerical method. (Received May 12, 1953.)

633. Lawrence Markus: *Asymptotically autonomous differential systems.*

A real, ordinary, first order differential system $\mathfrak{S}: dx_i/dt = f_i(x, t)$, $i = 1, \cdots, n$, in class $C^{(1)}$ in an open set of Euclidean n-space for all $t > t_0$, is asymptotic to the autonomous system $\mathfrak{S}_\infty: dx_i/dt = f_i(x)$ in case $\lim_{t\to\infty} f_i(x, t) = f_i(x)$, where the convergence is uniform on compact sets. By using the intermediate autonomous systems $\mathfrak{S}_{t^*}: dx_i/dt = f_i(x, t^*)$, for each fixed t^*, and the limit system $\mathfrak{S}_\infty$, one investigates the boundedness, periodicity, and other asymptotic properties of the solutions of $\mathfrak{S}$. In particular one generalizes the plane Poincaré-Bendixson theory. (Received July 15, 1953.)

634*t*. F. I. Mautner: *Ergodic double cosets.*

Let G be a locally compact topological group and H a closed subgroup such that the coset space G/H has an invariant measure whose total measure is finite. Assume also that G has no nonconstant finite-dimensional continuous unitary representations. Then the double cosets of H in G are ergodic in the sense that any measurable function $f(g)$ on G which satisfies (almost everywhere) $f(hgh') = f(g)$ for $h, h' \in H$, $g \in G$ is a constant. Examples: Let G be the group of 2×2 real matrices of determinant one and H the modular group, or for example H can be the fundamental group of any (compact) Riemann surface (of genus >1). More generally, let G be any of the noncompact classical (simple Lie) groups and H one of the discontinuous subgroups studied by C. L. Siegel (*Symplectic geometry*, Amer. J. Math. vol. 65 (1943) pp. 1–86). The above should be contrasted with a result of C. L. Siegel (*Discontinuous groups*, Ann. of Math. vol. 44 (1943) pp. 674–689) according to which under other conditions the double sets behave discontinuously, i.e. in the opposite manner from the above. (Received May 11, 1953.)

635t. F. I. Mautner: *Note on the Fourier inversion formula on groups.*

Properties of the generalized Peter-Weyl-Fourier transform on locally compact groups are studied in more detail. Among other things it is shown that this generalized Fourier transform maps the space $L_2(G)$ of square integrable functions on G onto a certain space of operator valued functions. This latter space can be described independently as a certain space of measurable operator valued functions. From this one obtains easily various formulations of the generalized Fourier-inversion formula. These results are needed to study the Fourier transform of more general objects than functions such as distributions on groups. This will be carried out in detail in another publication. (Received May 4, 1953.)

636t. Zeev Nehari: *On the zeros of solutions of second-order linear differential equations.*

This paper is concerned with the relations between the function $p(z)$ and the location of the zeros of the solutions of the differential equation $w''(z)+p(z)w(z)=0$, both in the case in which $p(z)$ is an analytic function of the complex variable z in the half-strip $\mathrm{Re}\,\{z\}\geqq 0$, $-1<\mathrm{Im}\,\{z\}<1$, and in that in which $p(z)$ is a continuous function of z for $z\geqq 0$. Example of the results obtained: If $\int_0^\infty x\left|p[x\pm i(1-\delta)y]\right|dx<\infty$ $(z=x+iy)$ for sufficiently small δ and $|p(x+iy)|=o(x)$ for large x and $-1<y<1$, then no solution of the equation can have an infinite number of zeros in the half-strip $x\geqq 0$, $-1+\epsilon<y<1-\epsilon$ $(0<\epsilon<1)$. (Received September 2, 1953.)

637. Emanuel Parzen: *Some conditions for uniform convergence of integrals.*

Let m be a measure on a space R, and for each $t\in T$ and $n=1, 2, \cdots$, let $f_n(x, t)$, $f(x, t)$ be functions measurable in $x\in R$. It is of some interest to determine conditions for statements of the form (1) $\int_R|f_n(x, t)-f(x, t)|\,dm\to 0$ uniformly in t as $n\to\infty$. In this paper, we define modes of uniform convergence of a family of sequences of measurable functions which permit us to obtain necessary and sufficient conditions for (1) to hold. The main notions we define are (I) $f_n(x, t)\to f(x, t)$ almost everywhere ***m*-uniformly** in t if, for every $\epsilon>0$ and every measurable set A of finite measure, as $N\to\infty$, $m\{x\in A:|f_n(x, t)-f(x, t)|>\epsilon$ for some $n>N\}\to 0$ uniformly in t, and (II) $f_n(x, t)\to f(x, t)$ in measure ***m*-uniformly** in t if, for every $\epsilon>0$, as $n\to\infty$, $m\{x: |f_n(x, t)-f(x, t)|>\epsilon\}\to 0$ uniformly in t. Most of the relations that hold in measure theory between convergence almost everywhere, convergence in measure, and convergence in mean continue to hold between (I), (II), and (1). (Received July 17, 1953.)

638. Pasquale Porcelli: *Concerning R. E. Lane's integral.*

If g is a bounded and nondecreasing function on the set of all numbers, then the ***outer* $\bar{g}$-*length*** l_gS of the number set S is the largest number k such that k

$\leq \sum_{(G)} [g(q-)-g(p+)]$ for every countable collection G of segments (p, q) covering S. Theorem. If $\epsilon>0$, $\delta>0$ and $l_g S>\delta$, then there exists a nondecreasing simple step-function h, such that $|h(x)-g(x)|<\epsilon$ for every number x and $l_h S>\delta$. By means of this theorem it is shown that if f is *g-summable* on the interval $[a, b]$, i.e., if f is the limit on $[a, b]-T$, $l_g T=0$, of a uniformly bounded sequence $\{h_n\}_{n=1}^{\infty}$ of simple step-functions and $h_n(x+)\to f(x+)$ if $g(x+)>g(x)$ and $h_n(x-)\to f(x-)$ if $g(x)>g(x-)$, then the sequence $\{\int_a^b h_n dg\}_{n=1}^{\infty}$ of Lane integrals converges to a limit $\int_a^b f dg$ depending only on f. The class of g-summable functions contains Lane's g-integrable functions, and Riemann integrable and Lebesgue summable functions ($g(x)=x$ on $[a, b]$). Integration by parts is retained in case $h_n(x)\to f(x)$ everywhere in $[a, b]$. (Received May 27, 1953.)

639t. M. H. Protter: *A singular initial value problem for a class of hyperbolic equations.*

The initial value problem for equations of the form $K(t)(u_{xx}+u_{yy})-u_{tt}+a(t)u_t=0$ is treated. The function $K(t)$ is a positive nondecreasing, continuously differentiable function, and the case $K(0)=0$ is allowed; $a(t)$ is a positive smooth function which is $O(1/t^{\alpha})$, $t\to 0$, $\alpha\leq 1$. Under these conditions the Cauchy problem is solved with initial data on the parabolic plane, $t=0$. The method used is a generalization of the method used by M. H. Martin for the wave equation. For $K(t)=1$, $a(t)=k/t$, this yields the solution of the Cauchy problem for the Euler-Darboux equation, while the case $a(t)=0$, $K(0)=0$, $K'(0)\geqq 0$ extends the results obtained by Bers for the equation $K(y)u_{xx}-u_{yy}=0$. (Received March 30, 1953.)

640. L. T. Ratner: *On approximately convex sets.*

D. H. Hyers and S. M. Ulam [Proc. Amer. Math. Soc. vol. 3 (1952) pp. 821–828] and J. W. Green [Duke Math. J. vol. 19 (1952) pp. 499–504] have considered the concept of approximately convex function. (A real-valued function $f(x)$ defined on a convex subset S of E_n, the n-dimensional Euclidean space, is said to be ϵ-convex provided that $f(kx+(1-k)y)\leq kf(x)+(1-k)f(y)+\epsilon$ for all x and y in S and for every k, $0\leq k\leq 1$.) Let M be a closed and bounded subset of E_n; $f(x)=d(x, M)$ is a convex function on the whole space if and only if M is a convex set, as is well known. In the present paper the concept of approximately convex sets is linked to that of approximately convex functions; M is said to be ϵ-convex provided that $f(x)$ is ϵ-convex. Then $d(M, HM)\leq k\epsilon$ (where HM is the convex hull of M, the distance is the Hausdorff distance between M and HM, and k depends upon the dimension of E_n). Conversely, if $d(M, HM)\leq\epsilon$ then $f(x)$ is ϵ-convex. Examples are given showing the necessity of the constant k. Approximately convex sets which also possess other properties of interest are characterized. (Received July 16, 1953.)

641t. R. M. Redheffer: *A theorem of Carlson.*

Let $F(z)=\Pi(1-z^2/\lambda_n^2)$, $\lambda_n>0$, $\Lambda(u)=$number λ's $\leq u$. With $p(v, x)=\Lambda(vx)/vx-\Lambda(x/v)/(x/v)$, condition A is $\lim_{\delta\to 0}\lim_{x\to\infty}\int_{\delta}^{1-\delta}p(v, x)dv/(1-v^2)=0$. Previously it was shown that A implies $\log^+|F(x)|=o(x)$ if $\Lambda(u)/u<M$. Now it is shown that $\log^+|F(x)|=o(x)$ implies A if $\lambda_{n+1}-\lambda_n>c>0$. The first result generalizes, and the second completes, a theorem of Carlson that $\log^+|F(x)|=o(x)$ if $\lambda_{n+1}-\lambda_n>c$ and $\lim\Lambda(u)/u$ exists. (Received February 12, 1953.)

642t. R. M. Redheffer: *Entire functions with uniformly distributed zeros.*

Let $F(x)=\Pi(1-x^2/\lambda_n^2)$ where $\lambda_1\leqq\lambda_2\leqq\lambda_3\cdots$ and let $\Lambda(u)$ be the number of λ's $\leqq u$. Suppose $|\Lambda(u)-Du|\leqq H$ for some constants $D>0$, $H>1/2$. Paley and Wiener showed $|F(x)|<C|x|^{4H-1}$ for some constant C. The author later showed $|F(x)|<C|x|^{4H-2}$, and gave a counterexample for each x. If $F(x)$ is fixed, it is now proved that $F(x)=o(|x|^{4H-2})$. Moreover, let $C(x)\rightarrow 0$. Then there is an $F(x)$ such that $F(x)>C(x)x^{4H-2}$ for an infinite sequence $x=x_i\rightarrow\infty$. The proof makes essential use of the fact that $\Lambda(u)$ is an integer. Let $|\Lambda(u)-Du|\leqq H(u)$ where $H''(u)\leqq 0$, $\lim H(u)=\infty$, $\lim H'(u)=0$. Then $\log|F(x)|\leqq G(x)$ where $G(x)\sim 4H(x)\log[x/H(x)]$. Moreover, there is a fixed function $F(x)$ and a sequence $x=x_i\rightarrow\infty$ such that $\log|F(x)|>G_1(x)$ where $G_1(x)\sim 2H(x)\log[x/H(x)]$. Finally, there is a function $H(u)$ of the above type for which the first result is best possible, and also a function for which the second is best possible. (Received February 19, 1953.)

643. R. M. Redheffer: *Sets of exponentials with finite deficiency.*

According to Paley and Wiener, the set $\{e^{\pm i\lambda_n x}\}$ has deficiency D if it becomes complete on a given interval when D but not when fewer functions $\{e^{inx}\}$ are adjoined to the set. If $D=0$ the set is exact. If λ_n is periodic, in the sense that $\lambda_{n+p}=\lambda_n+A$ for some fixed A and p, and if $\lim n/\lambda_n<\infty$, then $\{e^{i\lambda_n x}\}$ is exact on a finite interval. This results from the following generalization of a theorem of Paley and Wiener: Let $\sum 1/|\lambda_n|^2<\infty$, let $F(z)=\prod(1-z^2/\lambda_n^2)$, and let $\{e^{\pm i\lambda_n x}\}$ have finite (positive or negative) deficiency. Then the deficiency is D if and only if $\int_1^\infty x^{2D-2}|F(x)|^2<\infty$, $\int_1^\infty x^{2D}|F(x)|^2dx=\infty$. There exists a set λ_n and a set λ_n' such that $|\lambda_n-n|<2$, $|\lambda_n'-n|<2$, $\lambda_n<\lambda_n'$, $\{e^{\pm i\lambda_n x}\}$ is complete, and $\{e^{\pm i\lambda_n' x}\}$ is not complete. One can even have $\lambda_n<\lambda'_{n-1}$. One can construct sets exact on intervals D and D' such that the combined set has arbitrarily large deficiency on $D+D'$. (Received February 12, 1953.)

644. M. S. Robertson: *Schlicht solutions of the Sturm-Liouville equation.*

Let $z^2q(z)=\sum q_n z^n$ be nonconstant and regular in $|z|<1$, q_n real, $q_0\leqq 1/4$. Let $\Re z^2q(z)$ assume its maximum on $|z|=r$, for every $r<1$ at $z=r$. A unique universal constant $A=A(q)$ associated with $q(z)$ is determined so that the unique solution $W_A(z)=\sum\beta_n z^{n+\beta}$, $\beta_0=1$, of $W''+(A(q(z)-q_0z^{-2})+q_0z^{-2})W=0$, corresponding to the larger root β of the indicial equation, is such that, when raised to the power $1/\beta$, it is schlicht and star-like in $|z|<1$, the radius of univalency being one when $0<A<\infty$. For constants $C>A$ the corresponding solutions are not schlicht in $|z|<1$. Let $z^2p(z)$ be regular in $|z|<1$, γ real, $|\gamma|\leqq\pi/2$, for which, in $|z|<1$, $\Re(e^{i\gamma}z^2p(z))\leqq\cos\gamma(A(|z|^2q(|z|)-q_0)+q_0)$, $A=A(q)$. Let $W(z)=\sum a_n z^{n+\alpha}$, $a_0=1$, be the unique solution of $W''+p(z)W=0$, corresponding to the root α of the indicial equation with the larger real part. Then $W(z)$, raised to the power $1/\alpha$, is schlicht and spiral-like in $|z|<1$. The constant $A(q)$, constructively determined, is the largest possible one for a given $q(z)$. (Received July 10, 1953.)

645*t*. Walter Rudin: *A theorem on maximum modulus algebras.*

Let K be the closure of a plane domain D whose boundary C consists of n simple closed nonintersecting curves. Let A be an algebra of complex-valued functions f, continuous on K, such that $|f(z)|$ attains its maximum on C. Theorem I: If A contains all functions analytic and single-valued on K, then every member of A is analytic in D. This extends a result previously obtained for simply connected domains (Bull. Amer. Math. Soc. vol. 58 (1952) p. 646). The proof depends on the representation of linear functionals as measures, and on Theorem II: Suppose C consists of n analytic curves. If μ is a complex measure on C such that $\int_C g(t)d\mu(t) = 0$ whenever g is analytic and single-valued on K, then $d\mu(t) = f(t)dt$, where $f \in L_1$ on C; the function $F(s) = (1/2\pi i)\int_C d\mu(t)/(t-s)$, which is analytic in D, has nontangential boundary values f almost everywhere on C. Theorem II is proved by induction on n. (Received May 7, 1953.)

646. Charles Saltzer: *Discrete potential theory in two dimensions.* Preliminary report.

The results below are based upon the finite difference analogues of Green's first and second identities given by Courant, Friedrichs, and Lewy (Math. Ann. (1928)), and the fundamental solution of the five point Laplace difference equation given by McCrea and Whipple (Proc. Roy. Soc. Edinburgh (1939)), and Stohr (Math. Nach. (1950)). In this paper a theory of discrete harmonic functions, i.e. functions satisfying the Laplace difference equation, is developed within the framework of the difference equation which parallels classical potential theory. The analogues of Green's third identity, double and simple layer potentials as well as an analogue of the discontinuity of a double layer potential at the boundary are given. The last result is used to reduce the boundary value problems of the first and second kind to finite difference analogues of the integral equations of potential theory. The associated Neumann-Poincaré series are shown to converge for all regions, finite, infinite, simply or multiply connected provided the boundaries are finite. Regularity at infinity is defined and multiple-valued solutions are given. Generalizations and applications are indicated and existence and uniqueness theorems are given. (Received September 3, 1953.)

647. Francis Scheid: *Difference methods and the initial value problem for a parabolic equation of arbitrary even order.*

In this paper it is proved that the differential equation $u_t = \sum_0^{2p} a_n(x, t)u^{(2p-n)} + d(x, t)$ has a solution for $t \geq 0$, $-\infty < x < \infty$, satisfying $u(x, 0) = f(x)$, provided $f(x)$, $d(x, t)$, and all $a_n(x, t)$ are of class C^{4p} (uniformly continuous and bounded together with their first $4p$ derivatives with respect to x) and their first time derivatives are C^0, and where $a_0 = (-1)^p A_0$ with $A_0 \geq \delta > 0$. The solution is of class C^{2p}. A uniqueness theorem is also obtained. Fritz John has treated in a fundamental paper (Comm. on Pure and Appl. Math., May, 1952) the case $p = 1$. He obtained the solution as the limit of a convergent difference process. His methods prove to be adequate for the general case and are closely followed. Using the result here obtained a reduction of hypothesis will be affected in a second paper. (Received May 14, 1953.)

648t. J. T. Schwartz: *Analytic perturbations of operators in Banach spaces.*

Theorem: Let $A(s)$ be an analytic function of the complex variable s defined for s in a domain D and having values in the algebra $B(X)$ of bounded operators on a Banach-space X. Suppose that $0 \in D$, and that $A(0)$ has an eigenvalue λ of multiplicity m such that some circle $C_\delta = \{\mu \mid |\lambda - \mu| < \delta\}$ contains no points of $\sigma(A(0))$ other than λ. Then there exists an $\epsilon > 0$ such that for $|s| < \epsilon$, $\sigma(A(s)) \cap C_\delta$ consists of a finite number of points $\lambda_1(s) \cdots \lambda_k(s)$, $k \leq m$. Each $\lambda_i(s)$ can be expressed as a fractional power series in s, and each corresponding projection $E_i(s)$ as a fractional Laurent series in s with a singularity of finite order at $s = 0$. *Corollary*: If $A(s)$ is normal for all real $s \in D$, then $\lambda_i(s)$ and $E_i(s)$ depend analytically on s. The proof will appear in a forthcoming book on spectral theory. (Received May 15, 1953.)

649t. V. L. Shapiro: *Uniqueness of double trigonometric integrals.*

Let $c(u, v)$ in L^2 on any bounded domain be $o((u^2+v^2)^{-\epsilon})$, $\epsilon>0$. Suppose the double trigonometric integral $\int_{E_2} e^{i(xu+yv)}c(u, v)\,du\,dv$ is $(C, 1)$ circularly summable to $f(x, y)$. If f satisfies a Lipschitz condition of order α on every bounded domain (α depending on the domain), it is then shown in this paper that $(4\pi^2)^{-1}\int_{E_2} e^{-i(xu+yv)}f(x, y)\,dx\,dy$ is $(C, 1)$ circularly summable to $c(u, v)$ almost everywhere. Using the fact that the integral is equisummable with a trigonometric series, the proof is brought about by showing that the generalized Laplacian of the Riemann function associated with the integral is equal to f. (Received June 22, 1953.)

650. A. L. Shields: *Hausdorff dimension of a vector sum.* Preliminary report.

Let A and B be two sets of real numbers. Then their vector sum, $A+B$, is the set of all numbers $a+b$, a in A, b in B. Let Hd A denote the Hausdorff dimension of A. Salem conjectured (1) Hd $(A+B) \leqq$ HdA + Hd B. This is unproven. In an attempt to prove it we are led to the following. Let m be a non-negative, totally finite measure on the Borel sets on the line. Let $a(m) = \sup \alpha$ such that $\int\int (1/|x-y|^{\alpha})dm(x)dm(y) < \infty$. Then the following two statements would imply Salem's conjecture: (2) $a(m * n) \leqq a(m)+a(n)$, where $m * n$ denotes convolution; (3) given a measure 1 on $A+B$ (i.e. $A+B$ complement is a null-set for 1), then there exist measures m and n on A and B respectively, such that $(m * n)(E) \leqq 1(E)$ for every Borel set E. (2) is proven for the special case when m and n are the same. (Received July 20, 1953.)

651t. Maurice Sion: *On the existence of functions having given partial derivatives on a curve.*

Let m and n be positive integers. Let $A \subset E_n$ be a homeomorphic image of the unit interval. If functions on A, satisfying certain conditions depending on m and suggested by Taylor's formula, are given, then a sufficient condition on A is found for the existence of a function f of class $C^{(m)}$ on E_n whose partial derivatives through order m agree with the given functions on A. The condition depends on m in such a way that as m increases more curves satisfy it. For any m there are curves that satisfy this condition and yet possess infinite Carathéodory $(m-1)$-measure. If $m=1$, the condition reduces to having the ratio of the length of arc to chord between any two points uniformly bounded. These results are applied to certain curves and functions originally defined by H. Whitney to show that if m and n are integers, $n \geqq 2$, then there exists a function f of class $C^{(n+m-1)}$ on E_n and a homeomorphic image P of the unit interval, $P \subset E_n$, such that: all the partials of f of order greater than m vanish everywhere on P; all the partials of f vanish at one point of P; and yet f is not constant on P. (Received July 13, 1953.)

652. K. T. Smith: *Positive functions in spaces with reproducing kernel.*

The natural order determined by the positive functions in a functional class with reproducing kernel is discussed. The results obtained relate positivity of the kernel to properties of the functional class. Applications to Green's functions of second order elliptic partial differential equations are given. (Received July 17, 1953.)

653. H. H. Snyder: *Monogenic functions and residue theory in a harmonic algebra.*

The product $(1, i)\times(1, j)$ of the ordinary complex algebra with itself is a linear associative commutative algebra over the real field $(1, i, j, k)$. Multiplication is defined by $i^2=j^2=-1=-k^2$, $ij=k$, $ik=-j$, $jk=-i$. Let $w=x+iay+jbz+ku$ (where $a^2+b^2=1$) denote the variable whose range is the algebra. Developing ideas of P. W. Ketchum (Trans. Amer. Math. Soc. vol. 30), monogenic functions $f(w)=U_1+iU_2+jU_3+kU_4$, where the U_k are real-valued functions of the real variables x, y, z, u, are defined. A set of linear first order partial differential equations in the U_k which characterize monogenicity of $f(w)$ are obtained. *Theorem*: Let u be fixed but arbitrary, and let R be a region of three-space in which $f(w)$ is monogenic. Then $\nabla^2 U_k=0$ throughout R (where $\nabla^2(\ )=(\ )_{xx}+(\ )_{yy}+(\ )_{zz}$). Elementary (monogenic) functions of w are defined. Two analogues of the Euler polar representation are obtained. Singularities of functions (which have them) are found to consist of two lines through the origin. Line and surface integration are defined and generalizations of the Cauchy integral formulas obtained. The residue of a function at a singular line is defined, and applications are made to the evaluation of real line integrals in three dimensions. Geometrical properties are found which relate this class of harmonic functions to the real and imaginary parts of an ordinary analytic function $f(x+iy)$. (Received July 17, 1953.)

654. D. L. Thomsen, Jr.: *On an integral containing one small and two large parameters.*

The integral $J=\int_0^a \exp\left[-h\phi(t)+k\psi(t)\right]dt$ $(a>0)$ is considered as h, $k\to+\infty$ and $a\to 0^+$. The functions $\phi(t)$ and $\psi(t)$ satisfy suitable continuity conditions, and in addition it is assumed $\phi(0)=\phi'(0)=\psi(0)=0$, $\phi''(0)>0$, and $k=o(h)$. Then considering the various rates with which $a\to 0^+$, with which h, $k\to+\infty$, and the sign of $\psi'(0)$, the integral J is shown to be asymptotic to suitable forms of the error function. (Received July 17, 1953.)

655. H. L. Turrittin: *Convergent solutions for linear differential equations with analytic coefficients.*

The author has shown in the most general case that the divergent asymptotic series solutions representing a fundamental set of solutions corresponding to a singular point of a system of linear homogeneous differential equations with analytic coefficients can be summed and the divergent series solutions replaced by convergent generalized factorial series, thus completing the work of W. J. Trjitzinsky, Trans. Amer. Math. Soc. vol. 37 (1935) pp. 80–119. First the linear matrix differential equation is reduced to a suitable canonical form, an extension of that given by M. Hukuhara, J. Fac. Sci. Hokkaido Imp. Univ. (I) vol. 5 (1937) pp. 123–166. The analysis then proceeds in steps paralleling those given by Trjitzinsky, the computations always being carried out in matrix notation to minimize the algebraic complications; and convergence of the factorial series is finally proved by estimating the rate of growth of solutions of a related system of integral equations. For example Trjitzinsky's problem, p. 118, is easily handled by a change of independent variable $x=s^2$, whereupon it is found that his equation has a convergent generalized factorial series solution of the form $1+\sum_{s=0}^{\infty} a_s/[x^{1/2}(x^{1/2}-\gamma)(x^{1/2}-2\gamma)\cdots(x^{1/2}-s\gamma)]$ where the a's and γ are suitably chosen constants and x is limited to the proper curvilinear sector extending to infinity in the complex x-plane. (Received July 14, 1953.)

656. J. G. Wendel: *On uniqueness of Fourier-Stieltjes transforms.*

$G=\{g\}$ is a locally compact abelian group, $G^*=\{g^*\}$ its character group, $L=\{x\}$

its Banach algebra of complex-valued integrable functions, M its algebra of measures m of bounded variation $\|m\|$; as usual, L is identified with the subset of absolutely continuous $m\in M$. The well known fact that m is uniquely determined by its Fourier-Stieltjes transform $m(g^*)=\int g^*(g)m(dg)$ is here expressed algebraically as follows. Let M be endowed with any topology in which it becomes a topological algebra containing L as a dense subset, and in which the functions $m(g^*)$ are continuous. Then the *closed* maximal ideals of M are precisely the sets $M(g^*)=\{m \mid m(g^*)=0\}$. One such topology is: $m_\alpha \to m$ means $\|xm_\alpha - xm\| \to 0$, $x\in L$, and $\sup_\alpha \|m_\alpha\| < \infty$, where for $x\in L$, $m\in M$ we define $xm\in L$ by $(xm)(g)=\int x(g-h)m(dh)$. (Šreider [Mat. Sbornik vol. 27 (69) (1950) pp. 297–318)] viewed M as a Banach algebra and showed that it has a profusion of additional maximal ideals (all necessarily closed in the norm topology), so many more that the $M(g^*)$'s are neither dense in, nor the boundary of, the space of all maximal ideals.) (Received July 24, 1953.)

657. F. B. Wright: *A class of factors of type* II$_1$.

Let A be an AW^* algebra (Kaplansky, Ann. of Math. vol. 53 (1951) pp. 235–249) of finite class, and suppose that A has a trace defined in it. Let M be any maximal two-sided ideal of A. If A is of type II$_1$, then A/M is an AW^* factor of type II$_1$ for every M. In case A is of type I (in which case A necessarily has a trace), then A/M is an AW^* factor of type I(n) for all M except possibly for a closed nowhere-dense set in the space of maximal two-sided ideals of A. This set is empty if and only if A is the direct sum of a finite number of homogeneous algebras of type I. In case this set is not empty, then A/M is an AW^* factor of type II$_1$ for every M in the set. This shows the existence of a class of AW^* factors which are (1) not a priori weakly closed, and (2) not a priori related to the few previous known examples of such factors. (Received July 17, 1953.)

658. A. D. Ziebur: *On a double eigenvalue problem.*

In the study of the rigidity of surfaces of revolution one encounters the following "double" eigenvalue problem [E. Rembs, Math. Zeit. vol. 35 (1932) pp. 529–535]: Find those numbers $a>0$ for which there exists a not identically vanishing function y and an integer $m^2-1>0$ such that y is a solution of the system (*) $(py')'=\lambda p^2q(a+r)^{-1}y=0$, $y(0)=y(b)=0$, with $\lambda=m^2-1$. Here p, q, and r are analytic, non-negative functions on $(0, b)$ and p has a pole of the first order at the end points. Rembs demonstrated the existence of a countably infinite set of such "exceptional" values a, and Minagawa and Radó [to be published in Math. Zeit.] showed that these values are at most countable in number. For a fixed a, (*) is a regular singular Sturm-Liouville eigenvalue problem and the application of known theorems about the eigenvalues $\lambda_n(a)$ leads to the conclusion that the a's satisfying the original problem form a dense set on the positive real axis. (Received June 11, 1953.)

APPLIED MATHEMATICS

659. R. C. F. Bartels (p) and A. C. Downing, Jr.: *On surface waves generated by travelling disturbances with circular symmetry.*

Kelvin's ship wave problem for the surface waves produced by a distribution of pressure moving with constant velocity on the surface of a body of water with infinite depth is formulated in terms of a singular integrodifferential equation. It is shown that the surface elevation of the wave, $\eta(x, y)$, satisfies the equation $\eta(x, y)+p(x, y)+(1/2\pi)\iint_{-\infty}^{+\infty}(1/((x-x')^2+(y-y')^2)^{1/2})\partial^2/\partial x'^2\eta(x', y')dx'dy'=0$ where $p(x,y)$

is proportional to the surface pressure. To this is added the condition: $\psi(x, y)\to 0$ as $x\to+\infty$ and $y\to\pm\infty$. The solution of the equation for ψ, taken by itself, is not unique; the solutions of the homogeneous equation correspond to the "free" surface waves. However, for the pressure distributions with circular symmetry that are treated, the boundary condition is sufficient to specify the solution completely. The expression for the predominant part of the surface wave produced behind the moving source of disturbance is obtained with the aid of the Abelian theorems for the complex inversion integral in the theory of Laplace transforms. (Received July 13, 1953.)

660t. Nathaniel Coburn: *The intrinsic form of the characteristic relations in the steady supersonic motion of a fluid.*

The purpose of this paper is to write the characteristic relations for the steady supersonic motion of a fluid in terms of curvatures associated with the characteristic manifolds, and the rates of change of the magnitude of the velocity and that of the sound speed with respect to displacements along the normal to the characteristic surfaces and along two independent directions in these manifolds. This is done for steady three-dimensional non-isentropic rotational flows. Canonical forms of these relations are obtained and application is made to plane rotational flows. Here it is shown that simple waves (straight line bicharacteristics) exist for the limiting case when the Mach number of the flow is one. Then, the stream lines are orthogonal to the straight line bicharacteristics, and the magnitude of the velocity and the sound speed are constant along a stream line. For general plane rotational flows, necessary and sufficient conditions for the existence of simple waves are obtained. (Received July 16, 1953.)

661. C. L. Dolph, J. E. McLaughlin, and Immanuel Marx (p): *Symmetric linear transformations and complex quadratic form.*

MacFarlane (Cambridge Philos. Soc. vol. 43 (1947)) devised a nonlinear variational approach to the complex eigenvalue problem occurring in the theory of a anomalous propagation. This paper treats the finite-dimensional aspects of the corresponding linear Rayleigh-Ritz procedure. In particular, a spectral theorem is derived for a class of symmetric linear transformations on an n-dimensional vector space over an algebraically closed field of characteristic $p\neq 2$. If the vector space is nonsingular with respect to a symmetric bilinear form, a necessary and sufficient condition for such a theorem is that the transformations have minimum functions with simple roots. In physical applications the complex number field is used and the eigenvalues have the additional property that imaginary parts are restricted to one half-plane. Under these circumstances the imaginary parts are characterized as saddle points in suitably chosen subspaces in a manner similar to the minimum characterization of the eigenvalues in the case of real fields. (Received July 16, 1953.)

662. R. L. Evans: *Errors in the asymptotic solution of linear ordinary differential equations.*

A new method of estimating the error in an asymptotic approximation was suggested by a reading of T. Thorkelsson's *Divergent power series* (Reykjavik, 1934). Consider (1) $L(y)=\sum_{\nu=0}^{n} p_\nu(x)y^{(n-\nu)}=0$, where the p_ν's are polynomials such that for one solution $y(x)\sim\sum_{\alpha=0}^{\infty} b(\alpha)x^{\rho-\alpha}$ in a sector S. Then the recurrence relation for the b's can be written as (2) $\sum_{\nu=0}^{m} q_\nu(\alpha, \rho)b(\alpha+m-\nu)=0$. Let $u(x)=\sum_{\alpha=0}^{N-1} b(\alpha)x^{\rho-\alpha}$

and (3) error $v(x)=y(x)-u(x)$. Then if $\rho=\sigma+N$ and $b(\alpha+N)=B(\alpha)$, (2) becomes $\sum_{\nu=0}^{m} q_\nu(\alpha,\sigma+N)B(\alpha+m-\nu)$ which corresponds to a differential equation (4) $M(v)=0$. From (1), (3), and (4) one obtains (5) $M(v)-L(v)=L(u)$ which is a differential equation of order $(n-1)$ having $v(x)$ as a particular solution $\ni \lim_{x\to\infty \text{ in } S}|x^{N-1}v(x)|=0$. When $n=2$, (5) is readily solved but when $n>2$ it is usually better to use the author's other error estimate (Proc. Amer. Math. Soc. vol. 4 (1953)). The error depends on the last m terms of $u(x)$ and is not necessarily least when $b(N-1)x^{\rho+1-N}$ is the smallest term. For (6) $L(y)=x^4y''+(A_1x+A_2)x^2y'+(A_3x^2+A_4x+A_5)y=0$ $(A_1\neq 0)$ a suitable contour integration of (5) gives $|v(x)|<|(N-\rho-1)\cdot((N-\rho)-A_2(N-1))/N(A_1-2\rho-N+1)|\cdot|T_{N-1}|+|A_4/Nx(A_1+2\rho-N-1)|\cdot|T_{N-1}|+|A_5/Nx(A_1+2\rho-N+1)|\cdot|T_{N-2}|$ where the T's are the last terms in $u(x)$. Equation (6) has the confluent hypergeometric equation and a modified Bessel equation as special cases involving $A_5=0$. This work was supported by the Office of Ordnance Research. (Received July 13, 1953.)

663. G. E. Forsythe: *Arbitrarily close lower bounds for the fundamental frequency of a convex membrane.*

Let R be a convex plane region bounded by a piecewise sufficiently smooth curve C. Let λ be the least number such that $\Delta u=u_{xx}+u_{yy}=-\lambda u$ in R, with $u=u(x,y)=0$ on C. (Subscripts x, y denote differentiation.) Let a square net N_h of mesh-width h be formed by parallels to the axes. Let R_h denote the nodes of N_h inside R; let C_h be the intersection of N_h with C. A self-adjoint finite-difference operator Δ_h approximating Δ is introduced; it is the symmetric part of the operator (65.1) in W. E. Milne, *Numerical solution of differential equations*, Wiley, 1953. Let λ_h be the least number such that $\Delta_h v+\lambda_h v=0$ in R_h, where $v=v(x,y)$ is defined for $(x,y)\in R_h\cup C_h$ and vanishes on C_h. Theorem: As $h\to 0$, $\lambda_h/\lambda\leq 1-h^2(A+B)/D+o(h^2)$. Here $A=\iint_R(u_{xx}^2+u_{yy}^2)dxdy$; $B=\int_C u_n^2\sin^2 2\tau d\tau$; $D=12\iint_R(u_x^2+u_y^2)dxdy$; u_n is the normal derivative of u; τ is the angle between the tangent to u and the x axis; u is the fundamental solution of (*). Thus ultimately $\lambda_h\uparrow\lambda$. This extends the result for certain polygonal regions R announced and discussed in Bull. Amer. Math. Soc. Abstract 59-4-509. (Received July 13, 1953.)

664t. Isidor Heller: *On the problem of shortest path between points.* I.

The $n!$ closed paths connecting n given points are represented by the set P_n of n by n permutation matrices. These, interpreted as points in n^2-space, are the extreme points of the convex set of doubly stochastic matrices (Birkhoff, von Neumann). In problem above only the subset C_n consisting of the $(n-1)!$ cycles of order n is admitted. C_n has dimension $(n-1)^2-n$ so that the 6 points of C_4 form a 5-dimensional simplex (H. Kuhn in a letter to author). A first objective was to determine the extreme hyperplanes of the convex of C_5, which is an 11-dimensional polytope in 25-space with 24 vertices, the main difficulty opposing straightforward computation being the nearly 2.5 million possibilities of choosing 11 points out of 24. Results: The convex of C_5 is characterized by a nonredundant system of 224 hyperplanes of the following 6 types. (1) $X_{ij}\geq 0$ $(i\neq j)$; (2) $X_{ii}=0$; (3) sum of any row $=$ sum of any column $=1$ (one of the $2n$ equations to be omitted in order to avoid redundancy); (4) $X_{ij}+X_{ji}\leq 1$; (5) $X_{ij}+X_{jr}+X_{rs}-X_{st}-X_{tr}\leq 1$ for distinct (i,j,r,s,t); (6) $2X_{ij}+2X_{ji}-X_{ir}+X_{jr}-X_{si}+X_{sj}-X_{rs}\leq 2$ for distinct (i,j,r,s). (Received June 24, 1953.)

665t. Isidor Heller: *On the problem of shortest path between points.* II.

For general $n \geq 5$ the conditions characterizing doubly stochastic matrices with zeros in the main diagonal and the condition that the elements of any principal minor of order $k \leq n/2$ should sum to at most $k-1$ represent a relatively small number of the extreme hyperplanes of the convex of C_n. More numerous classes are obtained through a relation transforming certain types of hyperplanes of the cases $m < n$ into such of the case $m = n$; so that for instance the 120 hyperplanes of type (6) of the convex of C_5 (see paper above) determine 1800 extreme hyperplanes of the convex of C_6. The problem of shortest closed path connecting n points and the problem of shortest nonclosed path (with distinct beginning and end point) connecting $n-1$ points are equivalent in the sense that there exists a nonsingular linear transformation mapping the convex of all possible paths of the first kind onto the convex of all possible paths of the second kind. (Received June 22, 1953.)

666t. M. Z. Krzywoblocki: *Generalization of Coburn's "independent scalars" in homogeneous turbulence to compressible media.*

Coburn furnished a method for the construction of independent scalars in various types of homogeneous turbulence in incompressible fluids. By definition, these scalars must satisfy, in addition to other conditions, continuity relations identically, that is, these tensors must have zero divergence. Briefly, they are solenoidal tensors. But, as was proved by the author in many of his papers, the tensors in the theory of turbulence in compressible media are not any longer solenoidal, and their divergence must be equal "identically" to some given function. The present note discusses the changes which must be introduced into Coburn's paper in order to obtain a generalization of his method of the construction of independent scalars for homogeneous turbulence in compressible media. (Received July 6, 1953.)

667. F. M. Reza: *Some new applications of the theory of a polynomial in a complex variable to electrical network theory.*

The notions of derivative of a polynomial and location of its critical points have seldom been used in electrical engineering problems. The object of this paper is to provide a new connection between the mathematics of polynomials and the electrical network theory. The following two key theorems serve this purpose by connecting the class of positive real functions (Brune functions) used in electrical network theory, to the theory of the location of critical points of polynomials in a complex variable. Theorem 1: If $P(s)/Q(s)$ represents the driving-point impedance of a one terminal-pair lossless network (or a two-element type dissipative network), then $(d^nP(s)/ds^n)/(d^nQ(s)/ds^n)$ will also represent a one terminal-pair lossless network (or a two-element type dissipative network). Theorem 2: The logarithmic derivative of any Hurwitz polynomial is a positive real function. Network configurations corresponding to the above theorems are given. To demonstrate some introductory uses for the above theorems, a lemma of Laguerre is first proved by electrical analogy; then some formulas are discussed for generating RC polynomials from a given RC polynomial. (Received June 29, 1953.)

668t. H. E. Salzer: *New formulas for facilitating osculatory interpolation.*

Hermite's osculatory interpolation formula employing the function $f(x)$ and its derivative $f'(x)$ at n equally spaced arguments $x_i = x_0 + ih$, i ranging from $-[(n-1)/2]$ to $[n/2]$, is expressible in the form (1) $f(x_0+ph) = \sum_i (\{L_i^{(n)}(p)\}^2\{1-2L_i^{(n)\prime}(i)(p-i)\}f_i$

$+\{L_i^{(n)}(p)\}^2(p-i)hf_i')+R_{2n}(p)$, where $x=x_0+ph$, $f_i=f(x_i)$, $f_i'=f'(x_i)$, $L_i^{(n)}(p)=\prod'^{[n/2]}_{j=-[(n-1)/2]}(p-j)/\prod'^{[n/2]}_{j=-[(n-1)/2]}(i-j)$ ($j=i$ absent from $\prod'$) and $R_{2n}(p)=f^{(2n)}(\xi)h^{2n}\{\prod^{[n/2]}_{j=-[(n-1)/2]}(p-j)\}^2/(2n)!$ for $x_{-[(n-1)/2]}\leqq\xi\leqq x_{[n/2]}$. Formula (1) is very much more accurate than the corresponding n-point Lagrangian formula and considerably more accurate than even the $2n$-point Lagrangian formula for x_i at intervals of h. Also (1) is specially suited for interpolation in many functions (e.g., Bessel, probability) that are tabulated with their derivative. To avoid the tremendous amount of labor in calculating the coefficients of f_i and f_i' in the form given in (1), it is shown that we may express (1) as (2) $f(x_0+ph)\sim\sum_i(\alpha_i f_i+\beta_i hf_i')/\sum\alpha_i+R_{2n}(p)$, where $\alpha_i=a_i/(p-i)^2+b_i/(p-i)$, $\beta_i=a_i/(p-i)$, and where $a_i=k(n)/\{\prod'^{[n/2]}_{j=-[(n-1)/2]}(i-j)\}^2$, $b_i=-2L_i^{(n)\prime}(i)a_i$, the $k(n)$ being any suitably chosen constant of proportionality depending only upon n (here chosen to make a_i and b_i integers). The exact values of a_i and b_i are given for $n=2(1)11$, $i=-[(n-1)/2]$ to $[n/2]$, so that Hermite's formula can be applied exactly for any polynomial up to the 21st degree. A schedule gives approximate upper bounds for the coefficients of $f^{(2n)}(\xi)h^{2n}\sim\Delta^{2n}f(x)$ in $R_{2n}(p)$. (Received May 15, 1953.)

669. H. E. Salzer: *Orthogonal polynomials arising in the numerical evaluation of inverse Laplace transforms.*

In finding $f(t)$, the inverse Laplace transform of $F(p)$, where (1) $f(t)=(1/2\pi j)\int_{c-j\infty}^{c+j\infty}e^{pt}F(p)dp$, the function $F(p)$ may be either known only numerically or too complicated for evaluating $f(t)$ by Cauchy's theorem. When $F(p)$ behaves like a polynomial without a constant term, in the variable $1/p$, along $(c-j\infty, c+j\infty)$, one may find $f(t)$ numerically using new quadrature formulas (analogous to those employing the zeros of the Laguerre polynomials in the direct Laplace transform). Suitable choice of p_i yields an n-point quadrature formula that is exact when ρ_{2n} is any arbitrary polynomial of the $(2n)$th degree in $x=1/p$ without a constant term, namely: (2) $(1/2\pi j)\int_{c-j\infty}^{c+j\infty}e^p\rho_{2n}(1/p)dp=\sum_{i=1}^n A_i^{(n)}\rho_{2n}(1/p_i)$. In (2), $x_i=1/p_i$ are the zeros of the orthogonal polynomials $p_n(x)=\prod_{i=1}^n(x-x_i)$ where (3) $(1/2\pi j)\cdot\int_{c-j\infty}^{c+j\infty}e^p(1/p)p_n(1/p)(1/p)^idp=0$, $i=0, 1, \cdots, n-1$, and $A_i^{(n)}$ correspond to the Christoffel numbers. The normalization $P_n(1/p)=(4n-2)(4n-6)\cdots 6p_n(1/p)$, $n\geqq 2$, produces all integral coefficients. $P_n(1/p)$ is proved to be $(-1)^ne^{-p}p^nd^n(e^p/p^n)/dp^n$. The normalization factor is proved, in three different ways, to be given by (4) $(1/2\pi j)\int_{c-j\infty}^{c+j\infty}e^p(1/p)[P_n(1/p)]^2dp=(-1)^n/2$. Proofs are given for the recurrence formula (5) $(2n-3)P_n(x)=[(4n-2)(2n-3)x+2]P_{n-1}(x)+(2n-1)P_{n-2}(x)$, for $n\geqq 3$, and the differential equation (6) $x^2P_n''(x)+(x-1)P_n'(x)-n^2P_n(x)=0$. The quantities $p_i^{(n)}$, $1/p_i^{(n)}$, and $A_i^{(n)}$ were computed, mostly to $6S-8S$, for $i=1(1)n$, $n=1(1)8$. (Received June 4, 1953.)

670*t*. H. H. Snyder: *A class of Galerkin vectors.* Preliminary report.

Let $\bar\rho=i\xi+j\eta+k\zeta$ be the displacement vector of a point in the interior of a perfectly elastic body, and let $\bar K$ be the vector body force. The equation to determine $\bar\rho$ is $(\nabla^2+a\nabla\,\mathrm{div})\bar\rho+b\bar K=0$ (cf. Westergaard, *Theory of elasticity and plasticity*), where a and b are constants of the material. Galerkin and Papcovitch have shown that $\bar\rho=b[(1-\mu)\nabla^2-(1/2)\nabla\ \mathrm{div}]\bar F$ provided that $\nabla^4\bar F+\bar K/(1-\mu)=0$. If $K=0$, then $\bar F=if_1+jf_2+kf_3$ is a biharmonic vector function, and hence the $f_i(x, y, z)$ are biharmonic scalar functions. An algebra A is defined such that the components of monogenic functions over A are scalar biharmonic functions. Let V be the set of all vectors in three dimensions. Functions $\bar G$: $A\rightarrow V$ are defined in such a way that the $\bar G$ are all biharmonic vector functions. A theory of functions for this class of Galerkin vectors is developed, and the three-dimensional states of stress defined by them are analyzed. (Received July 17, 1953.)

671. W. S. Snyder: *A method of obtaining bounds for the solution of certain functional equations.*

If a Volterra integral equation has a non-negative kernel and if iteration decreases a function g at each point, then g and its iterate are upper bounds for the solution of the integral equation. An estimate for the difference of g from the solution is obtained. Various methods of obtaining such upper bounds are considered. Similar results for lower bounds are immediate and the method is extended to certain types of differential-difference equations. (Received July 20, 1953.)

672. Domina E. Spencer: *Field theory and the new electrodynamics.*

A single Galilean-relativistic equation for the force between moving charges has been derived. This equation provides the basis of a new electrodynamics which treats adequately the principal results of Maxwell's theory while obviating the necessity of introducing a theory of special relativity. The implications of the equation for field theory and the introduction of the scalar and vector potential are discussed in this paper. (Received July 17, 1953.)

673t. G. L. Spencer, II: *Existence of Taylor-Maccoll flow and related inequalities.*

The Taylor-Maccoll equations for the flow about a right circular cone immersed without yaw in a uniform flow such that an attached shock wave is formed are considered. From the local existence of solutions of the equations which meet the conditions on the cone the existence in the large of solutions which also meet the shock conditions is obtained provided that the angle θ, the normalized velocity u on the body, and the gas constant γ satisfy the inequalities $[\gamma-1+2\sin^2\theta]\geqq\gamma-1$ and $(\gamma-1)u+(\gamma+1)(1-u)^{1/2}\operatorname{arc}\cos\theta\geqq(\gamma+1)\theta(1-u^2)^{1/2}$. This work was sponsored by Army Office of Ordnance Research Contract No. DA-36-034-ORD-966. (Received July 10, 1953.)

674. G. L. Spencer, II: *The compressible flow about a pointed body of revolution of curved profile.*

The equations and boundary conditions for the rotational flow about a pointed body of revolution of curved profile which is immersed without yaw in a uniform flow such that an attached shock wave is formed are considered. If the body profile given in polar coordinates by $\theta=B(r)$ is cone-like in the sense that $B^{(2n-1)}(0)=0$, $n=1, 2, \cdots$, and certain inequalities are satisfied, then series for the velocity components, the entropy functions, and the shock profile are obtained. These series constitute a formal solution since it can be shown that if they converge they represent a solution. This work was sponsored by Army Office of Ordnance Research Contract No. DA-36-034-ORD-966. (Received July 10, 1953.)

GEOMETRY

675t. Louis Auslander: *Curvature in Finsler geometry.*

If a manifold M has a Finsler metric F, then the sphere bundle (space of line elements) S over M has a natural induced Riemannian metric R. The metric R has the following properties: (a) the distance between any two points in S is greater then or equal to the distance between the projections of these points in M; (b) by associating with a geodesic its tangent directions, one gets a curve in S which is a geodesic in S with respect to R and which has the same length in S as in M. By the use of a satisfactory definition of sectional and mean curvature one can prove the following: (1) A generalization of Myers' theorem holds, i.e., a complete n-dimensional Finsler manifold whose mean curvature is $\geqq e^2 > 0$ is compact and has diameter $\leqq \pi/e$. From this it follows that the fundamental group of M is finite. (2) A generalization of Synge's Theorem holds, i.e., an even-dimensional Finsler metric of positive sectional curvature is simply connected. (3) Let M be a complete simply connected manifold with sectional curvature $\leqq 0$; then M is homeomorphic to Euclidean space. This definition of negative curvature agrees with the one given by Busemann in Acta Math. vol. 80, pp. 259–310. (Received April 23, 1953.)

676t. L. M. Blumenthal: *On an extension of a theorem of Jordan and von Neumann.*

The theorem referred to proves that euclidean spaces of finite or infinite dimensions are characterized among all complete normed linear spaces L by the validity of the norm relation $\|f+g\|^2+\|f-g\|^2=2(\|f\|^2+\|g\|^2)$, $f, g \in L$. The relation is metrically equivalent in L to the following (feeble euclidean) four-point property: if $p, q, r \in L$ with q a middle-element of p, r, then for each $s \in L$, p, q, r, s are congruently contained in the euclidean plane E_2. It is shown in this paper that in complete, convex, externally convex, metric spaces $\{M\}$ this property is equivalent to the weak euclidean four-point property introduced by the writer in 1934, and hence it characterizes finite or infinite-dimensional euclidean spaces among the much larger class of spaces $\{M\}$ (see L. M. Blumenthal, *Theory and applications of distance geometry*, Oxford, Clarendon, 1953, pp. 123–129). (Received May 1, 1953.)

677. W. M. Boothby: *Some fundamental formulas for Hermitian manifolds.*

The *equations of structure* of a complex-analytic manifold M with Hermitian metric are: $d\omega^i=\omega^k\omega_k^i+\Omega^i$; $d\omega_j^i=\omega_j^k\omega_k^i+\Omega_j^i$ where the forms ω^i, ω_j^i are uniquely defined in the bundle B of orthonormal frames over M (S. S. Chern, Ann. of Math. vol. 47 (1946) pp. 85–121). Every tensor or tensorial form on M has well defined components relative to each frame, components which become functions on B if we allow the frame to vary over all orthonormal ones, giving components on B of each tensor of M. Using these components and the equations of structure, the tensor calculus of M may be transferred to B, a general technique of Chern which in this instance is developed in detail for the case of Hermitian manifolds. Formulas, including expressions for $d\phi$ and $\delta\phi$ which in the Kaehler case are similar to those given by H. Guggenheimer (Comment. Math. Helv. vol. 25 (1951) pp. 257–297), are derived and applications are given. (Received July 13, 1953.)

678t. John DeCicco: *Dynamical systems of ∞^3 conic sections.*

If a positional field of force possesses at least $3\infty^2$ proper conics as dynamical trajectories, then all are conics. If the lines of force are all straight, and if the field possesses at least $2\infty^2$ proper conics as trajectories, then all are conics. In the projective plane, a dynamical system of ∞^3 conics consists of either: (I) all the conics for which a fixed point and line are pole and polar, or: (II) all the conics tangent to two distinct lines, real or imaginary. In the Euclidean plane, a dynamical family of ∞^3

ellipses consists of four types. The period is obtained for every elliptic trajectory in each of the four associated parallel or central fields of force. In each, there are ∞^3 elliptic trajectories for which the square of the period is directly proportional to the cube of the mean distance. The Newtonian field of force is characterized by the following weakened form of Kepler's laws on planetary motion: (1) the field of force contains ∞^3 ellipses as trajectories, and (2) for each of the ∞^3 ellipses, the square of the period varies directly as the cube of the mean distance. (Received April 27, 1953.)

679t. David Gale: *Separation theorems for families of convex sets.*

The classical theorem on the separation of disjoint convex sets by a hyperplane may be rephrased as follows: If C_1 and C_2 are disjoint open (compact) convex sets in n-space, then there exist disjoint open (closed) halfspaces H_1 and H_2 such that $C_1 \subset H_1$ and $C_2 \subset H_2$. In this form the result can be generalized as follows. *Theorem*: Let $\{C_\alpha\}$ be an indexed family of open (compact) convex sets in n-space, $\alpha \in A$, such that $\bigcap_{\alpha\in A} C_\alpha = \emptyset$. Then there exists a family $\{H_\alpha\}$ of open (closed) halfspaces such that $C_\alpha \subset H_\alpha$ and $\bigcap_{\alpha\in A} H_\alpha = \emptyset$. The proof when the C_α are compact follows easily from the classical theorem. When the C_α are open and the family $\{C_\alpha\}$ has the finite intersection property, the proof is considerably more difficult. (Received July 8, 1953.)

680. W. R. Hutcherson: *Concerning some cyclic involutions with periods $p \leqq 157$.*

Lucien Godeaux [*Sur quelques surfaces algébriques représentant des involutions cycliques*, Bulletin de l'Academie royale de Belgique (1951) pp. 1106–1118] and other authors have studied cyclic involutions with certain prime periods $p \leqq 73$. The writer [W. R. Hutcherson, *Voisage du cinquième ordere d'une involution de période* 13, Bulletin Royale des Sciences de Liège, No. 11 (1952) pp. 483–487] has studied a fourth degree surface where the involutions contained thereon were of period $p \leqq 13$. However, this present paper exhibits a fifth degree surface invariant under a cyclic homography $x_1':x_2':x_3':x_4' = x_1:\epsilon x_2:\epsilon^{10}x_3:\epsilon^{3}x_4$, where $\epsilon^{13}=1$. When $p=157$, a surface of order fourteen was found. It was invariant under the homography $x_1':x_2':x_3':x_4' = x_1:\epsilon x_2:\epsilon^{14}x_3:\epsilon^{101}x_4$ where ϵ is the one hundred and fifty-seventh primitive root of unity. Only three points of coincidence appeared on this surface. Certain invariant space curves of order (157)(14) on this surface were found to pass through these three invariant points as twelve tuple points. (Received July 13, 1953.)

681. N. D. Lane (p) and Peter Scherk: *Characteristic and order of differentiable points in the conformal plane.*

In an earlier paper, the authors defined differentiable points on an arc A in the conformal plane. They described the various types of interior differentiable points s by means of a one-row matrix (a_0, a_1, a_2), the characteristic of s. In the present paper they prove: (1) The cyclic order of s, i.e., the maximum number of points which a small neighborhood of s can have in common with any circle, is not less than $a_0+a_1+a_2$. (2) If s decomposes A into two arcs of cyclic order three, then the order of s is equal to $a_0+a_1+a_2$. In order to prove (2) they establish a strong kind of differentiability at an end point of an arc of cyclic order three. (Received July 13, 1953.)

LOGIC AND FOUNDATIONS

682. P. C. Gilmore: *On axioms related to the principle of the excluded third.*

It has been shown by P. C. Gilmore (Neder. Akad. Wetensch. vol. 56, No. 2, Ser. A) that the effect of Griss' criticism of the intuitionistic logic can be expressed by adding $(Ex)p(x)\vee(Ax)\neg p(x)$, which we denote by A, as an axiom of the logic for every atomic proposition or predicate p. It is worthwhile discussing the relations between A and other expressions related to B, the principle of the excluded third, and the principle of the excluded third itself $((Ax)\cdot P(x)\vee\neg P(x)$ for any wff P). Consider also the expressions C, D and E which are defined to be respectively, $(Ax)\cdot p(x)\vee\neg p(x)$ and $(Ax)p(x)\vee(Ax)\neg p(x)$, for any atomic predicate or proposition p, and $(Ax)\cdot Q(x)\vee\neg Q(x)$, for any wff Q constructed from the atomic predicates and propositions without the aid of quantifiers. It is not difficult to prove that D implies B, that C implies E, and that B implies A, C, or E. One can also prove that A and C are independent. (Received July 10, 1953.)

683*t*. Theodore Hailperin: *Quantification theory for empty individual domains.*

A set of axioms for quantification formulas valid in all individual domains, including the empty domain, is presented. The system resulting is different from, and more "natural" than, that of Mostowski (Journal of Symbolic Logic vol. 16 (1951) pp. 107–111) in that far fewer of the usual quantification formulas become invalid. The difference between this system and Mostowski's arises from unlike treatments of vacuous quantification (quantifaction over formulas not containing the variable which appears in the quantifier). The axioms are so phrased that by the addition of one further axiom the usual quantification theory for formulas valid in all nonempty domains is obtained. A simple proof of completeness is given. (Received April 20, 1953.)

684*t*. S. C. Kleene and E. L. Post: *The upper semi-lattice of degrees of recursive unsolvability.*

The degrees of recursive unsolvability (Post, Bull. Amer. Math. Soc. Abstract 54-7-269) form an upper semi-lattice under a l.u.b. operation $\mathbf{a}\cup\mathbf{b}$, but not a lattice. To any degree $\mathbf{a}$ (including the degree O of a recursive set), let $\mathbf{a}'$ be the degree of the complete A-canonical set (Post, loc. cit.) or equivalently of the predicate $(Ex)T_1^A(a, a, x)$ (Kleene, *Introduction to metamathematics*, Chapter XI) where A is of degree $\mathbf{a}$; and let $\mathbf{a}^{(\omega)}$ be the degree e.g. of the predicate $M^A(a, k)$ (Kleene, loc. cit.). There exist between $\mathbf{a}$ and $\mathbf{a}'$ (between all $\mathbf{a}, \mathbf{a}', \mathbf{a}'', \cdots$ and $\mathbf{a}^{(\omega)}$) infinitely many degrees pairwise incomparable, and also e.g. infinitely many degrees ordered among themselves as the rational numbers. (Received July 17, 1953.)

685. H. G. Rice: *Recursive real numbers.*

The four definitions of Specker (J. Symbolic Logic vol. 14 (1949) pp. 145–158) are stated, using general recursive functions, and the fact of their equivalence mentioned (Robinson, J. Symbolic Logic vol. 16 (1951) p. 282). The following theorems are stated and proofs outlined. 1. There is no effective general method for deciding whether or not two recursive real numbers are equal. However, two unequal numbers can be effectively compared for size. 2. The recursive real numbers form a field. 3. The field of recursive real numbers is closed under sequences whose convergence is given by a recursive function. 4. The field of recursive complex numbers is algebraically closed. The last proof is adapted from Rosenbloom (Amer. Math. Monthly vol. 52 (1945) pp. 562–570). (Received July 15, 1953.)

686*t*. Hao Wang: *Constructive foundations of mathematics.* I.

A theory Σ is presented such that for every ordinal number α of Cantor's second number class, there are sets of level α. On the 0th level are the finite sets built up from the empty set. On the $(\beta+1)$th level are all the sets of level β plus all sets of them which can be defined by properties referring at most only to the totality of all sets of level β. If α is a limiting number, then sets of level α include all and only the sets of levels lower than α. The theory Σ is determined by a schema of axiomatization such that for each given α, a formal system Σ_α dealing with sets of level α is given. To put it in another way, Σ is the union of all such systems. A modelling proof of consistency and a proof-theoretic one are given; the latter is apparently "finitist." The theory includes strong axioms of limitation. (Received July 16, 1953.)

687t. Hao Wang: *Constructive foundations of mathematics.* II.

All sets in Σ are enumerable in Σ so that Skolem's paradox does not occur. Consistency of each system Σ_α is provable in $\Sigma_{\alpha+1}$. Therefore, Gödel's theorems do not apply to the whole theory Σ. Axioms of choice and forms of the continuum hypothesis are provable in Σ. Ordinary theories of natural and real numbers can be obtained in Σ, including such standard theorems as the least upper bound, the limit point, the finite covering. It is contended that, although this theory Σ would not satisfy Brouwer who seems to permit nothing but the effectively decidable, it should satisfy Hilbert who only asks to be finitist in the metamathematics which justifies the system actually used by a proof of consistency. Moreover, the theory Σ is natural in that it includes all the standard principles such as the tertium non datur and the axiom of extensionality. This is taken as a decisive advantage over most of the alternative reconstructions of mathematics. (Received July 16, 1953.)

STATISTICS AND PROBABILITY

688t. K. L. Chung: *Contributions to the theory of Markov chains.* II.

Let $\{X_n\}$, $n \geqq 0$, be a Markov chain with stationary transition probabilities and with the non-negative integers as states forming one recurrent class. Let f be a real-valued function on the non-negative integers. The main purpose of this paper is to study the sequence $\{f(X_n)\}$, $n \geqq 0$. We obtain, in turn, an ergodic theorem of the Hopf type, weak and strong laws of large numbers, the central limit theorem (due to Doblin), the law of the iterated logarithms, and the limit theorems for the maxima of $\sum f(X_n)$. These results are proved very simply by exploiting an idea of Doblin's and they are applicable to a finite (recurrent) chain without any condition whatever. The conditions of validity in the denumerable case deserve attention. Among other things it is shown by an example that a certain set of conditions, attributed to Kolmogorov, is in reality *not* sufficient for the validity of the central limit theorem. Furthermore, conditions of validity for the strong limit theorems and the limit theorems for the maxima are obtained by a natural strengthening of the corresponding conditions for the weak limit theorems. (Received June 24, 1953.)

689t. T. E. Harris: *On chains of infinite order.*

Let s_n, $n=0, \pm 1, \cdots$, be a stationary stochastic process with D possible values $0, 1, \cdots, D-1$. Completing work of Doeblin and Fortet, Onicescu and Mihoc, and others on "chaînes à liaisons complètes" it is shown that such a process exists satisfying $P(s_{n+1}=i \mid s_n=u_0, s_{n-1}=u_1, \cdots)=Q_i(u)$ with probability 1, where the Q_i are given functions of $u=(u_0, u_1, \cdots)$ satisfying certain conditions. The quantity $x_n=\sum_{k=0}^{\infty} s_{n-k}/D^{k+1}$ uniquely determines the past of s_n with probability 1. The distribution $G(x)$ of the past must be one of the three types (1) a step function with one step; (2) $G(x)=x$, $0\leqq x\leqq 1$, this being the case if and only if the x_n are independent for different values of n, each being equidistributed; (3) purely singular. Special results on the prediction problem are obtained for the following type of process: let y_n be a stationary Markov chain with a finite number of states divided into D groups of states G_i; let $x_n=i$ if $y_n\in G_i$. The methods depend upon the fact that the past, x_n, is a special type of Markov process recently used as a model for learning processes by a number of authors. (Received July 16, 1953.)

690*t*. M. S. Klamkin: *On the uniqueness of the distribution function for the Buffon needle problem.*

In the well known Buffon needle problem the probability that a needle of length L, when dropped on a board ruled with equidistant parallel lines of spacing D ($D\geqq L$), will intersect one of the lines is given by $P=2L/\pi D$. This result has been derived by Barbier in a very elegant method not depending upon a distribution function. It is then shown that the distribution function in x and ϕ (where x is the distance from the needle's midpoint to the nearest line and ϕ is the acute angle between this distance and one of the ruled lines) is not unique. However, if x and ϕ are assumed to be independent variables, then it is shown that the distribution in x is uniform and thus unique, and that the distribution in ϕ is not unique. If the problem is made slightly more involved by taking $D\leqq L$, then it can be shown that there is only one distribution function which leads to the probability one would obtain by assuming a uniform distribution in x and ϕ. (Received July 17, 1953.)

691*t*. Gerhard Tintner: *Stochastic linear programming.*

Linear programming deals with the maximization of a linear form under linear inequalities. These inequalities can be transformed into equalities by the use of slack variables. We maximize $\phi=a'x$ (a and x column vectors of n components) under the conditions: $Bx=c$ (B a matrix, c a column vector of m components) and $x\geqq 0$. Select m values of a, x, c and m columns of B. Denote the selection $By^{(k)}$. Then $\phi^{(k)}=a^{(k)\prime}x^{(k)}$ requiring $x^{(k)}=B^{(k)-1}c^{(k)}\geqq 0$. The x is chosen for which $\phi^{*}=\max \phi^{(k)}$. Let the probability distribution of the elements of a, B, c be $P(a, B, c; w)$ where w is a set of controlled parameters. From P we derive the probability distribution of the maximand: $Q(\phi^{*}; w)$. We maximize the functional $g[Q(\phi; w)]$ with respect to w. A special case is $g=E\phi^{*}$, the mathematical expectation of the maximand. (Received June 18, 1953.)

TOPOLOGY

692. R. D. Anderson: *Some upper semi-continuous collections of continuous curves filling up R^3.*

The author establishes that for any positive integer p, there exists an upper semi-continuous collection G of mutually exclusive locally connected continua filling up R^3 (the 3-cell) such that G with respect to its elements as points is homeomorphic to R^p. This result is a strengthening of (and is to be published with) the author's previously announced result (Bull. Amer. Math. Soc. Abstract 59-3-319) that for any positive integer p, there exists a monotone mapping of R^3 onto R^p. (Received July 17, 1953.)

693. R. G. Bartle: *Weak compactness in function spaces.*

A directed system (f_α) of functions converges *quasi-uniformly* on a set X to f_0 iff it converges on X and, given $\epsilon>0$, a_0 there exist $a_1, \cdots, a_n \geq a_0$ such that $X = \bigcup_{i=1}^{n} \{x; |f_{a_i}(x)-f_0(x)|<\epsilon\}$. If X is compact Hausdorff and $f_a \in C(X)$, then $f_0 \in C(X)$ iff the convergence is quasi-uniform on X. A set $\mathfrak{F} \subset C(X)$ is *quasi-equicontinuous* on X iff every directed system (x_a) in X converging to x_0 converges quasi-uniformly on $\mathfrak{F}$. *Theorem*: A set $\mathfrak{F}$ is conditionally compact in the pointwise topology of $C(X)$ iff $\mathfrak{F}$ is pointwise bounded and quasi-equicontinuous on X. A subset of the Banach space $C(X)$ is conditionally compact in the weak topology iff it is bounded and quasi-equicontinuous. Similar conditions apply to the Banach space of bounded functions on an abstract set. Applications are made to weakly compact operators. These notions and results stem from work of Arzelà, Tychonoff, Sirvent, and Šmulian. (Received May 28, 1953.)

694. W. R. Baum: *On the two-dimensional universal minimal cycles.*

Let M^n be an n-dimensional closed oriented manifold. A 2-dimensional cycle Z^2 in M^n is said to be minimal if the intersection cycle $U^{n-1} \cdot Z^2$ is null-homologous in M^n for all the $(n-1)$-dimensional cycles U^{n-1} in M^n. (All cycles taken with respect to the integers as coefficients.) H. Hopf has shown (Comment. Math. Helv. vol. 14 (1941) pp. 257–309) that the group $\mathfrak{M}^2$ of the homology classes of the minimal 2-cycles modulo the subgroup $\mathcal{L}^2$ of the homology classes of the cycles which are 2-sphere images is determined by the fundamental group $\mathcal{G}$ of M^n. If $U^{n-1} \cdot Z^2 \sim 0$ holds not only for all the integral cycles U^{n-1} but also for every U^{n-1} which is a cycle mod m $(m=2, 3, 4, \cdots)$, then the (integral) cycle Z^2 is called a 2-dimensional universal minimal cycle in M^n. Let $\mathcal{G} \cong \mathcal{F}/\mathcal{R}$ be a representation of the fundamental group $\mathcal{G}$ by a free group $\mathcal{F}$, and $\mathcal{C}^2$ the second commutator-subgroup of $\mathcal{F}$. Then one can prove the theorem: Z^2 is then and only then universal minimal if Z^2 has a homotopy boundary which represents an element of the group $\mathcal{C}^2$. From this theorem follows: The homology group $\widetilde{\mathfrak{M}}^2$ of the universal minimal cycles Z^2 modulo the group $\mathcal{L}^2$ depends only on the fundamental group $\mathcal{G}$. The dependence can be described explicitly by an isomorphism between the group $\widetilde{\mathfrak{M}}^2/\mathcal{L}^2$ and a certain group which depends only on $\mathcal{G}$. (Received July 13, 1953.)

695. T. A. Botts, E. E. Floyd (p), and E. J. McShane: *Lattices with pathological order topologies.*

If P is a partially ordered set, then, using the order topology, P can also be considered as a topological space. It is easy to show that P is then a T_1-space. Various examples are exhibited in which further separation properties do not hold. One example is of a complete, infinitely distributive lattice in which the open sets have the finite intersection property. Hence there exists a net in P which converges, in the order topology, to every point of P. It is also shown that there exists a lattice group which is not Hausdorff in its order topology. It then follows that the group operations are not continuous. (Received July 20, 1953.)

696*t*. E. R. Fadell: *Fixed point free mappings.* Preliminary report.

Let ϕ denote the fixed point property and $\bar{\phi}$ its negation. A topological space X is said to be strongly $\bar{\phi}$ if there exists a fixed point free mapping from X *onto* itself. Clearly strongly $\bar{\phi}$ implies $\bar{\phi}$. Examples show that strongly $\bar{\phi}$ and $\bar{\phi}$ are not equivalent on continua or on connected, locally connected spaces. It is well known that every multicoherent Peano continuum is $\bar{\phi}$. By a slight variation in the Kuratowski proof it is shown that such a space is strongly $\bar{\phi}$. Hence strongly $\bar{\phi}$ and $\bar{\phi}$ are equivalent on this class of spaces. The question as to the equivalence of strongly $\bar{\phi}$ and $\bar{\phi}$ for the general Peano continuum remains open. (Received July 16, 1953.)

697t. Jean B. Feidner: *Topological properties of a linear neighborhood space.*

In this investigation of a linear neighborhood space, any neighborhood U_x of x is defined thus: U_x contains x and at least three, but not all, elements of the space; if $x \neq y$, there exists one and only one U_{xy}; U_{xy} is simply ordered, consisting of x, y, and all other elements of the space between x and y; U_{xy} satisfies the Dedekind cut postulate and Cantor's linearity postulate; given p non-element of U_{xy}, $U_{xp} \cdot U_{xy} = x$ or U_{xy}. Define: x is a limit point of set M if $U_x \cdot (M - x) \neq 0$ implies $U_x \cdot (M - x)$ infinite. Other topological properties are defined as usual. The only connected sets are certain subsets of neighborhoods. Given set M, K_M consists of all points k of $S - M$ such that every $U_k \cdot M \neq 0$. This K-set facilitates the study of openness and closedness and methods of closing non-closed sets. The following subsets of M are studied in detail: L_M containing all points m such that every $U_m \cdot M$ is infinite; P_M containing remaining limit points of M; W_M containing all points m such that every $U_m \cdot M$ is finite; N_M containing remaining non-limit points of M. Further results are obtained by a similar classification of the elements of $S - M$ with reference to M. (Received July 13, 1953.)

698t. I. S. Gál: *On a computable looping invariant of knots.*

Let C be an arbitrary knot-cycle consisting of eléments a^{α}, $a^{-\alpha}$, b^{β}, $b^{-\beta}$, $\cdots$. Let $A(a^{\alpha})$ denote the linearily ordered sequence of elements obtained from C by writing down its elements starting with a^{α} and proceeding with respect to a fixed orientation of C. Assume that the linear ordering of the elements x^{ξ}, $x^{-\xi}$, y^{η}, $y^{-\eta}$ in $A(a^{\alpha})$ is $x^{\xi}y^{-\eta}x^{-\xi}y^{\eta}$ with $R\xi = R\eta$. Then the signature $\lambda_{xy}(a^{\alpha})$ is defined to be $R\xi I\xi R\eta I\eta$. In all other cases the signature is defined to be 0. Let $\Lambda(C, a^{\alpha}) = \sum \lambda_{xy}(a^{\alpha})$ where the summation is extended over all couples x, y of C. One can prove that $\Lambda(C)$ *is independent of the special choice of* a^{α} *and is an invariant of the knot.* This invariant is sufficient to distinguish all parallel knots and double knots (save for reflection) belonging to a fixed carrier knot. For example if C_M and C_W denote the alternating torus knot with $2n+1$ double points and Whitehead's double knot with the circle as the carrier knot, then $\Lambda(\pm C_M) = n(n+1)$ and $\Lambda(C_W) = 2n$. $\Lambda = 0$ does not characterize the circle, e.g. $\Lambda(3_1 + 4_1) = \Lambda(8_{II}) = 0$. (Received July 17, 1953.)

699. I. S. Gál: *The complete characterisation of knots by abstract cycles.*

It is known that a knot is completely determined by any of its regular normed projections. One can associate with any regular and normed projection an abstract cycle in the following way: Let a, b, c, $\cdots$ denote the double points of the projection and let an orientation be given. By the homeomorphism of the projection onto the unit circle there are two distinct points of the circle for every double point a, b, c, $\cdots$. The point of the circle corresponding to a on the upper (lower) line segment of the projection will be denoted by a^{1+i} or a^{1-i} (a^{-1-i} or a^{-1+i}) according as the crossing takes place from right to left or from left to right. For instance in the case of two trefoil knots the corresponding cycles are (for projections with three double points) $(a^{1+i}b^{-1-i}c^{1+i}a^{-1-i}b^{1+i}c^{-1-i})$ and $(a^{1-i}b^{-1+i}c^{1-i}a^{-1+i}b^{1-i}c^{-1+i})$. One can prove the following results: 1. *A knot is completely determined by any of its knot-cycles.* 2. *The (finite)* *set* $\{C\}$ *of all knot-cycles belonging to a fixed knot and having minimal number of elements is an invariant of the knot.* (Received June 15, 1953.)

700. Leonard Gillman (p) and Melvin Henriksen: *Linearly ordered paracompact spaces.*

A *linearly ordered space* is a linearly ordered set X in its interval topology. A Dedekind cut $u=(A|B)$, in which A has no last element and B no first, is called an *interior gap*, and is regarded as a "virtual" element satisfying $a<u<b$ for all $a\in A$ and $b\in B$. If X itself has no first (last) element, *a left* (*right*) *end-gap* l (r) is introduced, as a virtual element satisfying $l<x$ ($r>x$) for all $x\in X$. Let u be any gap of X other than l (r), and let $S=\{x_\xi\}_{\xi<\omega_\alpha}$ be an increasing (decreasing) sequence of points of X, in which ω_α is a regular initial ordinal, and such that $\lim_{\xi<\omega_\alpha} x_\xi=u$; S is called a *Q-sequence at* u if, for every transfinite limit ordinal $\lambda<\omega_\alpha$, $\lim_{\xi<\lambda} x_\xi$ is a gap. A gap u is a *Q-gap* if there exist both an increasing Q-sequence at u and a decreasing Q-sequence at u (or only the appropriate one, in case u is an end-gap). Theorem 1. A linearly ordered space X is paracompact if and only if every gap of X is a Q-gap. Theorem 2. Every linearly ordered space is countably paracompact. (For definitions, see, e.g., Canadian Journal of Mathematics vol. 3 (1951) pp. 219–224.) (Received July 17, 1953.)

701*t*. Leonard Gillman and Melvin Henriksen: *Linearly ordered Q-spaces.*

Terminology is as in preceding abstract. A cardinal $\aleph_\alpha$ is *nonmeasurable* if no countably additive two-valued measure can be defined on the family of all subsets of a set of power $\aleph_\alpha$, in such a way that every subset consisting of just one element has measure zero, and the entire set has measure one. Every $\aleph_\alpha$ smaller than the first strongly inaccessible cardinal $>\aleph_0$ is nonmeasurable (Ulam, Fund. Math. vol. 16 (1930) p. 146 ff.; Tarski, Fund. Math. vol. 30 (1938) p. 153). A Q-gap u of a linearly ordered space is *nonmeasurable* if the cardinal of each Q-sequence at u is nonmeasurable. Theorem. For every linearly ordered space X, the following statements are equivalent. (1) X is a Q-space (for definition, see Hewitt, Trans. Amer. Math. Soc. vol. 64 (1948) p. 85). (2) X is paracompact, and every closed subspace of X is a Q-space. (3) X is paracompact, and every closed discrete subspace of X is a Q-space. (4) Every gap of X is a nonmeasurable Q-gap. Corollary. Every linearly ordered Q-space is paracompact. Remark. (3) implies (1) for arbitrary Hausdorff spaces, not only for those that are linearly ordered (Katětov, Fund. Math. vol. 38 (1951) pp. 73–84; Shirota, Osaka Math. J. vol. 4 (1952) pp. 23–40). (Received July 17, 1953.)

702. Leonard Gillman, Melvin Henriksen (p), and Meyer Jerison: *On a theorem of Gelfand and Kolmogoroff.*

Let C denote the ring of all continuous real-valued functions on a completely regular space X. Theorem 1. The m-closure (Hewitt, Trans. Amer. Math. Soc. vol. 64 (1948)) of any ideal of C is an intersection of maximal ideals. (This was conjectured by Hewitt.) Let C^* denote the subring of bounded functions; C^* is m-closed. Theorem 2. The m-closure of every ideal of C^* is an intersection of maximal ideals (of C^*) iff $C=C^*$. The proofs are based on the theorem of the title (stated without proof in C.R. (Doklady) Acad. Sci. URSS. vol. 22 (1939), but proved herein): M is a maximal ideal of C iff there is a unique $p\in\beta X$ (Stone-Čech compactification) such that M is the set of all $f\in C$ for which $p\in\bar{Z}(f)$ (closure in βX of the set of zeros of f). This theorem is further applied to give alternate constructions of Hewitt's Q-space νX

(loc. cit.), as follows. Regard $f \in C$ as a continuous function on X into the one-point compactification of the reals, and denote by $\bar{f}$ its unique extension over βX. Theorem 3. νX is the set of all $p \in \beta X$ such that for every $f \in C$, $\bar{f}(p) \neq \infty$. Theorem 4. νX is the intersection of all the Q-spaces Y such that $X \subset Y \subset \beta X$. (Received July 17, 1953.)

703. S. I. Goldberg: *Representation theory of Lie algebras*. II. Preliminary report.

Let L be a Lie algebra over a field K of characteristic 0, M a subalgebra of L, and $\{V, P\}$ a representation module of L. Consider the groups $C^q(L, V)$ and $C^q(L, M, V)$ of q-dimensional cochains and relative cochains mod M and form the quotient $C^q(L, V)/C^q(L, M, V) = C^q_r(L, M, V)$ of reduced cochains. The author then displays the exact sequence $H^0(L, M, V) \xrightarrow{i^*} \cdots \xrightarrow{\delta^*} H^q(L, M, V) \xrightarrow{i^*} H^q(L, V) \xrightarrow{r^*} H^q_r(L, M, V) \xrightarrow{\delta^*} H^{q+1}(L, M, V) \xrightarrow{i^*} \cdots$. The following result is then obtained: Let L be a semi-simple Lie algebra over K, M a subalgebra of L, and P an irreducible and nontrivial representation of L, then $H^q(L, P) = H^q(L, M, P) = \{0\}$ for all dimensions q. In the case where $V = K$ the following results are established: Let $\mathfrak{g}$ be a compact connected semi-simple Lie group, $\mathfrak{h}$ a closed connected subgroup of $\mathfrak{g}$, and let L and M be the Lie algebras of $\mathfrak{g}$ and $\mathfrak{h}$ respectively. Then by using spectral sequences $H^1(M) \cong H^2(L, M)$ and $H^1(L, M) = \{0\}$. For all L and M over K (L semi-simple) and all P, $H^q_r(L, M, P) \cong H^q(L, M, P)$. (Received April 29, 1953.)

704t. Lillian Gough: *Logical relations between pairs of properties used to define endpoint.*

The logical relations between pairs of twenty-five properties which might be taken as definitions of endpoint of a connected set, continuum, or continuous curve are discussed in this paper. Properties 3 through 9 are due to H. M. Gehman (Trans. Amer. Math. Soc. vol. 30 (1928) pp. 63–84), property 10, to W. L. Ayres (Math. Zeit. (1931) pp. 161–178), properties 11, 12, 13 to R. H. Bing (Duke Math. J. vol. 18 (1951) pp. 653–663). Properties 3A, 4A, 5A, 6A, 8A, 9A, 10W, 10S, and 12A are obtained by substituting the concept of connected subsets for that of subcontinuum in the statements of the corresponding properties 3–12. Properties 11B, 12B, and 13B are concerned with a neighborhood of the point in question rather than with the entire set as are 11, 12, and 13. In 8C and 8D the concepts of subcontinuum and connected subset replace those of irreducible subcontinuum and irreducibly connected subset of 8 and 8A respectively. The properties concerned with subcontinua containing the point in question may be satisfied vacuously by a point of a connected set which is not considered an endpoint. Hence the "A" properties, property 4A particularly, seem better definitions of endpoint for a set which is connected but not closed. (Received July 20, 1953.)

705. Mary-Elizabeth Hamstrom: *Concerning the imbedding of upper semicontinuous collections of continua in continuous collections of continua.*

In the plane, let G be an upper semicontinuous collection of compact atriodic continuous curves which is an arc with respect to its elements. The collection G is said to be *imbedded* in a collection W provided it is true that each element of G is contained in some element of W and each element of W contains one and only one element of G. If H is a closed subcollection of arcs of G, $H^{(a)}$ denotes the collection of all elements g of H such that if ϵ is a positive number, then every segment of G

containing g contains an element of H which has a subarc the distance between whose endpoints is less than ϵ but whose diameter is not less than $1/n$. The notion of a K-curve of G is introduced and it is shown that there is a continuous collection W of mutually exclusive simple closed curves such that G is imbedded in W if and only if (1) each element of G is a K-curve of G, (2) if H is a closed subcollection of arcs of G then for no natural number n is it true that $H^{(n)}=H$, and (3) of each pair of simple closed curves of G one is interior to the other. Certain related theorems are also proved. (Received July 8, 1953.)

706. J. G. Hocking: *1-monotone images of 2-manifolds.*

It is the object of this note to establish the following result: *Let M be a 2-manifold with boundary B and let f be a compact, 1-monotone mapping of M such that $f|B$ is monotone. Then $f(M)$ is homeomorphic to M.* This is an extension (to the case of a manifold with boundary) of a theorem of Roberts and Steenrod (Theorem 1 of *Monotone transformations of 2-dimensional manifolds*, Ann. of Math. vol. 39 (1938) pp. 851–862) which was first generalized by V. Martin to the noncompact case without boundary in his paper *Monotone transformations of non-compact two-dimensional manifolds*, Duke Math. J. vol. 8 (1941) pp. 136–153 (see Theorem 1). The method of proof is to use a characterization of 2-manifolds due to G. S. Young in order to prove that $f(M)$ is a 2-manifold. Then a simple application of V. Martin's theorem completes the argument. As an application of this result above one proves the following: *Let M be a compact 2-manifold with boundary B and let f be a quasi-interior, monotone-1 mapping of M such that $f|B$ is monotone. Then f may be approximated arbitrarily closely by light interior mappings.* (Received July 17, 1953.)

707t. Meyer Jerison: *A property of extreme points of compact convex sets.*

Let $\{K_n\}$ be a uniformly bounded sequence of weak*-closed, convex sets in the conjugate space of a Banach space, and let A_n be the set of extreme points of K_n. Then $K=\bigcap_{n=1}^{\infty} K_n$ is contained in the weak*-closed convex hull of the set $A=\lim\sup A_n$ (in the topological sense). If, in addition, $\{K_n\}$ is a decreasing sequence, then A contains all of the extreme points of K. (Received July 13, 1953.)

708. Leo Katz (p) and J. H. Powell: *On the number of locally restricted directed graphs.*

A directed graph on n nodes P_i, $i=1, 2, \cdots, n$, is locally restricted in the sense that the numbers r_i, of joins $P_i\rightarrow x$, and s_i, of joins $y\rightarrow P_i$, are specified. The vectors r and s are two n-part partitions of the total number of joins. Using a matrix representation of the graph, it is shown that the number of directed graphs so restricted is expressible in terms of certain bipartitional functions tabulated by P. V. Sukhatme (Philos. Trans. Roy. Soc. London Ser. A vol. 237 (1938) pp. 375–409), and earlier by Cayley (Philos. Trans. Roy. Soc. London vol. 147 (1857) pp. 489–499). Sukhatme gives $Ag(s|r)$, the number of ways to fill the cells of a rectangular lattice so that the column sums form the partition s and the row sums, the partition r. The number of locally restricted directed graphs is $\eta(s|r)=Ag\{\prod_1^n(1+\delta_i)^{-1}(s|r)\}$, where Ag is linear in the partitions, δ_i operates on a pair of partitions to reduce the ith components of each by unity, $(1+\delta_i)^{-1}=1-\delta+\delta^2-\delta^3+\cdots$ and $Ag\ (S^{(n)}|r^{(n)})=0$ if any of the $(2n)$ arguments is negative. This work was sponsored by the Office of Naval Research. (Received July 16, 1953.)

709t. S. D. Liao: *On fourth obstruction of sphere bundles*. Preliminary report.

As application of certain homotopy properties of cyclic products of spheres (Liao, *On topology of cyclic products of spheres*, Bull. Amer. Math. Soc. Abstract 59-5-555), some formulas for fourth obstruction of sphere bundles may be deduced, using technique similar to that for the case of secondary obstructions (Bull. Amer. Math. Soc. Abstract 58-4-470). Let $\mathfrak{B}=\{B, E, \phi\}$ be a bundle with base space B a connected complex, total space E, projection ϕ, and fiber an n-sphere S_n. Assume that the fundamental group of B acts trivially on the nth homotopy groups of the fibers. Write the Gysin sequence of $\mathfrak{B}$ as $\cdots \to H^q(B, J) \xrightarrow{\phi^*} H^q(E, J) \xrightarrow{\beta} H^{q-n}(B, J) \to H^{q+1}(B, J) \to \cdots$ where J denotes the group Z of integers or the group of integers mod p. Suppose that the primary obstruction of $\mathfrak{B}$ vanishes. Then, ϕ^* is an into-isomorphism. Let f be a cross section of $\mathfrak{B}$ over B_n ($=$the nth skeleton of B) which extends over B_{n+3}. Then, for the fourth obstruction $Z^{n+4}(f)$ mod 2 and mod 3, we have: $\phi^*Z^{n+4}(f)=\theta(f)\cup\phi^*\beta Sq^4\theta(f)+Sq^4\theta(f)$ mod 2 for $n\geqq 5$, $\phi^*Z^{n+4}(f)=\theta(f)\cup\phi^*\beta\mathfrak{P}^1\theta(f)-\mathfrak{P}^1\theta(f)$ mod 3 for $n\geqq 3$ and $n\neq 4$, where $\theta(f)=a-\phi^*f^*a$ with $a\in H^n(E, Z)$ satisfying the condition $\beta(a)=$the unit of the integral cohomology ring of B, and $\mathfrak{P}^1$ is the cyclic reduced power operation mod 3. (Received July 8, 1953.)

710. E. A. Michael: *A selection theorem for continuous functions.*

Let X and Y be metric spaces, Y complete, let F be a function from X to the nonempty subsets of Y, and let n and k denote integers $\geqq -1$. F is called *lower semi-continuous* if $\{x\in X \mid F(x)\cap U\neq\emptyset\}$ is open in X for every open $U\subset Y$. A collection $\mathfrak{B}$ of subsets of Y is called *equi-LC*^{n} if it has the following property: If $y_0\in B_0\in\mathfrak{B}$, and if U is a neighborhood of y_0 in Y, then there exists a neighborhood V of y_0 in Y such that, for every $k\leqq n$ and every $B\in\mathfrak{B}$, every continuous mapping of the k-sphere into $B\cap V$ is homotopic to a constant mapping in $B\cap U$. The following theorem, which is related to results of Kuratowski [Fund. Math. vol. 24 (1935) p. 269], Montgomery and Zippin [Amer. J. Math. vol. 65 (1943) Theorem 3.1], and the author [Bull. Amer. Math. Soc. Abstract 59-2-262], can now be proved. Let F be lower semi-continuous with $\{F(x)\}_{x\in X}$ equi-LC^n, and let A be a closed subset of X with Lebesgue dim $(X-A)\leqq n+1$. Then every continuous $g: A\to Y$, such that $g(x)\in F(x)$ for every $x\in A$, can be extended to a continuous $f: U\to Y$, U a neighborhood of A, such that $g(x)\in\overline{F(x)}$ for every $x\in U$. If, moreover, every $F(x)$ has all its homotopy groups of order $\leqq n$ vanishing, then one can take $U=X$. (Received July 17, 1953.)

711. E. E. Moise: *Affine structures in 3-manifolds*. VII. *Disks which are pierced by intervals.*

Let D be a disk, and let e be an arc, in a triangulated 3-manifold K, such that $D\cap e$ is a point p which is an interior point both of D and of e. Suppose that e pierces D at p, in the sense that for each sufficiently small open neighborhood U of p, $U-D$ is the sum of two disjoint open sets each of which intersects the component of $U\cap e$ that contains p. *Theorem* 1. If D and e are tamely imbedded, then so also is $D\cup e$. *Theorem* 2. If D is locally polyhedral except at p, and e is a polyhedron, then $D\cup e$ is tamely imbedded. (Received July 17, 1953.)

712t. Mabel D. Montgomery: *Axioms of topology.*

The subject of this paper is a set of twelve axioms based on limit point as undefined concept. These axioms, referred to as the L-axioms, are expressed in terms of the relationship pLA where p is a point and A is a subset of a space S. The L-axioms do not form an independent set. Weaker forms of certain axioms are included in order to determine the properties of the relation pLA which are sufficient to yield most of the fundamental theorems of topology. It is shown that a certain four of the axioms form an independent set equivalent to the complete set. The question of equivalence of the L-axioms to some other sets of axioms for topology is considered. It is established that the set of L-axioms is equivalent to the first three Kuratowski axioms but is not equivalent to any subset of the Riesz or Hausdorff axioms. A weak upper semi-continuous decomposition of a space M and its associated decomposition space M^* are defined, and it is proved that if M satisfies the L-axioms, so also does M^*. (Received July 17, 1953.)

713t. R. S. Pierce: *Coverings of a topological space.*

Let P be a semi-lattice. Definition: a *P-space* is a set S of proper dual ideals of P, which is topologized by taking the sets of the form $\{X \in S \mid a \in X\}$ $(a \in P)$ as a neighborhood basis. Definition: a finite nonempty subset of P is called a *P-covering*. A *P-covering ideal* is a nonempty set $\mathfrak{a}$ of P-coverings such that (1) if $\alpha, \beta \in \mathfrak{a}$, then $\alpha \wedge \beta = \{a \wedge b \mid a \in \alpha, b \in \beta\} \in \mathfrak{a}$ and (2) if $\alpha \in \mathfrak{a}$ and α refines β (that is, for any $a \in \alpha$, there exists $b \in \beta$ with $a \leq b$), then $\beta \in \mathfrak{a}$. For any compact T_1 P-space S, denote by $\mathfrak{a}(S)$ the set of all P-coverings α which satisfy $\alpha \cap X \neq \emptyset$ for all $X \in S$. If $\mathfrak{a}$ is a P-covering ideal, let $S(\mathfrak{a})$ be the P-space of all dual ideals which are minimal in the set of dual ideals D satisfying $D \cap \alpha \neq \emptyset$ for all $\alpha \in \mathfrak{a}$. Then $\mathfrak{a} \to S(\mathfrak{a})$ and $S \to \mathfrak{a}(S)$ are reciprocal correspondences between the set of all P-covering ideals and the set of all compact T_1 P-spaces. (Received July 20, 1953.)

714. Everett Pitcher: *Chain groups and critical point inequalities.*

The construction of a lattice of exact sequences from filtered chain groups and a boundary homomorphism is inverted as follows. Suppose (*) $\cdots \to A^t_r \to B^t_r \to C_r \to A^t_{r-1} \to \cdots$, $r = \cdots, -1, 0, 1, \cdots$, is an exact sequence of discrete [or compact] abelian groups, $t = 1, \cdots, N$, and suppose $A^1_r = 0$, $A^t_r = B^{t-1}_r$. Examples: $X_1 \subset \cdots \subset X_N$, $B^t_r = H_r(X_t)$ [or $\pi_r(X_t)$], $C^t_r = H_r(X_t, X_{t-1})$ [or $\pi_t(X_t, X_{t-1})$]; in critical point theory of f: $X \to R$, $X_t = \{x \mid fx \leq a(t)\}$ where $a(t)$ is increasing sequence separating critical values of f. *Theorem: There are filtered chain groups J_r obtained from $C^1_r, \cdots, C^N_r$ by successive group extensions and homomorphisms $\partial: J_r \to J_{r-1}$ of order 2 such that* (*) *is the lattice of exact sequences. In particular* (**) $B^N_r = \partial^{-1}0/\partial J_{r+1}$ *and* $J_r/\partial^{-1}0 = \partial J_r$. In the critical point application, (**) is a strengthened group theoretical form of the Morse inequalities. If all critical points are nondegenerate and if M_q, R_q, η_q denote the number of critical points of index q and the Betti number and number of torsion coefficients in dimension q (integer coefficients), then $M_q \geq R_q + \eta_q + \eta_{q-1}$ and $M_q - M_{q-1} + \cdots + (-1)^q M_0 \geq R_q - R_{q-1} + \cdots + (-1)^q R_0 + \eta_q$. (Received July 20, 1953.)

715. Louis Robinson: *An extension of the Kneser inequality to complexes and a generalisation of the concept of the degree of a mapping.*

There is the theorem of H. Kneser (Math. Ann. vol. 103 (1930) p. 347) which states that if S_p and S_q are orientable closed two-dimensional manifolds of genus p and q respectively with $p \geq q \geq 1$, then the degree c of a continuous mapping of S_p into S_q satisfies the inequality $(p-1) \geq |c|(q-1)$. In this paper there is proved a generalization of the Kneser inequality to certain classes of complexes. Let K and $\bar{K}$ be connected complexes whose fundamental groups G and $\bar{G}$ are each given by a finite system of generators and one essential relation. To the groups G and $\bar{G}$, so given, there correspond the groups G_1^* and $\bar{G}_1^*$ introduced by H. Hopf (Comment. Math. Helv. vol. 14 (1941) p. 257) each of which is infinite cyclic or else consists of the identity element alone. A simplicial mapping of K into $\bar{K}$ induces a homomorphism of G_1^* into $\bar{G}_1^*$ which uniquely determines an integer which is called the pseudo-degree of the mapping. The generating element of G_1^* determines an integer which is called the pseudo-genus number of K. It is shown that between the pseudo-degree c and the pseudo-genus numbers p and q with $p \geqq q \geqq 1$ there exists the inequality $(p-1) \geqq |c|(q-1)$. (Received July 8, 1953.)

716t. Hans Samelson: *A connection between Whitehead and Pontryagin products.*

Let X be a 1-connected topological space, and let Ω be the space of loops in X, with base point x_0. Let $[\ ,\ ]$ denote Whitehead product, as usual, and let $*$ denote Pontryagin product in Ω [cf. e.g. R. Bott and H. Samelson, *On the Pontryagin product in spaces of paths*, to appear in Comment. Math. Helv.]. Let τ denote the composition of the standard isomorphism (lowering dimension by one) between the homotopy groups of X and of Ω with the natural map of the homotopy groups of Ω into the homology groups of Ω. With $\alpha \in \pi_{p+1}(X)$ and $\beta \in \pi_{q+1}(X)$ the following formula is shown to hold: $\tau[\alpha, \beta] = \pm(\tau\alpha * \tau\beta - (-1)^{pq}\tau\beta * \tau\alpha)$; special cases of this have been known to Hurewicz, Serre, G. W. Whitehead. One of the proofs uses the algebraic structure of the Pontryagin ring associated with a space consisting of two spheres joined at a point, as determined in the reference above. (Received March 31, 1953.)

717. D. E. Sanderson: *Isotopic deformation in 3-manifolds.*

If M and K are triangulated n-manifolds with boundary, a piecewise linear homeomorphism (p.l.h.) $h: M \rightarrow K$ maps simplexes of a subdivision of M linearly onto simplexes of a subdivision of K. A piecewise linear isotopy (p.l.i.) between two p.l.h.'s h and h' is a mapping $G(x, t)$ of $M \times [0, 1]$ into K, such that $G(x, 0) = h(x)$, $G(x, 1) = h'(x)$, and for every $t \in [0, 1]$, $G(x, t)$ is a p.l.h. If $M \subset K$ and h is the identity, G is an isotopic deformation of M. A shellable partition of M is a collection of closed ordered n-cells $\{C_i\}$ with disjoint interiors, where $M = \sum C_i$ and $C_i \cap \mathrm{Bd}(\overline{K - \sum_{j=1}^{i-1} C_j}) = \mathrm{Bd}(C_i)$ or is an $(n-1)$-cell ($\mathrm{Bd}(S)$ consists of points in S having no neighborhood homeomorphic to E_n). It is shown that if M is a 3-cell with triangulation T, and $M \cap \mathrm{Bd}(K) = \mathrm{Bd}(M)$ or is an $(n-1)$-cell, there is a subdivision of T whose (closed) simplexes form a shellable partition of M. There exist p.l.i.'s deforming M onto any simplex of T, or K onto $\overline{K-M}$. A 3-manifold N is a 3-sphere if and only if, for some 3-cell $C \subset N$, $\overline{N-C}$ has a finite shellable partition. There are exactly two isotopy classes of homeomorphisms of E_3 onto itself. Extensive use is made of articles by E. E. Moise (Ann. of Math. vol. 56 (1952) pp. 96–114 and vol. 55 (1952) pp. 172–176, 203–214) and S. S. Cairns (Ann. of Math. vol. 45 (1944) pp. 207–219). (Received July 17, 1953).

718t. A. D. Wallace: *A structure theorem for compact mobs.*

Recall that a mob is an associative map m: $S \times S \rightarrow S$ such that S is Hausdorff. Let S be compact, let E be the set of idempotents of S, and let P be the set of all those $x \in S$ such that $xS = S$. Recall that each $e \in E$ is the unit of a unique maximal algebraic subgroup of S with e as unit. It is shown that P is closed, that $P = \bigcup \{H(e) \mid e$

$\in E\cap P\}$, and that $E\cap P$ is the set of left units of S. Further $xy\in P$ implies $x\in P$ and $y\in P$, and $xa=xb$ implies $a=b$ if $x\in P$ and $a, b\in S$. Finally, if $e_0\in E\cap P$, then there is a (topological) isomorphism of $H(e_0)\times(E\cap P)$ onto P. This last result is close to an algebraic theorem of H. B. Mann. In proving the above we use some results of R. J. Koch. Let G be a compact Hausdorff group and X a compact Hausdorff space. In X define $xy=y$ for all x, y. If $S=G\times X$, then $P(S)=S$, $E(S)=a\times X$ (a the neutral element of G) and $H(a, x)=G\times x$, $x\in X$. This shows that the structural results above are "best possible" when $P(S)=S$, as noted by Mann. (Received December 15, 1952.)

719*t*. A. D. Wallace: *Geometric theorems on clans.*

A clan is a compact connected Hausdorff semigroup with unit. We use the Alexander-Kolmogoroff cohomology groups as developed by Spanier. Let S be a clan. If for some $p>0$ and some coefficient group G we have $H^p(S, G)\neq 0$ but $H^p(A, G)=0$ for each closed proper subset A of S, then S is a (topological) group. In particular a (classical) n-manifold is a clan if and only if it is a group. From a celebrated result of E. Cartan we see that an n-sphere is a clan only for $n=1, 3$. The results stated above are particularly concrete applications of general theorems. Among other consequences of these general theorems is the fact that a compact connected Hausdorff lattice is cohomologically trivial. This generalizes (in part) a result of Aumann and Eckmann. It is also shown that if an n-cell is a clan then the set of elements with inverses lie on the bounding $(n-1)$-sphere. (Received December 15, 1952.)

720*t*. C. T. Yang: *On theorems of Borsuk-Ulam, Kakutani-Yamabe-Yujobô, and Dyson.*

The results of this paper mainly rely on the concept of the order of a hyperplane which is a direct generalization of the order of a point as given in Alexandroff-Hopf, *Topologie*. Because of this concept, several theorems on spheres, such as Borsuk-Ulam's theorem (Fund. Math. (1933)), Kakutani-Yamabe-Yujobô's theorem (Osaka Math. J. (1950)), Dyson's theorem (Ann. of Math. (1951)), etc., are generalized. One generalization of Dyson's theorem is as follows: Let X be a compact subset of a euclidean space, symmetric with respect to an s-plane Θ. If there is an nth Θ-symmetric homology class (mod 2) of X with respect to which the order of Θ is not zero, then for any continuous function f on X there are n points $x_1, \cdots, x_n$ of X for which $f(x_1)=\cdots=f(x_n)=f(x_1')=\cdots=f(x_n')$ and the $(s+1)$-planes $\Theta x_1, \cdots, \Theta x_n$ are mutually orthogonal, where x_i' is the Θ-symmetric point of x_i and Θx_i is the $(s+1)$-plane determined by Θ and x_i, $i=1, \cdots, n$. In particular, if X is an n-sphere with center Θ, this gives Dyson's theorem on n-sphere. (Received June 22, 1953.)

L. W. COHEN,
Associate Secretary

THE CONFERENCE ON FIBER BUNDLES AND DIFFERENTIAL GEOMETRY IN ITHACA

A conference with the title *Fiber bundles and differential geometry* was held at Cornell University from May 3 to 7, 1953. It was supported by a grant from the National Science Foundation to Cornell University. The members of the organizational committee were S. S. Chern, University of Chicago, Samuel Eilenberg, Columbia University, N. E. Steenrod (chairman), Princeton University, R. J. Walker, Cornell University, and Hassler Whitney, Institute for Advanced Study.

In addition to the committee and the mathematicians at Cornell, the following twenty-two mathematicians attended the conference on invitation of the committee:

José Adem-Chahin, Warren Ambrose, Armand Borel, Eugenio Calabi, Alex Heller, F. E. P. Hirzebruch, S. T. Hu, Kunihiko Kodaira, E. G. Kundert, S. D. Liao, W. S. Massey, Deane Montgomery, J. C. Moore, G. D. Mostow, Katsumi Nomizu, Everett Pitcher, Hans Samelson, D. C. Spencer, P. A. Smith, R. L. Taylor, H. C. Wang, and G. W. Whitehead.

The program of the conference consisted of eight sessions with talks as follows:

1. Hans Samelson: *Survey of spectral sequences and the homology theory of bundles.*

W. S. Massey: *Exact couples.*

A. S. Shapiro: *Torsion in fiber bundles.*

2. Samuel Eilenberg: *Survey of the homology theory of groups and recent results on* $A(\pi, n)$, $K(\pi, n)$.

N. E. Steenrod: *Reduced powers, and homology theory of the symmetric groups.*

José Adem-Chahin: *Relations on iterated reduced powers.*

3. J. C. Moore: *Triad groups and the generalized isomorphism theorem of Hurewicz.*

G. W. Whitehead: *Survey of the methods of computing homotopy groups of spheres.*

Hans Samelson: *Pontrjagin products in spaces of loops.*

4. Paul Olum: *Survey of obstruction theory.*

S. D. Liao: *The secondary obstruction in sphere bundles.*

N. E. Steenrod: *The results of Thom and Wu on characteristic classes.*

5. Armand Borel: *Survey of the cohomology of Lie groups and their coset spaces.*

Alex Heller: *Universal bundles in singular homology.*

G. D. Mostow: *Solvable Lie groups and solvmanifolds.*

6. Warren Ambrose: *Generalized affine connections.*

S. S. Chern: *Infinite continuous pseudo-groups.*

7. D. C. Spencer: *Cohomology and Cousin's problem.*

Kunihiko Kodaira: *On complete continuous systems of divisors on algebraic varieties.*

F. E. P. Hirzebruch: *Complex structures on* $S^2 \times S^2$.

8. S. S. Chern: *The algebraic nature of the Chern classes.*

F. E. P. Hirzebruch, Kunihiko Kodaira, and E. G. Kundert: Three talks on "*Arithmetic genera.*"

H. C. Wang: *Closed manifolds with homogeneous complex structures.*

Some 17 years ago when Whitney isolated the concept of fiber bundle and observed its omnipresence throughout geometry, it became clear that he had opened a path for the large scale application of algebraic topology to the other branches of geometry. The conference program constituted a survey of the very extensive developments of the last six years based on this concept. The first half of the program was devoted to the upheaval within topology itself resulting from the use of fiber space techniques. The second half reviewed the applications to Lie groups, differential geometry, complex analytic manifolds, and algebraic geometry. The most striking feature of the conference was the frequent use of the same apparatus in two or more widely separated disciplines, with strong suggestions of a probable unification of geometry on some higher level. The discussions were marked by the presentations of numerous unsolved problems. These were recorded and a report on them is being prepared for publication.

N. E. STEENROD

BOOK REVIEWS

Algebraic projective geometry. By J. G. Semple and G. T. Kneebone. Oxford University Press, 1952. 8+404 pp. $7.00.

This work is similar in style and treatment to Semple and Roth's *Introduction to algebraic geometry*, but more elementary in scope; in fact the latter work, though it appeared earlier, can well be regarded as a sequel and continuation of the present one. On the other hand, it nearly coincides in scope, and in the fundamental point of view, with Todd's *Projective and analytical geometry*, but it uses a much simpler and more elementary technical apparatus.

A couple of introductory chapters deal with the parallel developments in the idea of geometry, on the one hand from descriptions of spatial phenomena, through the applications of algebraic technique, to the concept of geometric entities as defined in algebraic terms, which admits the extensions to any number of dimensions and to an arbitrary ground field (in the book, however, the field of complex numbers is the assumed ground field throughout); and on the other hand by extensions of the invariance group, from Euclidean through affine to projective geometry. This occupies about a tenth of the book. In the rest, complex homogeneous coordinates are the material, and the linear algebra of these the topic.

A chapter is devoted to one dimension, cross ratio, the homographic transformation, etc.; another chapter to the plane and the projective relations between ranges and pencils in it. The conic, and systems of conics, occupy the next four chapters; many metrical interpretations of projective results are pointed out, such as the treatment of confocals as a range or linear family of envelopes. This part of the book reaches its climax in an unusually luminous account of invariants. After a chapter on collineations, correlations, and the most elementary Cremona transformations (little of the last is treated except the quadratic transformations on three points; on the other hand collineations are carefully classified by means of the characteristic roots and vectors of their matrices) we turn to three dimensions.

Here after a preliminary chapter on the general relations between points, lines, and planes, the quadric locus and envelope are treated, classified, like the conics, by the rank of their coefficient matrices, and studied, for the general case, mainly in terms of the two generating reguli, and the appropriate parametrisations. Before considering families of quadrics there is a digression on the twisted cubic, and on the cubic surface, defined as the locus of intersections of corresponding planes of three related stars, which leads to the determinantal equation and the plane mapping—a treatment closely parallel to Reye's but expressed in algebraic terms. The pencil, range, and net of quadrics are then considered, with the cubic transformations determined by the latter, and Hesse's correspondence between its Jacobian sextic and a plane quartic—the last, only to the point of relating the 28 bitangents of the quartic to the 28 pencils of double contact in the net. Linear transformations in space are then treated, though nothing like a complete enumeration is attempted. There is here an interesting treatment of the collineations which leave a quadric invariant, and the applications of this theory to non-Euclidean geometry.

Last of all come a brief—too brief, perhaps—introduction to line geometry; and a final hint at the possibilities of n-dimensional geometry. The workings of duality in four and five dimensions, the general intersections relations in these spaces, the properties of the fifth associated plane in four dimensions, and the representation of conics in the plane and lines in ordinary space by points of five dimensions are offered to whet the appetite; and the student is left, a little abruptly perhaps, mature in outlook, and able to start in earnest on algebraic geometry.

P. DU VAL

Real functions. By Casper Goffman. New York, Rinehart, 1953. 12+263 pp. $6.00.

Principles of mathematical analysis. By Walter Rudin. New York, McGraw-Hill, 1953. 9+227 pp. $5.00.

Theory of functions of real variables. By Henry P. Thielman. New York, Prentice-Hall, 1953. 11+209 pp. $6.65.

These three books, each an introductory text on real function theory, have appeared almost simultaneously. This unusual situation has led the reviewer to write a single article comparing the three rather than to write a separate review of each. Thus, the grouping of the three books into one review is not to be taken as an indication that no one of them is of sufficient significance to merit a separate discussion. Rather, it is a recognition of the fact that they will be considered competitively so that a discussion of their relative merits would seem to be the most pertinent.

The following chart gives a brief summary of the contents of the three books. It lists the major topics considered in the union of the three books and gives under each author's name the reviewer's opinion of that author's treatment of the indicated topic.

Topic	Goffman	Rudin	Thielman
Real number system	good	good	good
Algebra of sets	good	good	good
Cardinal numbers	good	good	good
Ordinal numbers	good	none	incomplete
Elements of topology			
in Hausdorff spaces	none	none	good
in metric spaces	sketchy	sketchy	good
in Euclidean n-space	none	good	none
on real line	good	good	good
Limits superior and inferior	very sketchy	good	good
Continuity	good	elegant	good
Discontinuous functions	good	very sketchy	good
Infinite series	none	good	good
Fourier series	none	good	none
Uniform convergence	good	good	sketchy
Differentiation			
properties of derivatives	sketchy	good	very sketchy
existence theorems	very good	none	good
partial derivatives	none	good	none
Borel sets	good	very sketchy	good
Measure	specialized	abbreviated	very good
Riemann integrals	good	very good	good
Lebesgue integrals	involved	good	ineffectual
Differentiation of integrals	complicated	specialized	good

Some of these points call for more detailed discussion.

Each author obtains the real number system by extending smaller systems. Thielman starts with the positive integers (Peano postulates) and works through several extensions. Each of the others starts with the rational number system. Goffman characterizes it as a minimal ordered field and gives a uniqueness proof. Rudin merely appeals to the reader's intuition to accept the rational number system. Rudin and Thielman make the completeness extension by Dedekind cuts. Goffman does it first by Cauchy sequences, then does it again by Dedekind cuts. His proof of the equivalence of these procedures makes no mention of the fact that this equivalence depends on the Archimedean character of the rational field. He has mentioned informally (p. 30) that the rational field is Archimedean, but he has not proved it or even stated it as a theorem. Similarly, Rudin (p. 39) makes an informal statement which (while not inaccurate inasmuch as it refers to the real number system) seems to ignore the fact that there are non-Archimedean ordered fields which are Cauchy sequence complete but not Dedekind cut complete.

While on the subject, the reviewer would like to beat the drum for the idea that the discussion of extensions of number systems (involving, as it does, principally a verification of algebraic postulates in the extended system) belongs in an algebra course. For purposes of studying analysis it should suffice to describe the real number system as a complete ordered field and prove (or, preferably, refer to a proof of) uniqueness. Furthermore, the existence of least upper bounds is probably the most efficient completeness postulate—efficient in the sense of being directly applicable to the most theorems in real function theory.

Rudin avoids transfinite arguments altogether. Thielman mentions the linear ordering of the cardinals and develops the theory of ordinals to the point where he needs only the well-ordering theorem to prove linear ordering. He then states the well-ordering theorem informally, chooses not to prove it, and fails to point out its implications. Goffman gives a fairly complete discussion of ordinals including the amazing phenomenon of a proof of the well-ordering theorem with no mention of any transfinite axiom. The axiom of choice appears in the first paragraph of his proof in the following interesting way: "Let S be a set. Let there be a function $f(A)$ which associates an element $a \in S-A$ with every proper subset $A \subset S$, including the empty set." Perhaps this unusual wording makes it an axiom rather than a supposedly obvious statement; however it sounds more like the divine pronouncement, "Let there be light," in the first chapter of Genesis.

For an efficient discussion of uniform convergence and the reversal of order in iterated limits, the reviewer suggests the following theorems as a basis: (1) If both one-variable limits of $f(x, y)$ exist pointwise in the neighborhood of a point and if one of them is uniform (sometimes called subuniform) at the point, then the double limit exists at the point. (2) If both one-variable limits of $f(x, y)$ exist pointwise in the neighborhood of a point and if the double limit exists at this point, then (a) the iterated limits exist and are equal at this point, and (b) both one-variable limits are uniform at this point. (3) For a pointwise convergent sequence of continuous functions the set of points at which the convergence is not uniform is an F_σ set of the first category. Proofs of these theorems are not particularly long, and the immediate corollaries are too numerous to mention.

Thielman proves only that the uniform limit of a sequence of continuous functions is continuous. Rudin does not discuss category; so he is not interested in (3) above. He does prove (1) and (2a) with the stronger hypothesis in (1) that one of the one-variable limits is uniform on some fixed neighborhood of the point in question. He then makes very effective use of this result to get easy proofs of many corollaries, including that on order of summation of absolutely convergent double series. Goffman introduces the notions of category and of uniformity at a point and proves most of the interesting corollaries to theorems (1)–(3) above. However, he never actually states any one of the general theorems (1)–(3), with the result that his discussion is more circuitous than it needs to be.

Goffman's discussion of measure is limited to a direct development of Lebesgue measure on $[0, 1]$. Rudin discusses Lebesgue measure in Euclidean n-space, but he does it by abstract measure theory methods. That is, he discusses the extension of a completely additive function from a ring to a σ-ring. In describing the ring of elementary figures, he refuses to specify the closure properties of his intervals; so it is not at all clear that he has a ring. In defining an outer measure, he covers sets by intervals without specifying what kind. Later (p. 198) in order to prove regularity of the measure, he says that these covering intervals may be assumed to be open without affecting the value of the outer measure. This is true for continuous measures, therefore true for his only concrete example; but he pretends not to be restricting himself to this case. Rudin's whole discussion of measure, while elegant in outline, is sketchy in development and (as indicated above) somewhat careless. Thus, the last chapter tends to mar an otherwise beautifully written book. Thielman's discussion of measure is the best of the three. He develops Lebesgue measure in Euclidean n-space by the Carathéodory method of working with a postulationally defined outer measure.

Rudin defines the Lebesgue integral by means of approximations from below by simple functions. Goffman uses Fréchet's method of forming infinite series by considering infinite subdivisions of the entire range space. This leads to rather complicated looking proofs of the properties of the integral. Also, it restricts the discussion to domains with finite measure. Thielman uses Lebesgue's method of considering finite subdivisions of the range for bounded functions and then extending to the unbounded case by means of truncated functions. He then does it all over again by considering finite subdivisions of the domain into measurable subsets. For some unexplained reason this latter procedure is supposed to be particularly significant in connection with integrals defined over Borel sets. If there is a point to Thielman's discussion of integrals over Borel sets, the reviewer missed it.

Thielman proves practically none of the important properties of the integral, not even complete additivity. Only Goffman and Rudin prove the Lebesgue dominated convergence theorem. Only Goffman proves Egoroff's theorem, and only Rudin proves the Riesz-Fischer theorem. None of the three considers convergence in measure or gives necessary and sufficient conditions for mean convergence.

Rudin considers differentiation of integrals only for Riemann integrals of continuous functions. Goffman and Thielman both develop the Lebesgue theory. Thielman does a very neat job of it. Goffman makes it hard for himself by failing to prove and use the lemma that if $\overline{D}F(x) \geqq a$ almost everywhere in E, then $F(E) \geqq am(E)$.

Only Goffman and Thielman discuss Fubini's theorem. They are both rather vague about extending the proof from the case of non-negative functions to the general case. Neither points out the difficulty that if $\phi(y) = \int_X f(x, y)dx$, then it is not necessarily true that $\phi^+(y) = \int_X f^+(x, y)dx$. In fact, Thielman's statement (p. 180) that all his preceding results can be extended to functions with variable sign is incorrect.

Rudin is consistent in writing f for a function and $f(x)$ for one of its values. However, he slips up on sequences. He says (p. 19), "By a sequence we mean the values of a function f defined on the set J of all positive integers." Does this mean a sequence is a point set? Goffman speaks of a mapping f and a function $f(x)$. He says a sequence is a mapping and explains that the sequence $\{x_i\}$ and the point set $\{x_i\}$ are two different things. The notation is unfortunate, to say the least. Thielman is sound on the subject of sequences, but he varies from one place to another in his use of f and $f(x)$. Everyone seems to have trouble defining a series. Goffman gets out the easy way; he does not discuss series. Rudin does not commit himself to a precise definition, but he has a succession of remarks (p. 44) from which can be inferred first that the symbol $\sum_{n=1}^{\infty} a_n$ stands for a sequence (the sequence of partial sums), then that it stands for a number (the sum of the series). Thielman defines a series as a sequence (of partial sums) and then works himself into a verbal corner involving a series whose terms are its own partial sums!

Finally, an over-all view: Rudin's book is definitely the smoothest. He lists his theorems in the most effective order for facilitating his arguments, and he invariably comes up with extremely neat proofs. However, he does not push his investigations quite as far as the other authors do. A notable example of this is his failure to follow an excellent discussion of continuous functions by an investigation of category and Borel sets and their connection with real functions. With any topic he considers, Goffman pushes farther than either of the other two authors. In this sense, his book is the most complete of the three. Unfortunately, too many of his proofs are ungainly and complicated. Essentially, this stems from ineffective organization. In some places (compare his development of topology on the real line with Rudin's, for example) proofs could be simplified by rearranging the theorems. In other places (note the reviewer's comment on his treatment of differentiation of Lebesgue integrals) he repeats an argument several times because he fails to pull out an intermediate result which could be reused. Thielman falls in between. His proofs are not particularly striking, but standard and reasonably efficient and uncomplicated. He does not dig as deeply as Goffman, but more deeply than Rudin. The blight on Thielman's book is the surprising number of inaccurate statements and incomplete proofs. On a first reading the reviewer found ten examples, each of a flaw of one of the following types: informal statements of results that are not true, failure to consider all cases in a proof, use of concepts which have not been defined, use of results either without proof or before they have been proved. One such example was mentioned in the discussion of Fubini's theorem above. To cite one other, Thielman's proof that the union of a denumerable set of denumerable sets is denumerable tacitly assumes that the sets are disjoint.

Each of the three authors writes a very pleasing line of prose; so there is very little choice to be made on that score. Also, each book has an adequate supply of worthwhile exercises. The reviewer is unable to rate one above the other on this point.

There are a few typographical errors in each book. Those worth mentioning involve errors in cross references: Rudin—p. 74, line 22, for 4.1 read 4.2. Thielman—p. 97, line 6, for 3.2.2 read 3.2.1. Thielman—p. 184, line 7 from bottom, for 9.8.1 read 9.11.1.

M. E. MUNROE

Numerical solution of differential equations. By W. E. Milne. New York, Wiley, 1953. 11+275 pp. $6.50.

This book contains the first general treatment, in English, of numerical methods for solving differential equations. The author has been able to cover in the 275 pages only those classes of problems and methods which he considers most important. The methods are presented very clearly, with completely worked numerical examples, and should be easily mastered by the average reader. On the other hand it is evident from the choice of methods, particularly for problems involving latent roots of matrices and elliptic differential equations, that the emphasis is on the use of desk calculators rather than high speed computers. In fact, although the increased activity in the field of numerical analysis is largely due to the availability of high speed computers, these are mentioned only in the preface and in an appendix. Moreover, only one page is devoted to the treatment of round-off errors.

The first half of the book is devoted to methods for solving ordinary differential equations. Nine methods, some with variants, are presented for solving a single first-order equation. These include a method based on the trapezoidal rule, one based on Taylor's series, Picard's method, Adams' method, Moulton's method, Milne's method, and the Runge-Kutta method. Many of these methods involve the use of various numerical quadrature formulas which are derived in a special chapter on "analytic foundations." Additional methods are given for solving systems of first-order equations and initial value problems with equations of higher order. A special method is given for solving second-order equations with the first derivative absent. For each method, error estimates are derived which involve higher derivatives of the solution.

Six methods are presented for solving two-point boundary value problems. These include a method for linear equations where one solves a related initial value problem, a trial and error method, a variant of Picard's method, a method of differential variation, the Ritz method, and Galerkin's method.

The second half of the book is devoted to partial differential equations, and primarily to linear equations of second order. The author first considers problems of "explicit" type involving parabolic or hyperbolic equations where the numerical solution can be calculated step by step from the differential equation and the known initial and boundary conditions. A detailed study is made of difference equation methods for solving the heat and wave equations with two and three independent variables. Error estimates involving higher derivatives of the solution are given together with a discussion of the limitations on the size of the time interval which must be imposed for a given space interval for the sake of stability of the numerical procedure. In the reviewer's opinion the method given for treating curved boundaries is too complicated; the analysis of Collatz, reference 26, indicates that the use of four coefficients, as is usually done in practice, rather than ten, should be sufficiently accurate for most problems.

As a preliminary to the consideration of problems of "implicit" type involving elliptic equations where the unknowns are related by a system of simultaneous equations, the author devotes a chapter to methods for solving linear systems and for finding latent roots of matrices. After a brief review of matrix theory he discusses various methods of successive approximation which are based on an attempt to minimize a certain quadratic form. He considers a method involving changes in one unknown at a time, group changes, relaxation, a gradient method, and other iterative methods including one which uses Tchebycheff polynomials to accelerate the convergence. It might be well to mention that if the matrix of coefficients is positive definite, many of the methods can be simplified by the use of the quadratic form $Q(v)=(v\cdot Av)/2-(v\cdot b)$ rather than the one actually used. Among the methods for determining latent roots are the "escalator method," the use of iterated vectors, iteration and elimination, a method based on orthogonality properties of the latent vectors, and a gradient method. None of the methods given for finding latent roots seem to be suitable for large matrices.

Next, the author treats difference equation methods for solving elliptic equations in two dimensions including Laplace's equation, Poisson's equation, and the biharmonic equation. Both the usual five-point and the nine-point difference representations of the Laplacian operator are used. Special iterative methods are considered for solving the difference equations. Estimates of the accuracy of the solution of the difference equation for various mesh sizes are given involving higher derivatives of the solution. Although these estimates are not applicable in some cases, nothing better is usually available. The chapter is concluded with a nice discussion of the effect of discontinuities on the boundary.

The final chapter is devoted to methods for obtaining "characteristic numbers" both for ordinary and partial differential equations. These methods essentially involve the calculation of the characteristic equation for the associated difference equation and are not suited for use on computing machines. Three short appendices are given: (A) Round-off Errors, (B) Large Scale Computing Machines, (C) The Monte Carlo Method.

Although more emphasis on high speed computer methods would have increased the usefulness of the book, nevertheless the book is to be recommended as providing an excellent background in numerical methods. Examples for the student have been included and these, together with the clear presentation, the excellent format, and the relative freedom from misprints, make the book well adapted for classroom use.

DAVID YOUNG

Introduction to the foundations of mathematics. By R. L. Wilder. New York, Wiley; London, Chapman and Hall, 1952. 14+305 pp. $5.75.

For a number of years R. L. Wilder has given a very successful course in Foundations of Mathematics at the University of Michigan. This course has come to be well known, and the present book has grown out of it. The book promises to be as successful as the course. It is sound, thorough, modern, and readable, and well suited as a textbook for a course in foundations. It will probably be required reading for many other courses, and in addition should be valuable as a reference work for anyone interested in the philosophy or foundations of mathematics.

The book is divided into two parts. Part I, on Fundamental Concepts and Methods of Mathematics, presents the reader with actual instances of materials and methods of modern mathematics related to the foundations. For example, here is a list of some of the topics included in the seven chapters of Part I: The axiomatic method, independence, completeness, and consistency of axiom systems, axioms for simple order and equivalence, the theory of sets, paradoxes, the axiom of choice, cardinal and ordinal numbers, transfinite induction, the Hamel basis, Zorn's lemma, the real number system, Dedekind cuts, the Peano axioms for the integers, the complex number system, groups, semigroups, rings, ideals, integral domains, fields, vector spaces, group theory applied to algebra and geometry, and topology.

Part II of the book, called Development of Various Viewpoints on Foundations, takes up the broad questions that have arisen naturally from a consideration of the sort of situation met with in the first part. The beginning chapter of Part II traces the history of foundational questions up to about the year 1908. The next three chapters are devoted to three of the principal schools of thought on foundations: the Frege-Russell or logistic school, the intuitionist or Brouwer school, and the formalist or Hilbert school. In the last chapter, on the cultural setting of mathematics, the author presents some of his own views.

A feature of the book that will add greatly to its usefulness as a text is the inclusion, at the end of each chapter of Part I, of a list of suggested readings, followed by an extensive list of good problems. The eleven pages of bibliography, and the index of topics and of names, are other convenient features. Practically every topic the beginning student of foundations should be familiar with appears somewhere.

The chapter headings are: Chapter I, The Axiomatic Method;

Chapter II, Analysis of the Axiomatic Method; Chapter III, Theory of Sets; Chapter IV, Infinite Sets; Chapter V, Well-Ordered Sets; Ordinal Numbers; Chapter VI, The Linear Continuum and the Real Number System; Chapter VII, Groups and Their Significance for the Foundations; Chapter VIII, The Early Developments; Chapter IX, The Frege-Russell Thesis: Mathematics an Extension of Logic; Chapter X, Intuitionism; Chapter XI, Formalism; Chapter XII, The Cultural Setting of Mathematics.

More than enough material has been presented for a one-semester course, with the idea that some of it may be omitted. For this reason it would be unfair to call attention to other important topics that might have been included, but were not. As a matter of fact, this reviewer can think of but few such. There is but little emphasis on the distinction between axioms and rules of procedure. It is not surprising that the Church calculi of lambda-conversion and the related combinatory logic of Curry, both studied also by Rosser and Kleene, do not appear, although other works of these authors are treated. There is very little about semantics. There also seems to be lacking any sense of crisis in the foundations. One can deduce from the statements appearing in the book that we have no logical formulations which are certainly adequate for analysis and also provably consistent, although we do have formulations adequate for analysis and possibly consistent, and others adequate for parts of arithmetic and analysis and provably consistent. Whether this situation is inevitable, or may be improved, is one of the great challenges of the present state of the subject. This aspect does not seem to be emphasized.

The author's presentation of the three chief schools of logistics, intuitionism, and formalism is very fair and complete. He does not take sides, and tries to make the best possible case for each viewpoint. His own opinions, expressed in the last chapter, do not seem to touch on controversial issues. Rather, he presents a sort of sociological view of the development of mathematics, as determined by the culture of the age. This viewpoint is so broad and tolerant that each reader can deduce from it what he wishes. It seems unlikely that it will stir up any controversy or recriminations.

The style throughout is light and easy. One exception is the statement on page 278: "What great man, if honest with himself, has not observed the passing chimney-sweep without remarking to himself, 'There, but for the "concatenation of events," go I.' " If this sentence had been expressed in the notation of symbolic logic and then analyzed, it would probably have been amended, or perhaps omitted.

Altogether we have here a very fine addition to mathematical literature.

ORRIN FRINK

On the metamathematics of algebra. By A. Robinson. (Studies in Logic and the Foundations of Mathematics.) Amsterdam, North-Holland, 1951. 9+195 pp. 18.00 fl.

The purpose of this book is to show how the methods of symbolic logic may be applied to derive new results in algebra. This is a somewhat novel idea, since usually symbolic logic has been used to strengthen the foundations, rather than to add to the superstructure, of mathematical systems. The results obtained are in the nature of metatheorems rather than theorems; they deal with entire classes of algebraic systems rather than with one system at a time.

Some typical results are the following: 1. Any theorem formulated in the restricted calculus of predicates (in terms of equality, addition, and multiplication) which is true for all commutative fields of characteristic 0, is true for all commutative fields of characteristic $p>p_0$, where p_0 is a constant depending on the theorem. 2. Any theorem of the restricted calculus of predicates which is true for all non-Archimedean ordered fields is true for all ordered fields. 3. Any theorem of the restricted calculus of predicates which is true for the field of all algebraic numbers is true for any other algebraically closed field of characteristic 0.

A feature of the method is the following: since the results require only that the algebraic systems dealt with have certain broad, general properties, they naturally suggest significant generalizations of various algebraic notions. Examples of concepts generalized are: that of algebraic number, or a number algebraic with respect to a given commutative field; the notion of the polynomial ring obtained by adjoining n indeterminates to a commutative ring; and the concept of ideal. In general, the notions that may be handled by the author's method are those capable of being formulated within the restricted predicate calculus.

A number of the results obtained are significant for symbolic logic as well as for algebra. For example, it is shown that every model of an axiomatic system formulated in the restricted calculus of predicates can be extended. Consequently such a model cannot satisfy an axiom of completeness in Hilbert's sense. Since the concept of an ordered field can be formalized within the restricted calculus of predicates while the concept of an Archimedean field does in fact possess a model which is complete in Hilbert's sense (namely the field of all real numbers), it follows that Archimedes' axiom cannot be formalized within the restricted predicate calculus. This is interesting from the logical viewpoint.

The first chapter is a general introduction, summarizing the work. Chapter II, Construction of a formal language, describes the symbolism, rules of formation and procedure, and semantical interpretation of the object language. The resulting system is similar to the restricted predicate calculus of Hilbert. This chapter gives the logical machinery. The third chapter, which deals with the relation between deductive concepts on the one hand, and semantic or descriptive concepts on the other hand, contains also an important completeness theorem. One formulation of this theorem is: For every consistent set of statements K in a language L there exists a structure M such that all the statements of K hold in M, for some correspondence C. A consequence is a theorem of Lindenbaum, to the effect that every consistent set of statements K in a language L is included in a complete set K^*.

In the fourth chapter algebraic notions are introduced. Axioms are given for groups, rings, and fields in the logical notation used, and the notion of characteristic of a field and the Archimedean axiom are formalized. In Chapter V metamathematical theorems on algebraic fields such as those mentioned above are proved.

Chapter VI is more general, and deals with the broad notion of algebraic systems and algebras of axioms, rather than with special notions such as group, ring, and field. This chapter contains results on extensions of algebraic systems. The seventh and eighth chapters, on polynomials and algebraic predicates, deal with generalizations of the notions of polynomial and algebraic equation. The ninth chapter is about so-called convex systems. A system of axioms is convex, broadly speaking, if the intersection of any number of models of the system is a model. The last two chapters, on ideals and pre-ideals, are concerned with generalizations of the notion of ideal and of algebraic variety.

The author emphasizes that he has given more or less random samples of the type of result obtainable by his method. Many directions for further generalization are open. He has made a clear and compelling case for his principal contention, namely, that symbolic logic, in addition to its usefulness in investigating the foundations of algebra, is also a practical tool for obtaining new results in algebra of a very general sort, and results of a type not readily obtainable by other methods.

ORRIN FRINK

Abstract set theory. By A. A. Fraenkel. Amsterdam, North-Holland, 1953. 12+479 pp. $10.00.

This book is essentially a translated, revised, and expanded edition of the first three chapters of the author's well-known German treatise (*Einleitung in die Mengenlehre*, Berlin, 1st ed. 1919, 3d ed. 1928). In the preface the author mentions that he is preparing another book (*Foundations of set theory*). The list of topics to be discussed in that book ("antinomies of the transfinite, axiomatic methods of basing set theory, logistic attitudes from *Principia Mathematica* to the present day, intuitionism and neo-intuitionism, axiomatics in general and metamathematics") indicates that *Foundations* will bear the same relation to the last two chapters of *Einleitung* as does the present book to the first three.

Reviewing the first edition of *Einleitung*, G. A. Pfeiffer (Bull. Amer. Math. Soc. vol. 27 (1921) pp. 333–334) registered a mild complaint against the author's prolixity. In his review of the third edition, T. C. Benton (Bull. Amer. Math. Soc. vol. 36 (1930) pp. 27–28) chides the author for relegating the discussion of the axioms of set theory to the end of the book (i.e., to the part not contained in the version under review now).

If *Einleitung* is prolix, the present book is much more so. The introduction and first three chapters of *Einleitung* cover 209 pages; the same material now covers 331. In every discussion the author's aim is completeness; he mentions all the examples, all the philosophical interpretations and misinterpretations, and all the references that could conceivably be called relevant. While this might decrease the usefulness of the book as a text, it can only add to its value as a reference.

The treatment is intentionally non-axiomatic; the author wants to steer a course between naive carelessness and axiomatic formalism. He chooses to do this by the formulation of certain "principles." The principles are, in fact, highly informal statements of well-known set-theoretic axioms. There are seven of them: extension, subset, pairing, union, infinity, power, and choice. The principle of subsets, for instance, reads as follows: "given a set S and a property π meaningful for the elements of S, there exists the set containing those elements of S, and only those, which possess the property π."

In most respects there is no significant difference between the third edition of *Einleitung* and the present book. The subject matter, the style, the terminology, and the notation are all the same. (Sometimes this rigidity is mildly regrettable. Thus, for instance, union and intersection are still denoted by Σ and Π, and the power set of a set S is denoted by US; the U, presumably, stands for *Untermenge*.) Just as in the earlier edition, each section concludes with a useful collection of exercises.

A novel and laudable feature is the bibliography. This impressive list covers 137 pages; it is practically a separate publication in the same binding. (Possibly the publishers intend to make the bibliography and the main text available separately. The pages of the bibliography bear two numbers—one in continuation of the number system of the text and one independent of it.)

Readers not acquainted with *Einleitung* may gain an idea of the contents of the present book from the titles of its eleven sections; they are as follows. 1. Concept of set. Examples of sets. 2. The fundamental concepts. Finite and infinite. 3. Denumerable sets. 4. The Continuum. Transfinite cardinal numbers. 5. Ordering of cardinals. 6. Addition and multiplication of sets and cardinals. 7. Exponentiation of cardinals. The problem of infinitesimals. 8. Ordered sets. Similarity and order-types. 9. Linear sets of points. 10. General theory of well-ordered sets. Finite sets. 11. Ordinals and alephs. The well-ordering theorem.

PAUL R. HALMOS

Introduction to the theory of functions of a complex variable. By W. J. Thron. New York, Wiley, 1953. 10+230 pp. $6.50.

Here at last is a rigorous book on complex variables in the English language. There has long been a need for such a text and the author set for himself the task of filling this gap, and without doubt he has succeeded to a remarkable degree. The book is written in the Satz-Beweis style of Edmund Landau, each item being carefully labeled: Axiom, Theorem, Proof, Remark, etc., and in the reviewer's opinion this is of course the only possible style, where a rigorous exposition is the object in view. The book is divided into 31 short sections, instead of the more conventional longer chapters, and these shorter sections induce in the reader a comfortable feeling that not too much sustained effort will be required to read the book.

As every young instructor learns when he first strides into a class room resolved to make everything crystal clear to his pupils, the number system must first be put on a solid foundation, and this is not easy. Thron comes directly to grips with problem, and the allied problem of the foundations of geometry, and the first fourteen sections (more than one-third of the book) are devoted to these and related topics, and lie outside the domain proper of the theory of functions of a complex variable. These section headings are as follows:

1. Fundamental concepts, 2. Real numbers, 3. Cardinal numbers, 4. Complex numbers, 5. Sums and products, 6. Hausdorff spaces, 7. Metric spaces, 8. The plane of complex numbers, 9. Limits, continuity, differentiability, 10. Real functions of real variables, 11. Curves and regions in the plane of complex numbers, 12. Some combinatorial topology, 13. Jordan curves, 14. Rectifiable and directed curves. The material covered in these sections is just about what one would expect under these headings, and the only unusual item occurs in section 2. Here the author economizes on space and energy by advancing as an axiom the existence of a complete ordered field called the set of real numbers, and the reader is referred to Landau's Grundlagen for the proof that if there exists a set satisfying the Peano axioms then indeed there is a real number system.

Into the remaining space, devoted to complex variables proper, the author has packed a surprisingly large amount of material. In addition to all the standard topics one finds a proof of the Riemann mapping theorem, the construction of an elliptic modular function, and proofs of the Picard theorems. This large territory is covered by the simple device of leaving to the reader many of the easier proofs, and it is expressly stated in Remark 2.1 that if no proof is given for a theorem it means that the proof is left to the reader as an exercise. Despite this, the exercise lists at the end of the sections contain all too frequently the simple imperative "Prove theorem $X.Y.$" In any revision of this text, all such redundant exercises should certainly be replaced by problems which lead the reader further into the subject.

The author seems to belong to the group of extreme purists who feel that a picture should never be used to clarify a situation. For most of the book, no great harm is done, since the average reader already has some idea of the appearance of a circle, curve, etc. However when the author decomposes the set of linear fractional transformations into elliptic, parabolic, hyperbolic, and loxodromic subsets, he certainly should have added in Remark 28.4 that the reader should see figures 6, 7, and 8 of Ford's *Automorphic functions* for a clear understanding of the content of Theorem 28.20. Better still, these illustrations could easily have been reproduced. Again, in the proof of the Cauchy integral theorem, it is the author's intention to decompose a directed triangle into four similarly directed triangles, but misprints actually give one similarly directed triangle, two inversely directed triangles, and one line segment. This misprint is easily caught by anyone already familiar with the Goursat proof, but the poor student who is approaching it for the first time receives the full punishment for the author's sin of omitting a figure here.

The book is relatively free of misprints, and the only other important ones which the reviewer found occur in the proof of the Riemann mapping theorem where the function $\sqrt{(g(z)-\alpha)/(g(z)-1)}$ should be replaced by $\sqrt{(g(z)-\alpha)/(\bar{\alpha}g(z)-1)}$, and the function $h(z)$ should be read $(h(z)-\sqrt{\alpha})/(\sqrt{\bar{\alpha}}h(z)-1)$.

More serious errors are also present. (1) In the proof of Painlevé's theorem, the strong form of Cauchy's theorem is used, but this latter has not been proved (a pardonable omission) nor is any mention made of this omission (an unpardonable one). (2) The definition of the Farey series of order n is given in a sophisticated manner by requiring in the definition that two successive terms p_1/q_1 and p_2/q_2 satisfy the condition $p_2q_1-p_1q_2=1$. This of course spares the author the trouble of proving this result as a theorem, but in the opinion of the reviewer is logically unsound, because it leaves open the question of whether there might be two or more Farey series of order n (with his definition this can occur). If all fractions $0\leq p/q\leq 1$, with $q\leq n$, $(p, q)=1$, are to be included in the Farey series, this requirement assures us that there is at most one series of order n, but then this may turn out to be inconsistent with the condition $p_2q_1-p_1q_2=1$. (3) In Remark 23.3 the author states that his rather complicated definition of singular point is necessary "to avoid calling a point on a cut (other than a branch point) a singular point." As far as the reviewer could discover this is the only mention of the word cut in the entire book, and the remark only served to heighten the mystery, instead of elucidating it. (4) In the proof that on $|z|=r$, $\int z^{-1}dz=2\pi i$, the last assertion is unjustified, since no relationship has been developed between ϵ and η.

The author departs from standard usage in his definition of "conformal," and there seems to be no justification for this addition to the confusion in the meaning of words. On the other hand where a change should be made the author follows custom in defining a simple function in a region to be one that is single-valued holomorphic and takes no value more than once in the region. In the reviewer's opinion both the word simple and its international counterpart schlicht should be replaced by univalent for the simple reason that the latter is so suggestive of the more general concept of a multivalent function.

On the Jordan curve theorem, Thron states in his preface "The task I set myself demanded development of the basic tools of analysis and the inclusion of a proof of Jordan's curve theorem. . . ." On the other hand in a text appearing almost simultaneously Ahlfors states in his preface ". . . no proof is included of the Jordan closed curve theorem, which to the author's knowledge is never needed in function theory."

What is the place which Thron's book will occupy in the literature? Certainly it contains much valuable material, well organized and in convenient form for coordinated study. For this reason it belongs (a) in the library of every college which makes an attempt to teach mathematics, and (b) in the personal library of every specialist in function theory. However, just because it is so carefully written, with so much attention devoted to foundations, it should never (in the reviewer's opinion) be used as text either in a beginning or advanced course in function theory. The author is to be congratulated for his courage in writing such a book and his success in finding a publisher, and the publisher in turn has performed a real service for mathematics.

A. W. GOODMAN

Isoperimetric inequalities in mathematical physics. By G. Pólya and G. Szegö. (Annals of Mathematics Studies, no. 27.) Princeton University Press, 1951. 16+279 pp. $3.00.

The title of this book, as remarked by the authors in the preface (where the authors have admirably delineated the aims of the present work), suggests its connection with a classical subject of mathematical research, the "isoperimetric problem." This problem consists in seeking among all closed plane curves, without double points and having a given perimeter, the curve enclosing the largest area. The "isoperimetric theorem" gives the solution to the problem: of all curves with a given perimeter, the circle encloses the maximum area. If the perimeter of a curve is known, but the exact value of its enclosed area is not, the isoperimetric theorem yields a modicum of information about the area, an upper bound, an "isoperimetric inequality"; the area is not larger than the area of the circle with the given perimeter. There are, besides perimeter and area, many important geometrical and physical quantities (set functions, functionals) which depend upon the size and shape of a curve. There are many inequalities, similar to the isoperimetric inequality, which relate these quantities to each other. By extension, all these inequalities can be called "isoperimetric inequalities." Besides, there are analogous inequalities dealing with solids, pairs of curves (condenser, hollow beam), pairs of surfaces, and so forth. The present book is concerned with inequalities of this type.

An example of such an isoperimetric inequality, with which the subject matter of the book may be said to have begun, is the conjecture of B. de Saint-Venant (1856) concerning the torsion of elastic prisms: of all simply connected cross-sections with a given area, the circle has the maximum torsional rigidity. Inequalities of this sort are of practical value. Consider the inequality: of all triangular membranes with a given area the equilateral triangle has the lowest principal frequency. This furnishes a lower bound for the principal frequency of an arbitrary triangle in terms of its area. In doing this, a not readily accessible physical quantity (the principal frequency) is estimated in terms of easily accessible geometrical data (area, triangular shape). This illustrates the general trend of the present book: to estimate physical quantities on the basis of geometrical data, less accessible quantities in terms of more accessible ones.

This book is intended as a fairly complete account of investigations carried out by the authors for a number of years, and to which many scientists have contributed their efforts. The authors have succeeded in their task of achieving a unified, concise, easily readable presentation of the diversified material. In the course of the present review, references will be given only to papers not quoted in the book. Most of these have appeared since the publication of the book, which is a sign of the current interest in the subject matter under discussion.

A brief preface, which sets the stage for the rest of the book and presents concisely the authors' aims and point of view, has been summarized above. Chapter I, entitled "Definitions, methods and results," is a remarkably clear road map to the remainder of the book. As its title implies, this chapter fixes the notation to be employed throughout; it outlines the chief methods of attack employed: symmetrization, minimum principles, and expansion and variational methods; and it contains the precise statement of many of the results proved later. A great many results are compactly presented in the form of tables. The chapter ends with a brief survey of the following chapters.

Chapters II, III, and IV are concerned with the subject of capacity, and related topics. Chapter II is entitled "The principles of Dirichlet and Thomson." Let A be a closed surface, and $u(p)$ be a function harmonic outside A, assuming the constant value $u(p)=u_0$ on A and having the expansion $u(p)=X_0r^{-1}+X_1r^{-2}+X_2r^{-3}\cdots$ near infinity, where $X_0, X_1, X_2, \cdots$ denote surface harmonics of degrees 0, 1, 2, $\cdots$ respectively. The ratio $X_0/u_0=C$ depends only on the "conductor" A and is called its capacity. Clearly,

$$C = -\frac{1}{4\pi u_0}\iint \frac{\partial u}{\partial n_e}\,d\sigma = \frac{1}{4\pi u_0^2}\iiint |\operatorname{grad} u|^2 d\tau,$$

where the surface integral is taken over an arbitrary surface enclosing A (possibly A itself), $d\sigma$ and $d\tau$ are the surface and volume elements, respectively, and n_e is the outer normal. The principal purpose of the chapter is to find upper and lower bounds for C in terms of certain geometrical quantities associated with A. Upper bounds for C can be found by means of the principle of Dirichlet, which is formulated by the authors as follows: If $f(p)$ is an arbitrary scalar function defined on and outside A, with $f=u_0$ on A and $f=0$ at infinity, then

$$C \leqq (4\pi)^{-1}u_0^{-2}\iiint |\operatorname{grad} f|^2 d\tau \qquad \text{(Dirichlet's principle).}$$

Lower bounds for C can be found by means of Thomson's principle, which is formulated by the authors as follows: If $\mathbf{f}(p)$ is an arbitrary sourceless vector function defined on and outside A, that is,

$$\operatorname{div} \mathbf{f}(p) = 0, \qquad \text{outside } A,$$

and

$$(4\pi)^{-1}\iint \mathbf{f}\cdot n_e d\sigma = Q,$$

where Q is the total charge, then

$$1/C \leqq (4\pi)^{-1}Q^{-2}\iiint |\mathbf{f}|^2 d\tau \qquad \text{(Thomson's principle).}$$

It is clear that by choosing, in the above inequalities, particular functions $f(p)$ and $\mathbf{f}(p)$ satisfying the required conditions, one obtains upper and lower bounds for the capacity C of A. Direct formal proofs for the above inequalities are given, and it is shown that a certain minimum principle due to Gauss follows as a special case of Thomson's principle. The reviewer, however, has a predilection for the following derivation (patterned after J. B. Diaz and A. Weinstein, Journal of Mathematics and Physics vol. 26 (1947) pp. 133–136) which is based directly on Schwarz's inequality and clarifies the interrelation between the various minimum principles. For simplicity in writing, take $u_0=1$; then the problem of estimating C is that of finding upper and lower bounds for the Dirichlet integral

$$\iiint |\operatorname{grad} u|^2 d\tau \qquad (=4\pi C)$$

of the solution u of the Dirichlet problem:

$$\begin{aligned} u_{xx} + u_{yy} + u_{zz} &= 0, && \text{outside } A, \\ u &= 1, && \text{on } A, \\ \lim_{p \to \infty} u(p) &= 0. \end{aligned}$$

Schwarz's inequality states that

$$\left[\iiint \boldsymbol{\phi} \cdot \boldsymbol{\psi} d\tau\right]^2 \leq \left(\iiint \boldsymbol{\phi} \cdot \boldsymbol{\phi} d\tau\right)\left(\iiint \boldsymbol{\psi} \cdot \boldsymbol{\psi} d\tau\right),$$

the integrals being taken over the exterior of A. Let $f(p)$ be an arbitrary scalar function defined on and outside A, with $f=1$ on A and $f=0$ at infinity, and take $\boldsymbol{\phi} = \operatorname{grad} f$, $\boldsymbol{\psi} = \operatorname{grad} u$ in Schwarz's inequality. An application of Green's identity readily yields Dirichlet's principle

$$\iiint |\operatorname{grad} u|^2 d\tau \leq \iiint |\operatorname{grad} f|^2 d\tau.$$

Now let $\boldsymbol{f}(p)$ be an arbitrary (nonzero) vector function defined on and outside A, with $\operatorname{div} \boldsymbol{f}(p) = 0$, outside A, and take $\boldsymbol{\phi} = \boldsymbol{f}$, $\boldsymbol{\psi} = \operatorname{grad} u$ in Schwarz's inequality. An application of Green's identity

$$\iiint \boldsymbol{f} \cdot \operatorname{grad} u d\tau = \iiint [\operatorname{div} (u\boldsymbol{f}) - u \operatorname{div} \boldsymbol{f}] d\tau = -\iint u\boldsymbol{f} \cdot \boldsymbol{n}_e d\sigma$$

readily yields an inequality equivalent to Thomson's principle

$$\frac{\left(\iint \boldsymbol{f} \cdot \boldsymbol{n}_e d\sigma\right)^2}{\iiint |\boldsymbol{f}|^2 d\tau} \leq \iiint |\operatorname{grad} u|^2 d\tau.$$

If, in particular, $\boldsymbol{f} = \operatorname{grad} v$, where v is a nonconstant harmonic function, the last inequality yields (for the special Dirichlet problem under consideration here) E. Trefftz's (Verhandlungen Congress für Technische Mechanik, Zürich, 1927, p. 131) lower bound for the Dirichlet integral of the solution of Dirichlet's problem:

$$\frac{\left(\iint \frac{\partial v}{\partial n_e} d\sigma\right)^2}{\iiint |\operatorname{grad} v|^2 d\tau} \leq \iiint |\operatorname{grad} u|^2 d\tau.$$

If, in particular, the harmonic function v is given outside A by a potential of a single layer distribution, with density μ, over A, that is

$$v(p) = \iint \mu(q) \frac{1}{r_{pq}} d\sigma_q$$

(which serves to define v throughout space) then the last inequality (using the known jump condition for the normal derivative of a single layer potential) may be written

$$\frac{\left(\iint \frac{\partial v}{\partial n_e} d\sigma\right)^2}{4\pi \iint v\mu d\sigma - \iiint_{\text{interior of } A} |\operatorname{grad} v|^2 d\tau} \leqq \iiint |\operatorname{grad} u|^2 d\tau;$$

and, a fortiori, using Green's theorem to simplify the numerator,

$$4\pi\left(\iint \mu d\sigma\right)^2 \Big/ \iint v\mu d\sigma \leqq \iiint |\operatorname{grad} u|^2 d\tau,$$

which is equivalent to the principle of Gauss referred to above.

In order to make actual use of the inequality of Dirichlet's principle, suitable functions $f(p)$ must be chosen. The authors propose to choose first the level surfaces of f (as indicated in chapter I there are intuitive reasons of various kinds for preferring certain families of level surfaces) and then to find the "best function" possessing the prescribed level surfaces. Let $\psi(p) = \nu$ be the equation of the given level surfaces, call them $A(\nu)$, where $0 \leqq \nu < \infty$, and $A(0) = A$. Any function $f(p)$ having these prescribed level surfaces must be of the form $f(p) = \lambda(\psi(p))$ where $\lambda(t)$ is defined on $0 \leqq t < \infty$, and $\lambda(0) = u_0$, $\lambda(\infty) = 0$. It is shown that if

$$T(\nu) = \frac{1}{4\pi} \iint_{A(\nu)} |\operatorname{grad} \psi| \, d\sigma,$$

which depends only on the given surfaces $A(\nu)$, then upon choosing

$$\lambda(t) = \int_t^\infty [T(\nu)]^{-1} d\nu$$

one has

$$C \leqq \frac{1}{\int_0^\infty [T(\nu)]^{-1} d\nu},$$

which is the best upper bound for C obtainable from Dirichlet's principle once the level surfaces $A(\nu)$ are prescribed. In order to make actual use of the inequality of Thomson's principle, vector functions $f(p)$ must be chosen. Again, there are intuitive reasons for choosing a certain set of *lines of force* first. These lines do not determine $f(p)$ completely, but only its direction at each point p. The authors determine the best lower bound for C obtainable from Thomson's principle once the lines of force are prescribed in a certain way. A method of approximation given for the capacity is based on the theorem (when A is analytic, or, if not analytic, is star shaped with respect to an interior point p_0) that $1/C$ is the greatest lower bound of the quantities

$$-\frac{1}{4\pi}\iint_A V_0 \frac{\partial V_0}{\partial n_e}\, d\sigma = \frac{1}{4\pi}\iiint_{\text{exterior of } A} |\operatorname{grad} V_0|^2 d\tau$$

(compare with Trefftz's lower bound for C given above) where V_0 is an arbitrary harmonic function of the special type

$$V_0 = r^{-1} + H_1(x, y, z)r^{-3} + H_2(x, y, z)r^{-5} + \cdots + H_n(x, y, z)r^{-2n-1},$$

with r the distance from a fixed point p_0 in the interior of A, and $H_n(x, y, z)$ the most general homogeneous polynomial of degree n in x, y, z satisfying Laplace's equation. The principles formulated above are shown to apply to other charge distributions in which one has to deal with a pair of surfaces (a "condenser"). The analogous problems in two dimensions are also treated.

Chapter III bears the title: "Applications of the principles of Dirichlet and Thomson to estimation of the capacity." These principles are applied mainly in the form indicated in chapter II by choosing the level surfaces and the lines of force appropriately. Three classes of surfaces are considered: (1) convex surfaces, (2) surfaces which are star shaped with respect to a certain interior point p_0, and (3) surfaces of revolution. In (1) the exterior parallel surfaces and the normals to the given surface are chosen; in (2) the surfaces similar (with respect to p_0) to the given surface and the rays issuing from p_0 are chosen; while in (3) the surfaces of revolution obtained by rotating the level curves of the exterior conformal mapping of the meridian curve of the given surface onto a circle, and the curves in this mapping which correspond to the radii of the circle, are employed. As an example of the results obtained, consider the capacity C of a surface of revolution, representing the meridian curve in a complex w-plane and choosing the real axis as axis of symmetry. Let $w = f(z)$ be the mapping function of the exterior of this curve onto the exterior of the circle $|z| = \bar{r}$:

$$w = f(z) = z + c_0 + c_1 z^{-1} + \cdots + c_n z^{-n} + \cdots .$$

Then

$$\int_0^{\pi} \left\{ \int_{\bar{r}}^{\infty} [u(r, \theta)]^{-1} dr \right\}^{-1} d\theta \leqq C \leqq \left\{ \int_{\bar{r}}^{\infty} \left[\int_0^{\pi} u(r, \theta) d\theta \right]^{-1} dr \right\}^{-1},$$

where

$$r \operatorname{Im} f(re^{i\theta}) = 2u(r, \theta).$$

The equality signs hold for spheroids and only for spheroids. The same chapter also contains a proof of the Poincaré-Faber-Szegö inequality

$$C \geqq \left(\frac{3V}{4\pi} \right)^{1/3},$$

where C is the capacity of a closed surface and V is the volume of the solid bounded by the surface. The proof depends upon a combination of Dirichlet's principle and the process of symmetrization with respect to a point. As a particular, interesting special case of the inequalities obtained in this chapter, the following inequalities for the capacity C of a cube with edge a are given:

$$0.632a < C < 0.71055a.$$

Chapter IV is entitled "Circular plate condenser." A circular plate condenser consists of two congruent circular disks A_0 and A_1 with a common axis. Let the common radius be denoted by a, and the distance between the planes of the circular disks be c. Let $Q > 0$ and suppose Q and $-Q$ are the total charges of A_0 and A_1 respectively. In the position of equilibrium the potential will be constant on the disk and equal to $\pm V_0$. The problem of estimating the capacity $C = Q/2V_0$ of the condenser for small values of the ratio $c/a = q$ had been considered earlier by Kirchhoff and Ignatowsky. Using Gauss' principle, the authors obtain the following result:

$$\frac{C}{a} > \frac{1}{4q} + \frac{1}{4\pi} \log \frac{1}{q} + \frac{1}{4\pi}\left(\log 8 - \frac{1}{2} \right) + \epsilon,$$

where $\epsilon \to 0$ as $q \to 0$. The principle of Gauss is applied by assuming uniform charges of equal magnitude and opposite signs over the disks A_0 and A_1. A generalization of this line of reasoning, assuming an arbitrary circular-symmetrical charge distribution, leads not only to other lower bounds for C, but also, at least theoretically, to an exact formula for C. As a special case of the estimation of the capacity of a condenser consisting of two solids arising from each other by reflection in a plane, the chapter concludes with the following estimate for C for a circular plate condenser when c is large:

$$C = \frac{a}{\pi}\left(1 + \frac{2a}{\pi c}\right) + O\left(\frac{1}{q^2}\right).$$

Chapter V, entitled "Torsional rigidity and principal frequency," is divided into three main parts. The first part contains variational definitions for the torsional rigidity and the principal frequency, and consequences of these definitions. Let D be a bounded simply connected plane domain, C be its boundary curve, P its torsional rigidity, and Λ its principal frequency. Usually, the torsional rigidity of D is defined by the equation

$$P = 2\int\int_D v\,dx\,dy$$

where the function v (the stress function) is the solution of the following boundary value problem:

$$v_{xx} + v_{yy} + 2 = 0, \qquad \text{in } D;$$
$$v = 0, \qquad \text{on } C.$$

The variational definition for P used in this book is contained in the following inequality:

$$\frac{\displaystyle\int\int_D (f_x^2 + f_y^2)\,dx\,dy}{4\left(\displaystyle\int\int_D f\,dx\,dy\right)^2} \geq \frac{1}{P},$$

where f is a sufficiently smooth, not identically zero, real-valued function defined on $D+C$, satisfying the boundary condition $f=0$ on C. The equality sign holds if and only if $f=cv$, for some real number $c\neq 0$, where v is the stress function. (It may be remarked at this point that the reviewer has indicated, in the Proceedings of the Symposium on Spectral Theory and Differential Problems, Oklahoma A. and M. 1951, pp. 279–289, another variational definition for P, which is contained in the inequality

$$P \leq \int\int_D (g_x^2 + g_y^2)\,dx\,dy$$

where g is a sufficiently smooth function defined on $D+C$, satisfying the partial differential equation $g_{xx}+g_{yy}=-2$ in D. Here the "arbitrary" function g satisfies the same partial differential equation as the stress function v, and furnishes an upper bound for P, whereas in the authors' principle, the arbitrary function f satisfies the same boundary condition as the stress function v and yields a lower bound for P. This principle, which is related to Trefftz's lower bound (mentioned before) for the Dirichlet integral of the solution of Dirichlet's problem in terms of an arbitrary harmonic function, has been extended to the case of multiply connected domains and applied to the estimation of the torsional rigidity by H. F. Weinberger, Journal of Mathematics and Physics vol. 32 (1953) pp. 54–63.) The variational definition for Λ used in this book is contained in the inequality

$$\frac{\displaystyle\iint_D (f_x^2+f_y^2)\,dx\,dy}{\displaystyle\iint_D f^2\,dx\,dy} \geqq \Lambda^2,$$

where the arbitrary function f is as indicated above. The equality sign holds if and only if $f=cw$, for some real number $c\neq 0$, where w is the solution of the following boundary value problem:

$$\begin{aligned} w_{xx}+w_{yy}+\Lambda^2 w &= 0, \qquad w>0, &&\text{in } D,\\ w &= 0, &&\text{on } C. \end{aligned}$$

The variational definitions, together with Schwarz's inequality, are shown to yield the following interesting inequality connecting P, Λ and the area A of D:

$$P\Lambda^2 > 4A.$$

Assuming that C is star shaped, the method of choosing f having level lines which are similar to C is employed to obtain upper bounds for $1/P$ and Λ^2, in terms of an arbitrary function, which serves the purpose of assigning the values of f on the already prescribed level lines. Ingenious choices of this last arbitrary function lead to many interesting inequalities. The theory of conformal mapping is also employed to obtain further inequalities for P and Λ, for example $P\geqq(\pi/2)\mathring{r}^4$, where $\mathring{r}$ is the maximum inner radius of D, the equality sign holding only when D is a circle. A more general treatment of the problem of finding lower bounds for P is also given, when a set of curves is prescribed as the level curves of the arbitrary function f appearing in the variational definition of P. Let C_ρ, where $0\leqq\rho\leqq 1$, denote an arbitrary set of curves, "filling up the domain D," in such a way that C_0 is an interior point (or a finite set of interior points) of the domain D, C_1 is the boundary C of D, and C_ρ, for $0<\rho<1$, is a simple closed curve (or a finite set of mutually exclusive simple closed curves) such that C_ρ lies in the interior of $C_{\rho'}$ if $\rho<\rho'$. Then

$$P \geqq 4\int_0^1 [A(\rho)]^2[\lambda(\rho)]^{-1}d\rho$$

where $A(\rho)$ is the total area bounded by C_ρ and

$$\lambda(\rho) = \int_{C_\rho} |\operatorname{grad} \rho|\, ds.$$

The second part of chapter V contains the "inclusion lemma" and its application to deduce Saint Venant's approximate formula for torsional rigidity. The term "inclusion lemma" is used by the authors to refer to four lemmas whose content can be intuitively expressed, in the authors' words, as saying that "an arbitrary convex curve is only boundedly different from a suitable rectangle or a suitable ellipse." The application consists in showing that PIA^{-4} (where I is the polar moment of inertia of D with respect to its centroid) is contained between positive bounds for an arbitrary closed, bounded, convex domain D. The third part of chapter V contains applications of conformal mapping. Consider the conformal mapping

$$z = a_0 + a_1\zeta + a_2\zeta^2 + \cdots + a_n\zeta^n + \cdots, \qquad z = x + iy,$$

which maps D onto the interior of the circle $|\zeta|<1$. If $v(x, y)$ is the stress function, then the function Φ:

$$\Phi(x, y) = v(x, y) + \tfrac{1}{2}(x^2 + y^2)$$

is harmonic in D and is the real part of an analytic function (determined up to a purely imaginary constant)

$$F = \Phi + i\Psi$$

of the complex variable $z=x+iy$, which is regular in D. Since $z=z(\zeta)$ is analytic, one has

$$F(z(\zeta)) = u_0 + u_1\zeta + u_2\zeta^2 + \cdots + u_n\zeta^n + \cdots$$

and the problem of determining Φ (or v) is thus equivalent to determining the sequence of coefficients $u_0, u_1, u_2, \cdots$. Expansions for the polar moment of inertia are given in terms of the coefficients a_n of the mapping function $z=z(\zeta)$ and a proof of Saint Venant's theorem (of all simply connected sections with a given area, the circular cross section has the maximum torsional rigidity) is given. This theorem can be expressed by the inequality

$$2\pi P \leq A^2,$$

where A is, as usual, the area of the cross section. These considerations are based initially on the assumption that the series

$$|a_1| + 2|a_2| + \cdots + n|a_n| + \cdots$$

converges, but this assumption is later relaxed to that of the convergence of

$$|a_1|^2 + 2|a_2|^2 + \cdots + n|a_n|^2 + \cdots .$$

(In connection with the relationship between I, the polar moment of inertia of D with respect to its centroid, and the torsional rigidity P, the reviewer would like to recall at this point the simple inequality

$$P \leq I,$$

which was emphasized by the reviewer and A. Weinstein as an application of the formula, valid for simply or multiply connected D,

$$P = I - \iint_D (\phi_x^2 + \phi_y^2)dxdy.$$

where ϕ is the warping function in torsion.)

Chapter VI is entitled "Nearly circular and nearly spherical domains." The nature of the results contained here is perhaps best illustrated by considering the first example of this kind of investigation, due to Lord Rayleigh. Let $\bar{p}(\phi)$ be a fixed real valued function, $0 \leq \phi \leq 2\pi$, δ be a real number (in a sufficiently small neighborhood of zero) and consider the Fourier series expansion

$$\delta\bar{p}(\phi) = a_0 + 2\sum_{n=1}^{\infty} (a_n \cos n\phi + b_n \sin n\phi).$$

Rayleigh found that the principal frequency $\Lambda = \Lambda(\bar{p}, \delta)$ of the "nearly circular membrane" (the equation of whose boundary in polar coordinates r, ϕ is $r = 1 + \delta\bar{p}(\phi)$) is given by

$$\frac{j}{\Lambda} = 1 + a_0 - \sum_{n=1}^{\infty}\left(1 + \frac{2jJ_n'(j)}{J_n(j)}\right)(a_n^2 + b_n^2) + o(\delta^2)$$

as $\delta \to 0$, where j is the first positive root of the Bessel function $J_0(x)$, $j = 2.4048 \cdots$. Thus, Rayleigh expanded the variation $\delta\bar{p}$ of the circular boundary $r=1$ in a Fourier series and expressed the first and second variations of the physical quantity Λ^{-1} (dependent on the nearly circular boundary) in terms of the Fourier coefficients. The authors apply this method systematically, extending it by analogy from the plane to space, passing from Fourier expansions to expansions in spherical harmonics. A table in chapter I presents a concise survey of many of the results developed in chapter VI. The first part of the table lists physical quantities Q associated with a nearly circular curve C, i.e. expansions of the form

$$Q = 1 + a_0 + \sum_{n=1}^{\infty} R(n)\cdot(a_n^2 + b_n^2)$$

(neglecting terms of higher than second order in δ) where the Fourier coefficients a_n and b_n are as indicated above, and the $R(n)$ depend on the physical quantity Q. The second part of the table lists expansions of the form

$$Q = 1 + X_0 + \sum_{n=1}^{\infty} R(n)\cdot\frac{1}{4\pi}\int\int [X_n(\theta, \phi)]^2 d\omega,$$

for a nearly spherical surface whose equation, in spherical coordinates r, θ, ϕ, is

$$r = 1 + \delta\bar{p}(\theta, \phi),$$

with $\bar{p}(\theta, \phi)$ a fixed real valued function defined on the unit sphere, δ a real number in a sufficiently small neighborhood of zero, and

$$\delta\bar{p}(\theta, \phi) = \sum_{n=0}^{\infty} X_n(\theta, \phi)$$

the expansion of $\delta\bar{p}$ in spherical surface harmonics $X_n(\theta, \phi)$.

Chapter VII is entitled "On symmetrization." In the authors' own words, "the apparently scattered remarks of the present chapter are, in fact, carefully grouped around the idea of symmetrization." At the outset, the idea of symmetrization is connected with two general concepts (similar order and equimeasurability) concerning two real-valued functions of the n real variables $x_1, \cdots, x_n$. In terms of these two concepts, three kinds of symmetrization are defined in x, y, z space: with respect to a plane (Steiner), with respect to a straight line (Schwarz), and with respect to a point. Numerous results concerning the influence of symmetrization on many physical quantities follow. (L. E. Payne and A. Weinstein, Pacific Journal of Mathematics vol. 2 (1952) pp. 633–641, have recently obtained similar results for a generalized symmetrization which includes Steiner's and Schwarz's as special cases.) To mention only a few of the chapter's results: Schwarz symmetrization diminishes the polar moment of inertia with respect to the centroid; Schwarz symmetrization diminishes the capacity; of all conducting plates with a given area, the circle has the minimum electrostatic capacity (a conjecture of Lord Rayleigh). Many of the results obtained appear to be entirely inacessible to methods other than symmetrization. For example: of all tetrahedra with a given volume, the regular tetrahedron has the minimum surface area, integral of mean curvature, and capacity.

Chapter VIII is entitled "On ellipsoid and lens." The shape of an ellipsoid is characterized by its semi-axes a, b, c, with $a \geq b \geq c \geq 0$. Let α, β, γ be the eccentricities of the three principal sections of the ellipsoid, i.e.

$$1 - \alpha^2 = c^2 b^{-2}, \qquad 1 - \beta^2 = c^2 a^{-2}, \qquad 1 - \gamma^2 = b^2 a^{-2},$$

which satisfy the relation

$$1 - \beta^2 = (1 - \alpha^2)(1 - \gamma^2).$$

Approximating the electrostatic capacity by various geometric quantities is one of the principal aims. Let C' be an approximation to the capacity C of the ellipsoid. The relative error $(C' - C)/C$, which is a function of a, b, c, may be expanded in a power series about $(1, 1, 1)$, valid for small α, β, γ (i.e. for almost spherical ellipsoids). Written as an expansion in powers of β and γ, this expansion is a sum of homogeneous polynomials of different degrees; the non-identically vanishing polynomial of lowest degree is called the initial term of the relative error of the approximation C'. A table in Chapter I lists eight different approximations to C and their corresponding initial terms. In chapter VIII it is shown that one of these approximations:

$$\{11[a + b + c] + 4[(bc)^{1/2} + (ca)^{1/2} + (ab)^{1/2}]\}/45,$$

yields too large values for prolate spheroids and too small values for oblate spheroids; and the same holds for another approximation

$$\left[\frac{M}{4\pi} + \left(\frac{3V}{4\pi}\right)^{1/3}\right] \Big/ 2.$$

The authors illustrate very clearly in chapter I how the study of such particular examples as the ellipsoid and the lens could be useful in a continuation of the present study to lead eventually to a complete system of inequalities between the quantities C, V, S, M, $\cdots$.

The book concludes with seven notes and tables for some set functions of plane domains. The tables list (among others) the length L, the area A, the polar moment of inertia with respect to the centroid I, the maximum inner radius $\dot{r}$, the outer radius $\bar{r}$, the torsional rigidity P, and the principal frequency Λ for a circle, ellipse, narrow ellipse, square, rectangle, narrow rectangle, semicircle, sector, narrow sector, equilateral triangle, and regular hexagon. The notes contain new material and survey the more important contributions to the subject obtained by the authors, sometimes in collaboration with others, up to the date of appearance of the book. (For a recent, related result see G. Pólya, Journal of Mathematics and Physics vol. 31 (1952) pp. 55–57.) Their titles clearly indicate their connection with the various chapters described above. Note A, "Surface-area and Dirichlet's integral," deals with Steiner, Schwarz, and circular symmetrization and their effect on the volume, surface area, Dirichlet integral, and other quantities. Note B, "On continuous symmetrization," deals with the question of defining a transformation T_λ, depending continuously on a real parameter λ, with $0 \leq \lambda \leq 1$, such that T_0 is the identity, T_1 is Steiner symmetrization of a plane curve C, while T_λ, for $0 < \lambda < 1$, changes certain quantities associated with the curve C "in the same manner" as is done by Steiner symmetrization of C. Note C, "On spherical symmetrization," is concerned with the effect of this symmetrization, which is the three-dimensional analogue of the circular symmetrization of note A, on the capacity of a solid. Note D, "On a generalization of Dirichlet's integral," treats the generalized capacity defined by minimizing the integral

$$\iint_D \{ |\operatorname{grad} u|^2 + p(x, y)u^2 \} \, dxdy,$$

where $p(x, y)$ is a given function defined on D and the admissible functions u assume given values on the boundary of D. The classical Dirichlet principle arises if $p(x, y) \equiv 0$. Note E, "Heat conduction on a surface," considers the Dirichlet integral

$$\iint_D |\operatorname{grad} f|^2 d\sigma$$

taken over a connected domain of an open or closed surface in three-dimensional Euclidean space, f being a function defined on the surface and the gradient being defined in the sense of the metric of the surface. Lower bounds for the conductance of a "ring shaped" domain D on the surface are found in terms of geometrical quantities connected with D. Note F, "On membranes and plates" deals with three quantities Λ_1, Λ_2, Λ_3, defined by the following variational problems:

$$\Lambda_1^2 = \min \frac{\iint |\operatorname{grad} u|^2 d\sigma}{\iint u^2 d\sigma}, \qquad u = 0 \text{ on } C,$$

$$\Lambda_2^4 = \min \frac{\iint (\nabla^2 u)^2 d\sigma}{\iint u^2 d\sigma}, \qquad u = \frac{\partial u}{\partial n} = 0 \text{ on } C,$$

$$\Lambda_3^2 = \min \frac{\iint (\nabla^2 u)^2 d\sigma}{\iint |\operatorname{grad} u|^2 d\sigma}, \qquad u = \frac{\partial u}{\partial n} = 0 \text{ on } C,$$

where D is a bounded plane domain with a simple analytic boundary curve C. Λ_1 and Λ_2 are the principal frequencies of a membrane with fixed boundary and of a clamped plate, respectively. Λ_3 occurs in the study of the buckling of plates. Lord Rayleigh formulated the following conjecture (first proved by G. Faber and E. Krahn): of all membranes of a given area the circle has the gravest fundamental tone (lowest principal frequency). This note deals with the analogous problem for Λ_2 and Λ_3, under the hypothesis that the functions u for which the minima are attained never vanish in D. Note G, "Virtual mass and polarization," is dedicated to relating the quantities in the title to certain geometrical data of a solid.

This review only gives an idea of the nature and the variety of the problems discussed by the eminent authors, whose scientific accomplishments and lucid methods of presentation are well known to the mathematical public. In the brief time since its appearance this book has already become the standard reference text for workers in this field.

J. B. DIAZ

NOTES

The Second International Congress of the International Union for the Philosophy of Science will be held in Zurich, Switzerland, August 23–28, 1954. Communications should be addressed to the International Congress for the Philosophy of Science, International Forum of Zurich, Swiss Federal Institute of Technology, room 20d, Zurich 6, Switzerland.

The Department of Mathematics at Harvard University expects to appoint two new Benjamin Peirce Instructors in February, 1954. To be eligible for appointment, candidates must have received the Ph.D. by June or September, 1954, and should have high promise. Appointment is for two years, with re-appointment for a third year at the optinion of the instructor. Further particulars will be supplied on request.

The School of Mathematics of the Institute for Advanced Study will allocate a small number of grants-in-aid to gifted young mathematicians and mathematical physicists to enable them to study and to do research work at Princeton during the academic year 1954–1955. Candidates must have given evidence of ability in research comparable at least with that expected for the degree of Doctor of Philosophy. Blanks may be obtained from the School of Mathematics, Institute for Advanced Study, Princeton, New Jersey, and are returnable by January 1, 1954.

The Department of Mathematics at the Massachusetts Institute of Technology wishes to announce the availability of C. L. E. Moore Instructorships in Mathematics for 1954–1955, open to young mathematicians with doctorates who show definite promise in research. Applications should be filed not later than January 25, 1954, on forms obtained from the Department.

Mr. Thøger Busk of the University of Copenhagen is on leave of absence at the University of California, Berkeley; he has been awarded a Rockefeller fellowship.

Professor L. V. Ahlfors of Harvard University has been elected to membership in the National Academy of Sciences.

Professor C. C. Chevalley of Columbia University is on leave of absence at Nagoya University; he has received a Fulbright award.

Associate Professor D. E. Christie of Bowdoin College is on leave of absence at Princeton University; he has been awarded a Ford Foundation fellowship.

Professor Einar Hille of Yale University has been elected to membership in the National Academy of Sciences.

Associate Professor F. J. Massey, Jr., of the University of Oregon is on leave of absence at Harvard University; he has been awarded a Ford Foundation fellowship.

Assistant Professor Abraham Seidenberg of the University of California, Berkeley, is on leave of absence and has been awarded a Guggenheim fellowship.

Professor Alexander Weinstein of the University of Maryland has been elected a corresponding member of the Academia Nacional de Ciencias Exactas of Lima, Peru.

Dr. H. G. Eggleston of University College, Swansea, South Wales has been appointed a lecturer at Cambridge University.

Dr. S. H. Khamis of the United National Statistical Office has been appointed to an associate professorship at the Economic Research Institute, American University of Beirut.

Dr. Katsumi Nomizu of Columbia University has been appointed an attaché de recherches at the National Center for Scientific Research, Paris.

Mr. D. W. Pounder of A. V. Roe, Canada, Ltd., has accepted a position as aerodynamic engineer with de Havilland Propellers Ltd., Hatfield, Hertfordshire, England.

Mr. M. J. Aegerter of the University of Wisconsin has accepted a position as mathematician with the Naval Ordnance Test Station, China Lake, California.

Miss Florence R. Anderson of Northrop Aircraft, Inc., has accepted a position as research engineer with the North American Aviation Company, Los Angeles, California.

Assistant Professor J. W. Beach of the University of New Mexico has been appointed to an associate professorship at Northern Illinois State Teachers College.

Assistant Professor Helen P. Beard of Newcomb Memorial College, Tulane University of Louisiana, is on leave of absence at the Statistical Laboratory, University of California, Berkeley.

Dean Emeritus Samuel Beatty has been appointed chancellor of the University of Toronto.

Dr. S. E. Banach of the University of Illinois has accepted a position as research engineer with the Jet Propulsion Laboratory, California Institute of Technology.

Dr. Kurt Bing of Harvard University has been appointed to an assistant professorship at Rensselaer Polytechnic Institute.

Dr. D. W. Blackett of the Analytical Research Group, Princeton University, has been appointed to an assistant professorship at Boston University.

Mr. W. W. Bledsoe of the University of California, Berkeley, has accepted a position as mathematician with the Sandia Corporation.

Assistant Professor H. D. Block of the Iowa State College of Agriculture and Mechanic Arts has been appointed to a visiting assistant professorship at the University of Minnesota.

Dr. E. K. Blum of the University of Maryland has accepted a position as mathematician with the Naval Ordnance Laboratory, Silver Spring, Maryland.

Dr. C. W. Burrill of the University of Manchester, Manchester, England, has been appointed to an assistant professorship at New York University.

Professor L. E. Bush of the College of St. Thomas has been appointed to a professorship at Kent State University.

Dr. G. C. Byers has been appointed to an assistant professorship at the Michigan College of Mining and Technology.

Dr. J. R. Byrne of the University of Washington has been appointed to an assistant professorship at San Jose State College.

Dr. C. E. Capel of the Tulane University of Louisiana has been appointed to an assistant professorship at the University of Miami.

Dr. Buchanan Cargal of the Iowa State College of Agriculture and Mechanic Arts has accepted a position as senior aerophysics engineer with the Consolidated Vultee Aircraft Company, Ft. Worth, Texas.

Dr. C. C. Chang has been appointed to a research professorship at the Institute for Fluid Dynamics and Applied Mathematics, University of Maryland.

Mr. P. L. Chessin of the Cooper Union School of Engineering has accepted a position as associate engineer with the Westinghouse Electric Corporation, Baltimore, Maryland.

Mr. R. R. Christensen has accepted a position as research analyst with the Northrop Aircraft Company, Hawthorne, California.

Dr. D. R. Clutterham of the University of Illinois has accepted a position as senior aerophysics engineer with the Consolidated Vultee Aircraft Company, Ft. Worth, Texas.

Dr. H. S. Collins of the University of Maryland has been appointed to an assistant professorship at the University of South Carolina.

Professor J. A. Daum of the Agricultural and Mechanical College of Texas is on leave of absence and is engaged in research for the Research Foundation of the College.

Mr. R. B. Deal, Jr. of the University of Oklahoma has been appointed to an assistant professorship at the Oklahoma Agricultural and Mechanical College.

Dr. B. M. Drucker of the Oak Ridge National Laboratory has been appointed to an assistant professorship at the Georgia Institute of Technology.

Mr. J. E. Dutt of Columbia University has accepted a position as mathematician with the Remington Rand Corporation, South Norwalk, Connecticut.

Professor W. E. Edington of DePauw University has retired with the title professor emeritus.

Dr. W. S. Ericksen of the U. S. Forest Products Laboratory has been appointed to a professorship at the U. S. Air Force Institute of Technology.

Assistant Professor W. T. Fishback of the University of Vermont has been appointed to an assistant professorship at Ohio University.

Professor Edward Fleisher of Brooklyn College has retired with the title professor emeritus.

Assistant Professor J. B. Giever of the Massachusetts Institute of Technology has been appointed to an assistant professorship at the University of Oklahoma.

Professor Wallace Givens of the University of Tennessee is on leave of absence at the Institute of Mathematical Sciences, New York University.

Professor G. A. Hedlund of Yale University is on leave of absence at the Institute for Advanced Study.

Associate Professor L. A. Henkin of the University of Southern California has been appointed to an assistant professorship at the University of California, Berkeley.

Dr. I. N. Herstein of the Cowles Commission for Research in Economics, University of Chicago, has been appointed to an assistant professorship at the University of Pennsylvania.

Associate Professor R. C. Hildner of the University of New Mexico has accepted a position as staff member with the Sandia Corporation.

Assistant Professor P. G. Hodge, Jr., of the University of California, Los Angeles, has been appointed to an associate professorship at the Polytechnic Institute of Brooklyn.

Dr. Burrowes Hunt of the University of Pennsylvania has been appointed to an assistant professorship at Reed College.

Mr. W. R. Hydeman of the Defense Department has accepted a position as mathematician with the Engineering Research Associates, Arlington, Virginia.

Dr. H. G. Jacob of Yale University has been appointed to an assistant professorship at the Louisiana State University and Agricultural and Mechanical College.

Dr. Arno Jaeger of the University of Illinois has been appointed to an associate professorship at the University of Cincinnati.

Mr. J. W. Jewett of the University of Michigan has been appointed to an acting assistant professorship at the University of Alabama.

Dr. M. L. Juncosa of the Ballistic Research Laboratories, Aberdeen Proving Ground, Maryland, has accepted a position as mathematician with the Rand Corporation, Santa Monica, California.

Mr. Costas Kassimatis of the Ontario Research Foundation has been appointed a senior mathematics teacher at Cantab College, Toronto.

Mr. J. E. Kelley of the University of Michigan has accepted a position as analyst with the Defense Department, Washington, D. C.

Visiting associate Professor J. L. Kelley of the University of Kansas has been appointed to a professorship at the University of California, Berkeley.

Associate Professor E. S. Kennedy of the American University of Beirut is on leave of absence and has been appointed to a visiting professorship at Brown University.

Dr. Horace Komm of the University of the South has been appointed to an assistant professorship at Rensselaer Polytechnic Institute.

Dean Gillie A. Larew of Randolph-Macon Woman's College has retired with the title dean emeritus.

Dr. J. R. Lee of the University of Michigan has been appointed to an associate professorship at the College of William and Mary.

Mr. R. J. Lemelin of the Willow Run Research Center, University of Michigan, has accepted a position as analyst with the United Aircraft Corporation, East Hartford, Connecticut.

Associate Professor R. J. Levit of the University of Georgia is on leave of absence at the Stern Box Company, San Francisco, California.

Professor E. R. Lorch of Columbia University is on leave of absence at the University of Rome.

Assistant Professor G. G. Lorentz of the University of Toronto has been appointed to a professorship at Wayne University.

Dr. R. D. Luce of the Massachusetts Institute of Technology has been appointed director of the Behavioral Models Project, Bureau of Applied Social Research, Columbia University.

Assistant Professor R. C. Lyndon of Princeton University has been appointed to an assistant professorship at the University of Michigan.

Professor R. B. McClenon of Grinnell College has retired with the title professor emeritus.

Dr. F. A. McMahon of the Sperry Gyroscope Company has accepted a position as training supervisor with the General Precision Laboratory, Pleasantville, New York.

Assistant Professor L. E. Malvern of the Carnegie Institute of Technology has been appointed to an associate professorship at Michigan State College.

Dr. L. F. Meyers has been appointed to an acting assistant professorship at the University of Virginia.

Assistant Professor J. T. Moore of the Georgia Institute of Technology has been appointed to an associate professorship at the University of Florida.

Professor Emeritus E. E. Moots of Cornell College has been appointed a lecturer at Occidental College.

Miss Edith Moss has accepted a position as D.I.C. staff computer with the Massachusetts Institute of Technology.

Professor Emeritus E. J. Moulton of Northwestern University has been appointed to a visiting professorship at the University of Miami.

Associate Professor R. R. Otter of the University of Notre Dame is on leave of absence at Princeton University.

Visiting Assistant Professor T. K. Pan of the University of California, Berkeley, has been appointed to a visiting associate professorship at the University of Oklahoma.

Dr. Emanuel Parzen of the University of California, Berkeley, has accepted a position as research scientist at the Hudson Laboratories, Columbia University.

Dr. A. M. Peiser of Hydrocarbon Research, Inc., is now engaged in private consulting, with offices at 35 Wing Lane, Wantagh, New York.

Dr. C. L. Perry of the Oak Ridge National Laboratory has been appointed to an assistant professorship at the U. S. Naval Postgraduate School.

Professor R. S. Phillips of the University of Southern California is on leave of absence at Yale University.

Dr. R. L. Plunkett of the University of Virginia has been appointed to an assistant professorship at Vanderbilt University.

Assistant Professor M. H. Protter, of Syracuse University has been appointed as an assistant professorship at the University of California, Berkeley.

Mr. Valdemars Punga of the University of Massachusetts has been appointed to an assistant professorship at Rensselaer Polytechnic Institute.

Dr. Gustave Rabson of Purdue University has been appointed to an assistant professorship at Antioch College.

Mr. E. R. Rang of the University of Minnesota has accepted a position as mathematician with the Naval Ordnance Test Station, China Lake, California.

Dr. Mina S. Rees of the Office of Naval Research has been appointed dean of the faculty at Hunter College.

Professor H. E. Robbins of the University of North Carolina has been appointed to a professorship at Columbia University.

Dr. Louis Robinson of Syracuse University has accepted a position as mathematician with the International Business Machines Corporation, Boston, Massachusetts.

Professor R. A. Rosenbaum of Reed College has been appointed to a professorship at Wesleyan University.

Dr. Lawrence Rosenfeld of the Air Force Cambridge Research Center has accepted a position as mathematician with the Raytheon Manufacturing Company, Waltham, Massachusetts.

Professor E. V. Schenkman of the Louisiana State University and Agricultural and Mechanical College is on leave of absence at the Institute for Advanced Study.

Professor J. P. Scholz of Lebanon Valley College has been appointed to a professorship at Western College, Oxford, Ohio.

Assistant Professor B. L. Schwartz of Duquesne University has accepted a position as principal mathematician with the Battelle Memorial Institute, Columbus, Ohio.

Mr. B. M. Seelbinder of the University of North Carolina has been appointed to an acting assistant professorship at the University of Alabama.

Mr. J. M. Shaheen of the University of Cincinnati has been appointed to an assistant professorship at Tri-State College, Angola, Indiana.

Dr. Henry Sharp, Jr. has been appointed to an assistant professorship at the Georgia Institute of Technology.

Dr. R. J. Silverman of the Catholic University of America has accepted a position as operations analyst with the Armour Research Foundation, Illinois Institute of Technology.

Dr. V. R. Staknis of Boston University has been appointed to an assistant professorship at Northeastern University.

Assistant Professor W. L. Stamey of the University of Georgia has been appointed to an assistant professorship at the Kansas State College of Agriculture and Applied Science.

Mr. R. N. Tompson of Brown University has been appointed to an assistant professorship at Florida State University.

Mr. W. J. Turansky has accepted a position as mathematician with the Remington Rand Corporation, Philadelphia, Pennsylvania.

Dr. F. C. Warner of the University of Buffalo has been appointed to an assistant professorship at St. Lawrence University.

Mr. J. N. Williams of Trinity College has accepted a position as analyst with the Machine Computing Group, United Aircraft Corporation, East Hartford, Connecticut.

Assistant Professor G. N. Wollan of Memphis State College has been appointed to an assistant professorship at the Purdue University Center, Ft. Wayne, Indiana.

Dr. Henry Zatzkis of the University of Connecticut has been appointed to an assistant professorship at the Newark College of Engineering.

The following promotions are announced:

D. L. Arenson, Cook Research Laboratory, Skokie, Illinois, to technical director of the Aerophysics Section.

H. C. Bennett, Syracuse University, to an assistant professorship.

Gerald Berman, Illinois Institute of Technology, to an assistant professorship.

G. M. Bloom, Miami University, to an associate professorship.

Fred Brafman, Wayne University, to an assistant professorship.

H. N. Carter, University of Tulsa, to an associate professorship.

Max Coral, Wayne University, to a professorship.

S. P. Diliberto, University of California, Berkeley, to assistant dean of the College of Letters and Science.

J. F. Foster, Jr., University of Arizona, to an assistant professorship.

J. R. Garrett, Georgia Institute of Technology, to an associate professorship.

B. H. Gere, Hamilton College, to a professorship.

Leonard Gillman, Purdue University, to an assistant professorship.

A. M. Gleason, Harvard University, to an associate professorship.

Frank Harary, University of Michigan, to an assistant professorship.

A. R. Harvey, San Diego State College, to an associate professorship.

Melvin Henriksen, Purdue University, to an assistant professorship.

W. M. Hirsch, New York University, to an assistant professorship.

W. N. Huff, University of Oklahoma, to an associate professorship.

T. A. Jeeves, University of California, Berkeley, to an assistant professorship.

P. S. Jones, University of Michigan, to an associate professorship.

C. E. Langenhop, Iowa State College of Agriculture and Mechanic Arts, to an associate professorship.

C. C. Lin, Massachusetts Institute of Technology, to a professorship.

M. S. MacPhail, Carleton College, to a professorship.

Paul Meier, Johns Hopkins University, to an assistant professorship.

W. I. Miller, Bucknell University, to a professorship.

Morris Morduchow, Polytechnic Institute of Brooklyn, to an associate professorship.

D. C. Murdoch, University of British Columbia, to a professorship.

M. W. Oliphant, Georgetown University, to an assistant professorship.

L. L. Pennisi, University of Illinois, to an assistant professorship.

P. M. Pepper, Ohio State University, to an associate professorship.

T. J. Pignani, Loyola University, to an assistant professorship.

C. J. Pipes, Jr., Southern Methodist University, to an associate professorship.

George Piranian, University of Michigan, to an associate professorship.

E. D. Rainville, University of Michigan, to a professorship.

L. L. Rauch, University of Michigan, to a professorship.

H. L. Royden, Stanford University, to an associate professorship.

Charles Saltzer, Case Institute of Technology, to an assistant professorship.

Sister Mary Charles Schaffer, Immaculate Heart College, to an associate professorship.

R. H. Sorgenfrey, University of California, Los Angeles, to an associate professorship.

R. L. Westhafer, New Mexico College of Agriculture and Mechanic Arts, to a professorship.

L. A. Zadeh, Columbia University, to an associate professorship.

H. S. Zuckerman, University of Washington, to a professorship.

The following appointments to instructorships are announced: Amherst College: Dr. A. B. Willcox; University of British Columbia: Dr. R. J. Wisner; University of Colorado: Mr. E. B. McLeod, Jr.; University of Connecticut: Dr. O. I. Litoff, Mr. John Rausen; Cornell University: Mr. Walter Feit, Dr. I. S. Gál; Dartmouth College: Mr. R. G. Kuller; Duke University: Dr. C. M. Petty; University of

Maryland: Dr. Gertrude Ehrlich; Massachusetts Institute of Technology: Dr. D. L. Wallace; University of Minnesota: Dr. C. Y. Wang; Ohio State University: Dr. D. W. Dubois; University of Oklahoma: Dr. G. M. Petersen; Princeton University: Dr. Felix Haas; Purdue University: Dr. H. E. Chrestenson, Dr. Chester Feldman; San Jose State College: Mr. V. E. Hoggatt, Jr.; Stevens Institute of Technology: Mr. M. E. White; Trinity College: Mr. R. E. Montgomery; Tulane University of Louisiana: Mr. J. S. Griffin, Jr.; State College of Washington: Dr. M. T. Wechsler; Wayne University: Mr. H. T. Slaby; Willamette University: Mr. J. A. Nickel.

Professor Emeritus J. G. Hardy of Williams College died on September 5, 1953 at the age of seventy-nine years. He had been a member of the Society for fifty-one years.

Professor Emeritus W. P. Russell of Pomona College died on January 10, 1953 at the age of ninety years. He had been a member of the Society for forty-seven years.

Reverend D. A. Steele of Fordham University died on June 18, 1953 at the age of fifty-nine years.

NEW PUBLICATIONS

Abdelhay, J. *Curso de análise matemática.* Vol. I. 2d ed. Rio de Janeiro, Universidade do Brasil, 1953. 16+232 pp.

Abdelhay, J. See Dieudonné, J.

Artobolevskiĭ, I. I. *Teoriya mehanizmov i mašin.* 2d ed. Moscow, Gosudarstvennoe Izdatel'stvo Tehniko-Teoretičeskoi Literatury, 1952. 704 pp. 15.75 rubles.

Atti del Quarto Congresso dell' Unione Mathematica Italiana, Taormina, 1951. Vol. I, 324 pp.; Vol. II, 684 pp. Rome, Casa Editrice Perrella, 1953. 9000 lire.

Bagemihl, F. See Knopp, K.

Barriol, J. *Mécanique quantique.* Paris, Presses Universitaires de France, 1952. 12+257 pp. 1,500 fr.

Birkhoff, G., and MacLane, S. *A survey of modern algebra.* Rev. ed. New York, Macmillan, 1953. 11+472 pp. $6.50.

Bourbaki, N. *Éléments de mathématique.* XVI. Part I. *Les structures fondamentales de l'analyse.* Book III. *Topologie générale (fascicule de résultats).* (Actualités Scientifiques et Industrielles, no. 1196.) Paris, Hermann, 1953. 95 pp. 1500 fr.

Bruins, E. M. *Nouvelles découvertes sur les mathématiques babyloniennes.* Université de Paris, 1952. 29 pp.

Contributions to the theory of Riemann surfaces. (Annals of Mathematics Studies, no 30.) Princeton University Press, 1953. 264 pp. $4.00.

Diaz, J. B. *Inequalities and minimal principles in mathematical physics.* College Park, Md., The Institute for Fluid Dynamics and Applied Mathematics, University of Maryland, 1952. 2+55 pp. (mimeographed)

Dieudonné, J. *Análise harmônica.* Notes prepared by J. Abdelhay. Rio de Janeiro, Universidade do Brasil, 1952. 3+212 pp.

Dubnov, Y. S. *Osnovy vektornogo isčisleniya.* Part II. *Lineĭnye funkcii vektora. Vektornyĭ analiz (teoriya poleĭ). Načala tenzornogo isčisleniya.* Moscow, Gosudarstvennoe Izdatel'stvo Tehniko-Teoretičeskoi Literatury, 1952. 415 pp. 10.35 rubles.

Duschek, A. *Vorlesungen über höhere Mathematik.* Vol. III. *Gewöhnliche und partielle Differentialgleichungen. Variationsrechnung. Funktionen einer komplexen Veränderlichen.* Vienna, Springer, 1953. 9+512 pp. $9.25.

Ekstein, M. G. See Parrish, W.

Faddeev, D. K., and Sominskiĭ, I. S. *Sbornik zadač po vysšeĭ algebre.* 3d ed. Moscow, Gosudarstvennoe Izdatel'stvo Tehniko-Teoretičeskoi Literatury, 1952. 308 pp. 7.20 rubles.

Flora, F. See Galilei, G.

Friedrichs, K. O. *Mathematical aspects of the quantum theory of fields.* New York, Interscience, 1953. 8+272 pp. $5.00.

Funk, P., Sagan, H., and Selig, F. *Die Laplace-Transformation und ihre Anwendung.* Vienna, Franz Deuticke, 1953. 7+106 pp. $2.40.

Galilei, G. *Opere.* Ed. by F. Flora, Milan, Riccardo Ricciardi, 1953. 29+1139 pp.

Gel'fond, A. O. *Transcendentnye i algebraičeskie čisla.* Moscow, Gosudarstvennoe Izdatel'stvo Tehniko-Teoretičeskoi Literatury, 1952. 224 pp. 7.20 rubles.

Grodzinski, P. *Getriebelehre.* I. *Geometrische Grundlagen.* 2d ed. (Sammlung Göschen, vol. 1061.) Berlin, de Gruyter, 1953. 159 pp. 2.40 DM.

Haag, J. *Les mouvements vibratoires.* Paris, Presses Universitaires de France, 1952. 268 pp. 1,600 fr.

Hedrick, E. R. See Klein, F.

Hildebrand, F. B. *Methods of applied mathematics.* New York, Prentice-Hall, 1952. 11+523 pp. $7.75.

HILTON, P. J. *An introduction to homotopy theory*. (Cambridge Tracts in Mathematics and Mathematical Physics.) Cambridge University Press, 1953. 8+136 pp.

HOBSON, E. W., HUDSON. H. P. SINGH, A. N., and KEMPE, A. B. *Squaring the Circle and other monographs*. New York, Chelsea, 1953. 4+57+4+143+7+110+3+51 pp. $3.25.

HUDSON, H. P. See HOBSON, E. W.

IRWIN, B. W. See PARRISH, W.

JOHN, F. *On behavior of solutions of partial differential equations*. College Park, Md., The Institute for Fluid Dynamics and Applied Mathematics, University of Maryland, 1953. 1+24 pp. (mimeographed)

KAPLAN, W. *A first course in functions of a complex variable*. Cambridge, Addison-Wesley, 1953. 7+485–619 pp. $3.50.

KAPLAN, W. *Advanced calculus*. Cambridge, Addison-Wesley, 1952. 13+679 pp. $8.50.

KELLEY, J. L. See MCSHANE, E. J.

KEMPE, A. B. See HOBSON, E. W.

KIMBALL, W. S. *Calculus of variations by parallel displacement*. London, Butterworths Scientific Publications, 1952. 8+543 pp. $7.50.

KINOKUNIYA, Y. *A course of Radonian calculus*. Sapporo, Tanne Trading Co., 1953. 1+36 pp. $0.30.

KLEIN, F. *Elementary mathematics from an advanced standpoint. Arithmetic-algebra-analysis*. Trans. by E. R. Hedrick and C. A. Noble. New York, Dover, 1953. 9+274 pp. Cloth, $3.25; paper, $1.50.

KNOPP, K. *Problem book in the theory of functions*. Vol. II. *Problems in the advanced theory of functions*. Trans. by F. Bagemihl. New York, Dover, 1953. 138 pp. Cloth, $2.50; paper, $1.25.

KUESSNER, H. G. *A review of the two-dimensional problem of unsteady lifting surface theory during the last thirty years*. College Park, Md., The Institute for Fluid Dynamics and Applied Mathematics, University of Maryland, 1953. 1+22 pp. (mimeographed)

KUESSNER, H. Q. *On the mathematical treatment of movements of the earth's crust*. College Park, Md., The Institute for Fluid Dynamics and Applied Mathematics, University of Maryland, 1953. 1+20 pp. (mimeographed)

LANGER, S. K. *An introduction to symbolic logic*. 2d ed. New York, Dover, 1953. 367 pp. Cloth, $3.50; paper $1.60.

LEFSCHETZ, S. *Algebraic geometry*. Princeton University Press, 1953. 9+233 pp. $5.00.

LENSE, J. *Reihenentwicklungen in der mathematischen Physik*. 3d ed. Berlin, de Gruyter, 1953. 216 pp. 26 DM.

LICHNEROWICZ, A. *Etude mathématique des théories relativistes de la gravitation et de l'électromagnétisme*. I. *Relativité générale classique*. Paris, 1953. 149 pp. (mimeographed)

MCCREA, W. H. *Analytical geometry of three dimensions*. 2d ed. New York, Interscience, 1953. 7+144 pp. $1.25.

MCSHANE, E. J. *Order-preserving maps and integration processes*. (Annals of Mathematics Studies, no. 31.) Princeton University Press, 1953. 6+136 pp. $2.75.

MCSHANE, E. J., KELLEY, J. L., and RENO, F. V. *Exterior ballistics*. University of Denver Press, 1953. 12+834 pp., 1 plate. $12.00.

MACLANE, S. See BIRKHOFF, G.

NOBLE, C. A. See KLEIN, F.

PARRISH, W., EKSTEIN, M. G., and IRWIN, B. W. *Data for X-ray analysis*. Vol. II. *Tables for computing the lattice constant of cubic crystals*. Eindhoven, Philips' Technical Library, 1953. 81 pp.

PARRISH, W., and IRWIN, B. W. *Data for X-ray analysis*. Vol. I. *Charts for solution of Bragg's equation* (*d versus* Θ *and* 2Θ. Eindhoven, Philips' Technical Library, 1953. 99 pp.

PÉRÈS, J. *Mécanique générale*. Paris, Masson, 1953. 4+407 pp. 2545 fr.

PICKERT, G. *Analytische Geometrie*. (Mathematik und ihre Anwendungen in Physik und Technik, Series A, vol. 24.) Leipzig, Geest and Portig, 1953. 10+398 pp. 26.00 DM.

POLUBARINOVA-KOČINA, P. YA. *Teoriya dviženiya gruntovyh vod*. Moscow, Gosudarstvennoe Izdatel'stvo Tehniko-Teoretičeskoĭ Literatury, 1952. 676 pp. 13.85 rubles.

Probability tables for the analysis of extreme-value data. (National Bureau of Standards Applied Mathematics Series, no. 22.) Washington, Government Printing Office, 1953. 3+32 pp. $0.25.

RENO, F. V. See MCSHANE, E. J.

ROMIG, H. G. 50–100 *binomial tables*. New York, Wiley, 1953. 27+172 pp. $4.00.

SAARNIO, U. *Die Wohlordnung einer nichtabzählbaren Menge und die Lösung des Kontinuumproblems*. (Abhandlungen der Gesellschaft für Logik und ihre Anwendungen, no. 1.) Helsinki, 1953. 59 pp.

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Melvin Henriksen, *On the prime ideals of the ring of entire functions*, p. 66, abstract 82.

Line 1 of the abstract: For "nonmaximal ideals" read "nonmaximal prime ideals."

Line 6 of the abstract: For "The class P of all prime ideals" read "The class P_m of all prime ideals P."

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